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GROUND STATE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH ZERO MASS IN $\mathbb{R}^N$

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Abstract. In this article, we study the semilinear elliptic equation

$$\begin{gathered} -\Delta u = |u|^{p(x)-2}u, \quad x \in \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N), \end{gathered}$$

where $N \geq 3$, $p(x) = \begin{cases} p, & x \in \Omega \\ 2^*, & x \notin \Omega, \end{cases}$ with $2 < p < 2^* := 2N/(N-2)$, $\Omega \subset \mathbb{R}^N$ is a bounded set with nonempty interior. By using the Nehari manifold, we obtain a positive ground state solution.

1. Introduction and statement of main result

Considering the semilinear elliptic equation

$$\begin{gathered} -\Delta u = |u|^{2^*-2}u + \chi_\Omega(x)(|u|^{p-2}u - |u|^{2^*-2}u), \quad x \in \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $N \geq 3$, $2 < p < 2^* := 2N/(N-2)$, $\Omega \subset \mathbb{R}^N$ is a bounded set with nonempty interior and

$$\chi_\Omega(x) = \begin{cases} 1, & x \in \Omega \\ 0. & x \notin \Omega \end{cases} \tag{1.2}$$

The well-known semilinear elliptic equation with zero mass is

$$\begin{gathered} -\Delta u = |u|^{2^*-2}u, \quad x \in \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N), \end{gathered} \tag{1.3}$$

where $N \geq 3$, which has been studied very intensely (see [6,10,18]) and the explicit expression of positive solutions was given. Of course, the semilinear elliptic equation with zero mass whose nonlinear term with subcritical growth has also been investigated by many authors, for example [4,7,8,9].

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By a transformation, (1.1) is equivalent to the following elliptic equation with variable exponent
$$
\begin{gathered}
-\Delta u=|u|^{p(x)-2}u, \quad x\in\mathbb{R}^N \\
u\in D^{1,2}(\mathbb{R}^N),
\end{gathered} \tag{1.4}
$$
where
$$
p(x)=\begin{cases} p, & x\in\Omega \\ 2^*. & x\notin\Omega \end{cases}
$$
The equations with variable exponent appear in various mathematical models, for example: electrorheological fluids [1, 17], nonlinear Darcy's law in porous medium [5], image processing [12]. Recently, these equations have been investigated by many authors, see for example, [2, 11, 13, 14, 16]. However, they did not consider problem (1.4); thus we study it in this article. Our main result reads as follows.

Theorem 1.1. *Assume that $N\geq 3$, $2<p<2^*$ and χ_Ω satisfies* (1.2). *Then equation* (1.1) *has a positive ground state solution.*

If $p\in C(\mathbb{R}^N,[2,2^*])$ and $p\not\equiv 2^*$, we do not yet know whether equation (1.1) has solution. We shall consider it in the future.

This article is organized as follows. Section 2 contains some preliminaries. Section 3 gives the proof of theorem 1.1.

2. Preliminaries

In what follows, we use the following notation.

- $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm
$$
\|u\|^2_{D^{1,2}(\mathbb{R}^N)}=\int_{\mathbb{R}^N}|\nabla u|^2\,dx.
$$
- $L^t(\mathbb{R}^N)$, $2\leq t<+\infty$, denotes a Lebesgue space endowed with the norm
$$
|u|_t^t=\int_{\mathbb{R}^N}|u|^t\,dx.
$$
- S denotes the best constant of Sobolev embedding $D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$; that is,
$$
S|u|^2_{2^*}\leq\|u\|^2_{D^{1,2}(\mathbb{R}^N)} \quad \text{for all } u\in D^{1,2}(\mathbb{R}^N).
$$
- D^{-1} is the dual space of $D^{1,2}(\mathbb{R}^N)$.
- C, C_i denote various positive constants.

For equation (1.1), the energy functional $I:D^{1,2}(\mathbb{R}^N)\to\mathbb{R}$ is defined by
$$
\begin{aligned}
I(u)&=\frac12\|u\|^2_{D^{1,2}(\mathbb{R}^N)}-\frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\,dx-\frac1p\int_{\mathbb{R}^N}\chi_\Omega(x)|u|^p\,dx \\
&\quad+\frac{1}{2^*}\int_{\mathbb{R}^N}\chi_\Omega(x)|u|^{2^*}\,dx \\
&=\frac12\|u\|^2_{D^{1,2}(\mathbb{R}^N)}-\frac1p\int_\Omega|u|^p\,dx-\frac{1}{2^*}\int_{\mathbb{R}^N\setminus\Omega}|u|^{2^*}\,dx.
\end{aligned}
$$
The Hölder and Sobolev inequalities imply
$$
\begin{aligned}
\int_\Omega|u|^p\,dx&\leq|u|^p_{2^*,\Omega}(\operatorname{meas}\Omega)^{\frac{2^*-p}{2^*}}\leq|u|^p_{2^*}(\operatorname{meas}\Omega)^{\frac{2^*-p}{2^*}} \\
&\leq S^{-\frac p2}(\operatorname{meas}\Omega)^{\frac{2^*-p}{2^*}}\|u\|^p_{D^{1,2}(\mathbb{R}^N)},
\end{aligned} \tag{2.1}
$$

