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LYAPUNOV-TYPE INEQUALITIES FOR FRACTIONAL BOUNDARY-VALUE PROBLEMS

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Abstract. In this article, we establish some Lyapunov-type inequalities for fractional boundary-value problems under Sturm-Liouville boundary conditions. As applications, we obtain intervals where linear combinations of certain Mittag-Leffler functions have no real zeros. We deduce also nonexistence results for some fractional boundary-value problems.

1. Introduction

The well-known Lyapunov result [9] states that if a nontrivial solution to the boundary-value problem

$$\begin{gathered} u''(t) + q(t)u(t) = 0, \quad a < t < b, \\ u(a) = u(b) = 0, \end{gathered}$$

exists, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}\,.$$

This result found many practical applications in differential and difference equations (oscillation theory, disconjugacy, eigenvalue problems, etc.); see [1, 2, 11, 13, 14, 15] and references therein.

The search for Lyapunov-type inequalities in which the starting differential equation is constructed via fractional differential operators has begun very recently. The first work in this direction is due to Ferreira [4], where he derived a Lyapunov-type inequality for differential equations depending on the Riemann-Liouville fractional derivative; that is, for the boundary-value problem

$$\begin{gathered} ({}_aD^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2, \\ u(a) = u(b) = 0, \end{gathered}$$

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where ${}_aD^\alpha$ denotes the Riemann-Liouville fractional derivative of order α. Precisely, the author proved that if the above problem has a nontrivial solution, then

$$\int_a^b |q(s)|\,ds > \Gamma(\alpha)\big(\frac{4}{b-a}\big)^{\alpha-1}.$$

Clearly, if we let $\alpha = 2$ in the above inequality, one obtains Lyapunov's standard inequality. In [5], a Lyapunov-type inequality was obtained by the same author for the Caputo fractional boundary-value problem

$$\begin{gathered}({}_a^CD^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2,\\ u(a) = u(b) = 0,\end{gathered}$$

where ${}_a^CD^\alpha$ denotes the Caputo fractional derivative of order α. In this work, Ferreira proved that if the above problem has a nontrivial solution, then

$$\int_a^b |q(s)|\,ds > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}.$$

For other works on Lyapunov-type inequalities for fractional boundary-value problems we refer the reader to [6, 7].

Motivated by the above works, we consider a Caputo fractional differential equation with Sturm-Liouville boundary conditions. More precisely, we consider the fractional boundary-value problem

$$({}_a^CD^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha < 2, \tag{1.1}$$

with the boundary conditions

$$pu(a) - ru'(a) = u(b) = 0, \tag{1.2}$$

where $p > 0$, $r \ge 0$ and $q : [a, b] \to \mathbb{R}$ is a continuous function. We distinguish two cases: the case $\frac{r}{p} > \frac{b-a}{\alpha-1}$ and the case $0 \le \frac{r}{p} \le \frac{b-a}{\alpha-1}$. For each case, a Lyapunov-type inequality is derived. The obtained results recover several existing inequalities from the literature. As applications, we obtain intervals where linear combinations of certain Mittag-Leffler functions have no real zeros. We deduce also nonexistence results for some fractional boundary-value problems.

Before presenting our main results, let us start by recalling the concepts of the Riemann-Liouville fractional integral and the Caputo fractional derivative of order $\alpha \ge 0$. For more details, we refer to [8].

Let $\alpha \ge 0$ and let f be a real function defined on a certain interval $[a, b]$. The Riemann-Liouville fractional integral of order α is defined by

$$({}_aI^0 f)(t) = f(t)$$

and

$$({}_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\,ds, \quad \alpha > 0,\ t \in [a, b].$$

The Caputo fractional derivative of order $\alpha \ge 0$ is defined by

$$({}_a^CD^0 f)(t) = f(t)$$

and

$$({}_a^CD^\alpha f)(t) = ({}_aI^{m-\alpha} D^m f)(t), \quad \alpha > 0,$$

where m is the smallest integer greater or equal to α.

2. Main results

2.1. Integral representation of the solution. We start by writing (1.1)-(1.2) in its equivalent integral form.

Lemma 2.1. *$u \in C[a,b]$ is a solution to* (1.1)-(1.2) *if and only if u is a solution to the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)\,ds, \quad t \in [a,b],$$

where G, the Green function associated to (1.1)-(1.2)*, is given by*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(\frac{r}{p}+t-a)(b-s)^{\alpha-1}}{\gamma} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ \frac{(\frac{r}{p}+t-a)(b-s)^{\alpha-1}}{\gamma}, & a \le t \le s \le b, \end{cases}$$

where $\gamma = \frac{r}{p} + b - a$.

Proof. The general solution to (1.1) is

$$u(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)u(s)\,ds,$$

where c_0 and c_1 are real constants. Taking the derivative of $u(t)$, we obtain

$$u'(t) = c_1 - \frac{(\alpha-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} q(s)u(s)\,ds.$$

Using the boundary condition $pu(a) - ru'(a) = 0$, we obtain

$$pc_0 - rc_1 = 0. \tag{2.1}$$

The boundary condition $u(b) = 0$ gives us

$$c_0 + c_1(b-a) - \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} q(s)u(s)\,ds = 0. \tag{2.2}$$

Then (2.1) and (2.2) yield

$$c_0 = \frac{r}{p} c_1 = \frac{r}{p\gamma\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} q(s)u(s)\,ds\,.$$

Therefore,

$$\begin{aligned} u(t) &= \frac{r}{p\gamma\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} q(s)u(s)\,ds + \frac{(t-a)}{\gamma\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} q(s)u(s)\,ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)u(s)\,ds, \end{aligned}$$

which concludes the proof. □

2.2. Green function estimates. Let

$$g_1(t,s) = \frac{(\frac{r}{p}+t-a)(b-s)^{\alpha-1}}{\gamma} - (t-s)^{\alpha-1}, \quad a \le s \le t \le b,$$

$$g_2(t,s) = \frac{(\frac{r}{p}+t-a)(b-s)^{\alpha-1}}{\gamma}, \quad a \le t \le s \le b.$$

We distinguish two cases.

Case $\frac{r}{p} > \frac{b-a}{\alpha-1}$.

Lemma 2.2. *Suppose that*

$$\frac{r}{p} > \frac{b-a}{\alpha-1}\,.$$

Then

$$0 \le G(t,s) \le G(s,s), \quad (t,s) \in [a,b]\times[a,b],$$
$$\max_{a\le s\le b} G(s,s) = \frac{1}{\Gamma(\alpha)}\frac{\frac{r}{p}(b-a)^{\alpha-1}}{(\frac{r}{p}+b-a)}\,.$$

Proof. Obviously, the function g_2 satisfies the following inequalities:

$$0 \le g_2(t,s) \le g_2(s,s), \quad a \le t \le s \le b.$$

Now, let us compute the derivative of $g_2(s,s)$ on (a,b). After some simplifications, we obtain

$$(g_2(s,s))' = \frac{(b-s)^{\alpha-2}}{\gamma}\Big(-\alpha s + (1-\alpha)(\frac{r}{p}-a)+b\Big).$$

Then $(g_2(s,s))'$ has a unique zero, attained at the point

$$s^* = \frac{b+(1-\alpha)(\frac{r}{p}-a)}{\alpha}\,.$$

It is easy to see that $(g_2(s,s))' > 0$ on $(-\infty, s^*)$ and $(g_2(s,s))' < 0$ on (s^*,b). On the other hand, from the condition $\frac{r}{p} > \frac{b-a}{\alpha-1}$, we obtain easily that $s^* < a$. By continuity of g_2, we deduce that

$$\max_{a\le s\le b} g_2(s,s) = g_2(a,a) = \frac{\frac{r}{p}(b-a)^{\alpha-1}}{(\frac{r}{p}+b-a)}\,.$$