where
$$|u|_{s,\Omega} = \Big(\int_\Omega |u|^t\,dx\Big)^{1/t}, \quad \forall t \in [1,+\infty).$$
Thus the functional I is well defined. By Lemma 3.1 in next section, I is of class $C^1(D^{1,2}(\mathbb{R}^N),\mathbb{R})$ and satisfies
$$\begin{aligned}
\langle I'(u),v\rangle &= \int_{\mathbb{R}^N} \nabla u\cdot\nabla v\,dx - \int_{\mathbb{R}^N} |u|^{2^*-2}uv\,dx - \int_{\mathbb{R}^N} \chi_\Omega(x)|u|^{p-2}uv\,dx \\
&\quad + \int_{\mathbb{R}^N} \chi_\Omega(x)|u|^{2^*-2}uv\,dx \\
&= \int_{\mathbb{R}^N} \nabla u\cdot\nabla v\,dx - \int_\Omega |u|^{p-2}uv\,dx - \int_{\mathbb{R}^N\setminus\Omega} |u|^{2^*-2}uv\,dx,
\end{aligned} \tag{2.2}$$
for all $u,v \in D^{1,2}(\mathbb{R}^N)$. Hence weak solutions of (1.1) correspond to the critical point of the functional I. Define
$$\mathcal{N} := \{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\} : J(u) = 0\}, \quad m := \inf_{u\in\mathcal{N}} I(u),$$
where
$$J(u) = \|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \int_\Omega |u|^p\,dx - \int_{\mathbb{R}^N\setminus\Omega} |u|^{2^*}\,dx.$$
Since all solutions of (1.1) belong to the manifold $\mathcal{N}$, first we seek for the minimizer u for m and then we prove u is a solution of equation (1.1).

3. Proof of Theorem 1.1

The proof relies on the following lemmas.

Lemma 3.1. *Assume that $N \ge 3$, $2 < p < 2^*$ and χ_Ω satisfies (1.2). Then the functional I is of class $C^1(D^{1,2}(\mathbb{R}^N),\mathbb{R})$ and $I'(\cdot)$ satisfies (2.2).*

Proof. Define
$$\psi(u) = \frac{1}{p}\int_\Omega |u|^p\,dx + \frac{1}{2^*}\int_{\mathbb{R}^N\setminus\Omega} |u|^{2^*}\,dx,$$
we need only to prove $\psi \in C^1(D^{1,2}(\mathbb{R}^N),\mathbb{R})$. Let $u,h \in D^{1,2}(\mathbb{R}^N)$. Given $x \in \Omega$ and $0 < |t| < 1$, by the mean value theorem, there exists $\lambda_1 \in (0,1)$ such that
$$\frac{\big||u+th|^p - |u|^p\big|}{t} = p|u+\lambda_1 th|^{p-1}|h| \le p(|u|+|h|)^{p-1}|h|.$$
Similarly, given $x \in \mathbb{R}^N\setminus\Omega$ and $0 < |t| < 1$, there exists $\lambda_2 \in (0,1)$ such that
$$\frac{\big||u+th|^{2^*} - |u|^{2^*}\big|}{t} = 2^*|u+\lambda_2 th|^{2^*-1}|h| \le 2^*(|u|+|h|)^{2^*-1}|h|.$$
The Hölder inequality implies that
$$\begin{aligned}
\int_\Omega (|u|+|h|)^{p-1}|h|\,dx &\le \Big(\int_\Omega (|u|+|h|)^p\,dx\Big)^{\frac{p-1}{p}} |h|_{p,\Omega} \\
&\le (|u|_{p,\Omega} + |h|_{p,\Omega})^{p-1}|h|_{p,\Omega} < +\infty
\end{aligned}$$
and
$$\int_{\mathbb{R}^N\setminus\Omega} (|u|+|h|)^{2^*-1}|h|\,dx \le \Big(\int_{\mathbb{R}^N\setminus\Omega} (|u|+|h|)^{2^*}\,dx\Big)^{\frac{2^*-1}{2^*}} |h|_{2^*,\mathbb{R}^N\setminus\Omega}$$

$$\le (|u|_{2^*,\mathbb{R}^N\setminus\Omega} + |h|_{2^*,\mathbb{R}^N\setminus\Omega})^{2^*-1}|h|_{2^*,\mathbb{R}^N\setminus\Omega} < +\infty .$$

It follows from the Lebesgue theorem that

$$\begin{aligned}
\langle \psi'(u), h\rangle &= \lim_{t\to 0} \frac{\psi(u+th)-\psi(u)}{t}\\
&= \lim_{t\to 0}\int_\Omega \frac{|u+th|^p - |u|^p}{pt}\,dx + \lim_{t\to 0}\int_{\mathbb{R}^N\setminus\Omega} \frac{|u+th|^{2^*}-|u|^{2^*}}{2^* t}\,dx\\
&= \int_\Omega \lim_{t\to 0}\frac{|u+th|^p - |u|^p}{pt}\,dx + \int_{\mathbb{R}^N\setminus\Omega}\lim_{t\to 0} \frac{|u+th|^{2^*}-|u|^{2^*}}{2^* t}\,dx\\
&= \int_\Omega |u|^{p-2}uh\,dx + \int_{\mathbb{R}^N\setminus\Omega} |u|^{2^*-2}uh\,dx.
\end{aligned}$$

Assume that $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$, then $u_n \to u$ in $L^{2^*}(\mathbb{R}^N)$ and $L^p(\Omega)$. If follows from [19, Theorem A.2 and A.4] that $|u_n|^{p-2}u_n \to |u|^{p-2}u$ in $L^{\frac{p}{p-1}}(\Omega)$ and $|u_n|^{2^*-2}u_n \to |u|^{2^*-2}u$ in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N\setminus\Omega)$. Hence combining the Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}
&\|\psi'(u_n)-\psi'(u)\|_{D^{-1}}\\
&\le \sup_{\|\varphi\|_{D^{1,2}(\mathbb{R}^N)}=1,\varphi\in D^{1,2}(\mathbb{R}^N)} \Big|\int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)\varphi\,dx\Big|\\
&\quad + \sup_{\|\varphi\|_{D^{1,2}(\mathbb{R}^N)}=1,\varphi\in D^{1,2}(\mathbb{R}^N)} \Big|\int_{\mathbb{R}^N\setminus\Omega} (|u_n|^{2^*-2}u_n - |u|^{2^*-2}u)\varphi\,dx\Big|\\
&\le C\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|_{\frac{p}{p-1},\Omega} + C\big||u_n|^{2^*-2}u_n - |u|^{2^*-2}u\big|_{\frac{2^*}{2^*-1},\mathbb{R}^N\setminus\Omega}\\
&= o(1).
\end{aligned}$$

Thus ψ is C^1. It is obvious that $I'(\cdot)$ satisfies (2.2). The proof is complete. □

Lemma 3.2. *Assume that $N \ge 3$, $2 < p < 2^*$ and χ_Ω satisfies* (1.2). *Then for any $u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$, there exists $t_u > 0$ such that $t_u u \in \mathcal{N}$.*

Proof. For any $u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$, define

$$f(t) := I(tu) = \frac{t^2}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{t^p}{p}\int_\Omega |u|^p\,dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N\setminus\Omega} |u|^{2^*}\,dx, \quad \forall t \in (0,+\infty).$$

Then one has

$$f'(t)t = \langle I'(tu), tu\rangle = t^2\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - t^p\int_\Omega |u|^p\,dx - t^{2^*}\int_{\mathbb{R}^N\setminus\Omega}|u|^{2^*}\,dx.$$

Combining $2 < p < 2^*$, we have $f'(t)t > 0$ for $t > 0$ small enough and $f'(t)t < 0$ for $t > 0$ large enough. Thus there exists $t_u > 0$ such that $f'(t_u)t_u = \langle I'(t_u u), t_u u\rangle = 0$. That is $t_u u \in \mathcal{N}$. The proof is complete. □

Lemma 3.3. *Assume that $N \ge 3$, $2 < p < 2^*$ and χ_Ω satisfies* (1.2). *Then $m > 0$.*

Proof. For any $u \in \mathcal{N}$, one has

$$\begin{aligned}
\|u\|^2_{D^{1,2}(\mathbb{R}^N)} &= \int_\Omega |u|^p\,dx + \int_{\mathbb{R}^N\setminus\Omega}|u|^{2^*}\,dx\\
&\le C\|u\|^p_{D^{1,2}(\mathbb{R}^N)} + C\|u\|^{2^*}_{D^{1,2}(\mathbb{R}^N)},
\end{aligned}$$

which implies that there exists $\alpha > 0$ such that

$$\|u\|_{D^{1,2}(\mathbb{R}^N)} \geq \alpha, \quad \forall u \in \mathcal{N}. \tag{3.1}$$