Thus

$$0 \le g_2(t,s) \le \frac{\frac{r}{p}(b-a)^{\alpha-1}}{(\frac{r}{p}+b-a)}, \quad a \le t \le s \le b.$$

Now, we turn our attention to the function $g_1(t,s)$. Let $s \in [a,b)$ be fixed. Differentiating $g_1(t,s)$ with respect to t, we obtain

$$\partial_t g_1(t,s) = \frac{(b-s)^{\alpha-1}}{\gamma} - (\alpha-1)(t-s)^{\alpha-2}, \quad s < t.$$

It follows from the above equality that $\partial_t g_1(t,s) = 0$ if and only if

$$t = t^* = s + \big[\frac{(b-s)^{\alpha-1}}{\gamma(\alpha-1)}\big]^{\frac{1}{\alpha-2}},$$

provided $t^* \le b$, i.e. as long as $a \le s \le b-(\alpha-1)\gamma$. However, from the condition $\frac{r}{p} > \frac{b-a}{\alpha-1}$, we observe easily that $b-(\alpha-1)\gamma < a$. Then we deduce that $s > b-(\alpha-1)\gamma$, i.e. $t^* > b$. In this case, $\partial_t g_1(t,s) < 0$, i.e. $g_1(\cdot,s)$ is strictly decreasing and, since $g_1(b,s) = 0$, we conclude that

$$0 \le g_1(t,s) \le g_1(s,s) = g_2(s,s) \le g_2(a,a) \le \frac{\frac{r}{p}(b-a)^{\alpha-1}}{(\frac{r}{p}+b-a)} \quad a \le s \le t \le b,$$

which concludes the proof. □

Case $0 \le \frac{r}{p} \le \frac{b-a}{\alpha-1}$.

Lemma 2.3. *Suppose that*
$$0 \le \frac{r}{p} \le \frac{b-a}{\alpha-1}\,.$$
Then
$$\Gamma(\alpha)|G(t,s)| \le \max\{\mathcal{A}(\alpha,r/p),\mathcal{B}(\alpha,r/p)\}, \quad (t,s)\in[a,b]\times[a,b],$$
where
$$\mathcal{A}(\alpha,r/p) = \frac{(b-a)^{\alpha-1}}{(\frac{r}{p}+b-a)}\Big(\Big(\frac{(b-a)^{\alpha-1}}{(\frac{r}{p}+b-a)(\alpha-1)^{\alpha-1}}\Big)^{\frac{1}{\alpha-2}}(2-\alpha)-\frac{r}{p}\Big),$$
$$\mathcal{B}(\alpha,r/p) = (\frac{r}{p}+b-a)^{\alpha-1}\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\,.$$