Thus for any $u \in \mathcal{N}$, we have

$$\begin{aligned} I(u) &= I(u) - \frac{1}{p}\langle I'(u), u\rangle \\ &= \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \big(\frac{1}{p} - \frac{1}{2^*}\big)\int_{\mathbb{R}^N\setminus\Omega} |u|^{2^*}\,dx \\ &\geq \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} \\ &\geq \big(\frac{1}{2} - \frac{1}{p}\big)\alpha^2. \end{aligned} \tag{3.2}$$

Hence $m > 0$. The proof is complete. □

Lemma 3.4. *Assume that $N \geq 3$, $2 < p < 2^*$ and χ_Ω satisfies* (1.2). *Then for any $u \in \mathcal{N}$, $J'(u) \neq 0$.*

Proof. By (3.1), for any $u \in \mathcal{N}$, one has

$$\begin{aligned} \langle J'(u), u\rangle &= \langle J'(u), u\rangle - pJ(u) \\ &= (2-p)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - (2^* - p)\int_{\mathbb{R}^N\setminus\Omega} |u|^{2^*}\,dx \\ &\leq (2-p)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} \\ &\leq (2-p)\alpha^2 < 0. \end{aligned} \tag{3.3}$$

Hence the proof is complete. □

Lemma 3.5. *Assume that $N \geq 3$, $2 < p < 2^*$ and χ_Ω satisfies* (1.2). *Suppose that $u \in \mathcal{N}$ and $I(u) = m$. Then u is a solution of* (1.1).

Proof. Assume that $u \in \mathcal{N}$ and $I(u) = m$. Then by the Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that $I'(u) = \lambda J'(u)$, which implies that $0 = \langle I'(u), u\rangle = \lambda\langle J'(u), u\rangle$. By Lemma 3.4, we obtain $\lambda = 0$. Hence $I'(u) = 0$. The proof is complete. □

Lemma 3.6. *Assume that $N \geq 3$, $2 < p < 2^*$ and χ_Ω satisfies* (1.2). *Then there exists a bounded sequence $\{u_n\} \subset \mathcal{N}$ satisfying $I(u_n) \to m$ and $I'(u_n) \to 0$ in D^{-1}.*

Proof. By the Ekeland variational principle in [19], there exist $\{u_n\} \subset \mathcal{N}$ and $\{\lambda_n\} \subset \mathbb{R}$ such that $I(u_n) \to m$ and $I'(u_n) - \lambda_n J'(u_n) \to 0$ in D^{-1}. By (3.2), one has

$$I(u_n) = I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \geq \Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)},$$

which implies $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Then we have

$$0 = \langle I'(u_n), u_n\rangle = \lambda_n\langle J'(u_n), u_n\rangle + o(1).$$

Combining (3.3), we obtain $\lambda_n \to 0$. For any $\varphi \in D^{1,2}(\mathbb{R}^N)$ with $\|\varphi\|_{D^{1,2}(\mathbb{R}^N)} = 1$, it follows from (2.1), the Hölder and Sobolev inequalities that

$$\Big|\int_\Omega |u_n|^{p-2}u_n\varphi\,dx\Big| \leq \Big(\int_\Omega |u_n|^p\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |\varphi|^p\Big)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\le \Big(S^{-\frac{p}{2}}(\operatorname{meas}\Omega)^{\frac{2^*-p}{2^*}}\|u_n\|^p_{D^{1,2}(\mathbb{R}^N)}\Big)^{\frac{p-1}{p}}\Big(|\varphi|^p_{2^*}(\operatorname{meas}\Omega)^{\frac{2^*-p}{2^*}}\Big)^{\frac{1}{p}}\\
&\le C\|u_n\|^{p-1}_{D^{1,2}(\mathbb{R}^N)}\|\varphi\|_{D^{1,2}(\mathbb{R}^N)}\le C.
\end{aligned}
$$

Thus combining the Hölder and Sobolev inequalities, we have

$$
\begin{aligned}
\|J'(u_n)\|_{D^{-1}} &= \sup_{\|\varphi\|_{D^{1,2}(\mathbb{R}^N)}=1,\varphi\in D^{1,2}(\mathbb{R}^N)} |\langle J'(u_n),\varphi\rangle|\\
&= \sup_{\|\varphi\|_{D^{1,2}(\mathbb{R}^N)}=1,\varphi\in D^{1,2}(\mathbb{R}^N)} \Big|2\int_{\mathbb{R}^N}\nabla u_n\cdot\nabla\varphi\,dx - p\int_\Omega |u_n|^{p-2}u_n\varphi\,dx\\
&\quad - 2^*\int_{\mathbb{R}^N\setminus\Omega}|u_n|^{2^*-2}u_n\varphi\,dx\Big|\\
&\le \sup_{\|\varphi\|_{D^{1,2}(\mathbb{R}^N)}=1,\varphi\in D^{1,2}(\mathbb{R}^N)} [2\|u_n\|_{D^{1,2}(\mathbb{R}^N)}\|\varphi\|_{D^{1,2}(\mathbb{R}^N)} + C\\
&\quad + C\|u_n\|^{2^*-1}_{D^{1,2}(\mathbb{R}^N)}\|\varphi\|_{D^{1,2}(\mathbb{R}^N)}] \le C.
\end{aligned}
$$

Hence we obtain

$$
\|I'(u_n)\|_{D^{-1}} \le \|I'(u_n)-\lambda_nJ'(u_n)\|_{D^{-1}} + |\lambda_n|\|J'(u_n)\|_{D^{-1}} = o(1).
$$

The proof is complete. □

If $p(x)\equiv 2^*$, equation (1.4) reduces to (1.3). It is well known that (1.3) has ground state solution

$$
v(x) = \frac{C_N}{(1+|x|^2)^{\frac{N-2}{2}}}, \tag{3.4}
$$

where $C_N := [N(N-2)]^{\frac{N-2}{4}}$ and v satisfies

$$
\int_{\mathbb{R}^N}|\nabla v|^2\,dx = \int_{\mathbb{R}^N}|v|^{2^*}\,dx = S^{N/2}.
$$

Let the energy functional of (1.3) be

$$
I_\infty(u) = \frac{1}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\,dx.
$$

Then we have

$$
I_\infty(v) = I_\infty(v) - \frac{1}{2^*}\langle I'_\infty(v),v\rangle = \frac{1}{N}\int_{\mathbb{R}^N}|\nabla v|^2\,dx = \frac{1}{N}S^{N/2}.
$$

Now for the energy m, we make the following estimation.

Lemma 3.7. *Assume that $N\ge 3$, $2<p<2^*$ and χ_Ω satisfies* (1.2). *Then $m<\frac{1}{N}S^{N/2}$.*

Proof. Inspired by the idea in [3,15]. For ground state solution v of equation (1.3), we define $v_n(x) := v(x+x_n)$, where $x_n := (0,0,\dots,0,n)$. Thus $\|v_n\|_{D^{1,2}(\mathbb{R}^N)} = \|v\|_{D^{1,2}(\mathbb{R}^N)} = S^{\frac{N}{4}}$ and then $v_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$, $v_n\to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$, $v_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$. Since for any $x\in\mathbb{R}^N$, $v_n(x)\to 0$, $u=0$. By Lemma 3.2, there exists $t_n\in(0,+\infty)$ such that $t_nv_n\in\mathcal{N}$. Then one has

$$
\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} = t_n^{p-2}\int_\Omega |v_n|^p\,dx + t_n^{2^*-2}\int_{\mathbb{R}^N\setminus\Omega}|v_n|^{2^*}\,dx. \tag{3.5}
$$

By (2.1) and (3.5), one has

$$\begin{aligned}
\|v\|^2_{D^{1,2}(\mathbb{R}^N)} &= \|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} \\
&\le C(t_n^{p-2}\|v_n\|^p_{D^{1,2}(\mathbb{R}^N)} + t_n^{2^*-2}\|v_n\|^{2^*}_{D^{1,2}(\mathbb{R}^N)}) \\
&= C(t_n^{p-2}\|v\|^p_{D^{1,2}(\mathbb{R}^N)} + t_n^{2^*-2}\|v\|^{2^*}_{D^{1,2}(\mathbb{R}^N)}),
\end{aligned}$$

which indicates that t_n cannot appraoch zero, that is $t_n \ge t_0$ for some $t_0 > 0$. Since Ω is bounded, there exists $R > 0$ such that $\Omega \subset B_R := \{x \in \mathbb{R}^N : |x| < R\}$. Since for n large enough,

$$\int_{|x-x_n|<R} \frac{1}{(1+|x|^2)^N}\,dx \le \int_{|x-x_n|<R} \frac{2^N}{n^{2N}}\,dx = \frac{2^N}{n^{2N}} \operatorname{meas} B_R = o(1), \tag{3.6}$$

we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |v|^{2^*}\,dx &= \int_{\mathbb{R}^N} |v_n|^{2^*}\,dx \\
&\ge \int_{\mathbb{R}^N\setminus\Omega} |v_n|^{2^*}\,dx \\
&\ge \int_{|x|\ge R} |v_n|^{2^*}\,dx \\
&= C_N^{2^*} \int_{|x|\ge R} \frac{1}{(1+|x+x_n|^2)^N}\,dx \\
&= C_N^{2^*} \int_{|x-x_n|\ge R} \frac{1}{(1+|x|^2)^N}\,dx \\
&= C_N^{2^*} \int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)^N}\,dx - C_N^{2^*} \int_{|x-x_n|<R} \frac{1}{(1+|x|^2)^N}\,dx \\
&= \int_{\mathbb{R}^N} |v|^{2^*}\,dx + o(1).
\end{aligned}$$

Thus one has

$$\int_{\mathbb{R}^N\setminus\Omega} |v_n|^{2^*}\,dx = \int_{\mathbb{R}^N} |v|^{2^*}\,dx + o(1) = S^{N/2} + o(1). \tag{3.7}$$

It follows from (3.5) that

$$\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} \Big(\int_{\mathbb{R}^N\setminus\Omega} |v_n|^{2^*}\,dx\Big)^{-1} \ge t_n^{2^*-2},$$

which implies

$$\limsup_{n\to\infty} t_n^{2^*-2} \le \|v\|^2_{D^{1,2}(\mathbb{R}^N)} S^{-\frac{N}{2}} = 1.$$

Thus up to a subsequence, one has $t_n \to T \in (t_0, 1]$. Notice that

$$\int_\Omega |v_n|^p\,dx = o(1).$$

By (3.5) and (3.7), one has $S^{N/2} = T^{2^*-2}S^{N/2}$. Thus $T = 1$. From (3.7) it follows that

$$\int_\Omega |v_n|^{2^*}\,dx = o(1).$$

We claim that

$$\frac{\frac{t_n^{2^*}}{2^*}\int_\Omega |v_n|^{2^*}\,dx}{\frac{t_n^p}{p}\int_\Omega |v_n|^p\,dx}\to 0.$$

Indeed, by (3.4) and (3.6), for n large enough, one has

$$\begin{aligned}
\int_\Omega |v_n|^{2^*}dx &\le C_N^{2^*}\int_{B_R}\frac{1}{(1+|x+x_n|^2)^N}\,dx\\
&= C_N^{2^*}\int_{|x-x_n|<R}\frac{1}{(1+|x|^2)^N}\,dx\\
&= \frac{C_N^{2^*}2^N}{n^{2N}}\operatorname{meas} B_R.
\end{aligned}$$

Since the interior of Ω is nonempty, there exist $z_0\in\mathbb{R}^N$ and $r>0$ such that $B_r(z_0):=\{x\in\mathbb{R}^N:|x-z_0|<r\}\subset\Omega$. Thus for n large enough, one has

$$\begin{aligned}
\int_\Omega |v_n|^p\,dx &\ge C_N^p\int_{B_r(z_0)}\frac{1}{(1+|x+x_n|^2)^{\frac{(N-2)p}{2}}}\,dx\\
&\ge C_N^p\int_{B_r(z_0)}\frac{1}{2^{\frac{(N-2)p}{2}}n^{(N-2)p}}\,dx\\
&= \frac{C_N^p 2^{\frac{(2-N)p}{2}}}{n^{(N-2)p}}\operatorname{meas} B_r.
\end{aligned}$$

Then we obtain

$$\frac{\int_\Omega |v_n|^{2^*}\,dx}{\int_\Omega |v_n|^p\,dx}\le C_N'\frac{1}{n^{2N-(N-2)p}}=o(1),$$

since $p<\frac{2N}{N-2}$. Combining $t_n\to 1$, we implies the claim holds. By calculations, one has

$$\frac{\frac{1-t_n^2}{2}}{\frac{1-t_n^{2^*}}{2^*}}=\frac{2^*(1-t_n^2)}{2(1-t_n^{2^*})}\to 1.$$

Recall that $t_nv_n\in\mathcal{N}$. Hence for n large enough, one has

$$\begin{aligned}
m &\le I(t_nv_n)\\
&= \frac{t_n^2}{2}\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)}-\frac{t_n^p}{p}\int_\Omega |v_n|^p\,dx-\frac{t_n^{2^*}}{2^*}\int_{\mathbb{R}^N\setminus\Omega}|v_n|^{2^*}\,dx\\
&= \frac{t_n^2}{2}\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)}-\frac{t_n^{2^*}}{2^*}\int_{\mathbb{R}^N}|v_n|^{2^*}\,dx-\frac{t_n^p}{p}\int_\Omega |v_n|^p\,dx+\frac{t_n^{2^*}}{2^*}\int_\Omega |v_n|^{2^*}\,dx\\
&= I_\infty(v_n)-\frac{1-t_n^2}{2}\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)}+\frac{1-t_n^{2^*}}{2^*}\int_{\mathbb{R}^N}|v_n|^{2^*}\,dx\\
&\quad -\frac{t_n^p}{p}\int_\Omega |v_n|^p\,dx+\frac{t_n^{2^*}}{2^*}\int_\Omega |v_n|^{2^*}\,dx\\
&= S^{N/2}+\Big(\frac{1-t_n^{2^*}}{2^*}-\frac{1-t_n^2}{2}\Big)S^{N/2}-\frac{t_n^p}{p}\int_\Omega |v_n|^p\,dx+\frac{t_n^{2^*}}{2^*}\int_\Omega |v_n|^{2^*}\,dx\\
&< S^{N/2}.
\end{aligned}$$

The proof is complete. $\square$

Lemma 3.8. *Assume that $N \geq 3$, $2 < p < 2^*$ and χ_Ω satisfies* (1.2). *Suppose that the sequence $\{u_n\} \subset \mathcal{N}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and satisfies $I(u_n) \to m < \frac{1}{N} S^{N/2}$ and $I'(u_n) \to 0$ in D^{-1}. Then there exists $u \in D^{1,2}(\mathbb{R}^N)$ such that up to a subsequence, $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$.*

Proof. Since $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ is a bounded, up to a subsequence, there exists $u \in D^{1,2}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$, $u_n \to u$ in $L^p(\Omega)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. For any $v \in D^{1,2}(\mathbb{R}^N)$, by $I'(u_n) \to 0$ in D^{-1}, one has

$$\begin{aligned}
0 &= \langle I'(u_n), v\rangle + o(1) \\
&= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla v \, dx - \int_\Omega |u_n|^{p-2} u_n v \, dx - \int_{\mathbb{R}^N \setminus \Omega} |u_n|^{2^*-2} u_n v \, dx + o(1) \\
&= \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - \int_\Omega |u|^{p-2} u v \, dx - \int_{\mathbb{R}^N \setminus \Omega} |u|^{2^*-2} u v \, dx \\
&= \langle I'(u), v\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{p}\langle I'(u), u\rangle \\
&= \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \big(\frac{1}{p} - \frac{1}{2^*}\big) \int_{\mathbb{R}^N \setminus \Omega} |u|^{2^*} \, dx \\
&\geq \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|^2_{D^{1,2}(\mathbb{R}^N)} \geq 0.
\end{aligned}$$

Define $v_n = u_n - u$. Thus one has

$$\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} = \|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} + \|u\|^2_{D^{1,2}(\mathbb{R}^N)} + o(1).$$

The Brezis-Lieb lemma implies

$$\int_\Omega |u_n|^p \, dx = \int_\Omega |v_n|^p \, dx + \int_\Omega |u|^p \, dx + o(1) = \int_\Omega |u|^p \, dx + o(1)$$

and

$$\int_{\mathbb{R}^N \setminus \Omega} |u_n|^{2^*} \, dx = \int_{\mathbb{R}^N \setminus \Omega} |v_n|^{2^*} \, dx + \int_{\mathbb{R}^N \setminus \Omega} |u|^{2^*} \, dx + o(1).$$

Combining this with $I(u_n) \to m$, we obtain

$$\begin{aligned}
m &= \frac{1}{2}\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{1}{2^*} \int_{\mathbb{R}^N \setminus \Omega} |v_n|^{2^*} \, dx + I(u) + o(1) \\
&\geq \frac{1}{2}\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{1}{2^*} \int_{\mathbb{R}^N \setminus \Omega} |v_n|^{2^*} \, dx + o(1).
\end{aligned} \tag{3.8}$$

It follows from $\langle I'(u_n), u_n\rangle = 0$ and $I'(u) = 0$ that

$$\begin{aligned}
0 &= \|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N \setminus \Omega} |v_n|^{2^*} \, dx + \langle I'(u), u\rangle + o(1) \\
&= \|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N \setminus \Omega} |v_n|^{2^*} \, dx + o(1).
\end{aligned}$$

Up to a subsequence, we assume that

$$\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)} + o(1) = b = \int_{\mathbb{R}^N \setminus \Omega} |v_n|^{2^*} \, dx + o(1).$$

Thus we have

$$\begin{aligned} Sb^{2/2^*} &= S\Big(\int_{\mathbb{R}^N\setminus\Omega}|v_n|^{2^*}\,dx\Big)^{2/2^*}+o(1)\\ &\le S\Big(\int_{\mathbb{R}^N}|v_n|^{2^*}\,dx\Big)^{2/2^*}+o(1)\\ &\le \|v_n\|^2_{D^{1,2}(\mathbb{R}^N)}+o(1)=b. \end{aligned}$$

Assume that $b\neq 0$. Then one has $b\ge S^{N/2}$. From (3.8), we obtain

$$m\ge (\frac{1}{2}-\frac{1}{2^*})b\ge \frac{1}{N}S^{N/2},$$

which is a contradiction. Hence $b=0$, and the proof is complete. □

Proof of Theorem 1.1. By Lemmas 3.3, 3.6 and 3.7, there exists a bounded sequence $\{u_n\}\subset\mathcal{N}$ satisfying $I(u_n)\to m\in(0,\frac{1}{N}S^{N/2})$ and $I'(u_n)\to 0$ in D^{-1}. Lemma 3.8 implies that there exists $u\in D^{1,2}(\mathbb{R}^N)$ such that up to a subsequence, $u_n\to u$ in $D^{1,2}(\mathbb{R}^N)$. Then $I(u)=m$ and $J(u)=0$. That is, m is achieved by a function $u\in D^{1,2}(\mathbb{R}^N)$. Since $I(|u|)=I(u)$ and $J(|u|)=J(u)$, we can assume that u is nonnegative. Lemma 3.5 implies that $u\in D^{1,2}(\mathbb{R}^N)$ is a solution of equation (1.1). It follows from the definition of m that $u\in D^{1,2}(\mathbb{R}^N)$ is a ground state solution of equation (1.1). It follows from the strongly maximum principle that $u>0$. This completes the proof. □

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References

[1] E. Acerbi, G. Mingione; *Regularity results for stationary electrorheological fluids*, Arch. Ration. Mech. Anal. 164 (2002) 213–259.
[2] C. O. Alves; *Existence of solution for a degenerate $p(x)$-Laplacian equation in $\mathbb{R}^N$*, J. Math. Anal. Appl. 345 (2008) 731–742.
[3] C. O. Alves, P. C. Carrião, E. S. Medeiros; *Multiplicity of solutions for a class of a quasilinear problem in exterior domains with Neumann conditions*, Abstr. Appl. Anal. (2004) 251–268.
[4] C. O. Alves, M. A. S. Souto, M. Montenegro; *Existence of solution for two classes of elliptic problems in $\mathbb{R}^N$ with zero mass*, J. Differential Equations 252 (2012) 5735–5750.
[5] S. N. Antontsev, S. I. Shmarev; *Elliptic equations and systems with nonstandard growth conditions: Existence, uniqueness and localization properties of solutions*, Nonlinear Anal. 65 (2006) 722–755.
[6] T. Aubin; *Problémes isopérimétriques et espaces de Sobolev*, J. Differential Geom. 11 (1976) 573–598.
[7] A. Azzollini, A. Pomponio; *On a "zero mass" nonlinear Schrödinger equation*, Adv. Nonlinear Stud. 7 (2007) 599–628.
[8] A. Azzollini, A. Pomponio; *Compactness results and applications to some "zero mass" elliptic problems*, Nonlinear Anal. 69 (2008) 3559–3576.
[9] H. Berestycki, P. L. Lions; *Nonlinear scalar field equations, I-Existence of a ground state*, Arch. Ration. Mech. Anal. 82 (1983) 313–346.
[10] L. A. Caffarelli, B. Gidas, J. Spruck; *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42 (1989) 271–297.
[11] L. Calota; *On some quasilinear elliptic equations with critical Sobolev exponents and nonstandard growth conditions*, Bull. Belg. Math. Soc. Simon Stevin 15 (2) (2008) 249–256.

[12] A. Chambolle, P.L. Lions; *Image recovery via total variation minimization and related problems*, Numer. Math. 76 (1997) 167–188.
[13] R.A. Mashiyev, B. Cekic, M. Avci, Z. Yucedag; *Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition.* Complex Var. Elliptic Equ. 57 (2012) 579–595.
[14] M. Mihailescu, V. Radulescu; *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent*, Proc. Am. Math. Soc. 135 (2007) 2929–2937.
[15] M. Montenegro, R. Abreu; *Existence of a ground state solution for an elliptic problem with critical growth in an exterior domain.* Nonlinear Anal. 109 (2014) 341–349
[16] P. Pucci, Q. Zhang; *Existence of entire solutions for a class of variable exponent elliptic equations*, J. Differential Equations 257 (2014) 1529–1566.
[17] M. Ruzicka; *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000.
[18] G. Talenti; *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[19] M. Willem; *Minimax Theorems*, Birkhäuser, 1996.

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