Proof. Following the proof of Lemma 2.2, we have
$$0 \le g_2(t,s) \le g_2(s,s), \quad a\le t\le s\le b$$
and $(g_2(s,s))'$ has a unique zero, attained at the point
$$s^* = \frac{b+(1-\alpha)(\frac{r}{p}-a)}{\alpha}\,.$$
Under the condition $0 \le \frac{r}{p} \le \frac{b-a}{\alpha-1}$, it is easy to observe that $s^*\in[a,b]$. Moreover, $(g_2(s,s))'>0$ on $(-\infty,s^*)$ and $(g_2(s,s))'<0$ on (s^*,b). Then
$$\max_{a\le s\le b} g_2(s,s) = g_2(s^*,s^*) = \mathcal{B}(\alpha,r/p).$$
Thus we have
$$0\le g_2(t,s)\le \mathcal{B}(\alpha,r/p), \quad a\le t\le s\le b.$$
Following the proof of Lemma 2.2, for a fixed $s\in[a,b)$, $\partial_t g_1(t,s)=0$ if and only if
$$t=t^*=s+\big[\frac{(b-s)^{\alpha-1}}{\gamma(\alpha-1)}\big]^{\frac{1}{\alpha-2}},$$
provided $t^*\le b$, i.e. as long as $a\le s\le b-(\alpha-1)\gamma$. So, if $s>b-(\alpha-1)\gamma$ (i.e. $\partial_t g_1(t,s)$ has no zeros), then $\partial_t g_1(t,s)<0$, i.e. $g_1(\cdot,s)$ is strictly decreasing and, since $g_1(b,s)=0$, we obtain
$$\max_{s\le t\le b} g_1(t,s) = g_1(s,s) = g_2(s,s), \quad s\in(b-(\alpha-1)\gamma,b).$$
It is easy to check that
$$s^*\in(b-(\alpha-1)\gamma,b).$$
Thus we have
$$0\le g_1(t,s)\le g_2(s^*,s^*)=\mathcal{B}(\alpha,r/p), \quad b-(\alpha-1)\gamma<s\le t\le b.$$
Now, we have to check the case when $a\le s\le b-(\alpha-1)\gamma$; i.e., $t^*\le b$. It is easy to see that $\partial_t g_1(t,s)<0$ for $t<t^*$ and that $\partial_t g_1(t,s)\ge 0$ for $t\ge t^*$. This together with the fact that $g_1(b,s)=0$ implies that $g_1(t^*,s)\le 0$ and, therefore, we only have to show that
$$|g_1(t^*,s)|\le \max\{\mathcal{A}(\alpha,r/p),\mathcal{B}(\alpha,r/p)\}, \quad s\in[a,b-(\alpha-1)\gamma].$$

After some simplifications, we obtain

$$|g_1(t^*,s)| = \frac{(b-s)^{\frac{(\alpha-1)^2}{\alpha-2}}(2-\alpha)}{\gamma^{\frac{\alpha-1}{\alpha-2}}(\alpha-1)^{\frac{\alpha-1}{\alpha-2}}} - \frac{(b-s)^{\alpha-1}}{\gamma}(s-a+\frac{r}{p}).$$

Let us define the function

$$h(s) = \frac{(b-s)^{\frac{(\alpha-1)^2}{\alpha-2}}(2-\alpha)}{\gamma^{\frac{\alpha-1}{\alpha-2}}(\alpha-1)^{\frac{\alpha-1}{\alpha-2}}} - \frac{(b-s)^{\alpha-1}}{\gamma}(s-a+\frac{r}{p}), \quad s \in [a, b-(\alpha-1)\gamma].$$

Now, we differentiate h in the interior of $[a, b-(\alpha-1)\gamma]$. We obtain

$$h'(s) = \frac{(b-s)^{\frac{(\alpha-1)^2}{\alpha-2}-1}}{(\alpha-1)^{\frac{3-\alpha}{\alpha-2}}\gamma^{\frac{\alpha-1}{\alpha-2}}} + \frac{(\alpha-1)(s-a+\frac{r}{p})(b-s)^{\alpha-2}}{\gamma} - \frac{(b-s)^{\alpha-1}}{\gamma}.$$

It is clear that h' is an increasing function in $[a, b-(\alpha-1)\gamma]$. Then we have

$$h'(s) \le h'(b-(\alpha-1)\gamma).$$

On the other hand, after some simplifications, we obtain

$$h'(b-(\alpha-1)\gamma) = 0,$$

which yields $h'(s) \le 0$. Therefore,

$$\max_{a\le s\le b-(\alpha-1)\gamma} h(s) = h(a) = \mathcal{A}(\alpha, r/p),$$

which concludes the proof. □

2.3. **Lyapunov-type inequalities.** We are ready to state and prove our main results.

Theorem 2.4. *If there exists a nontrivial continuous solution of the fractional boundary-value problem*

$$({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha < 2,$$
$$pu(a) - ru'(a) = u(b) = 0,$$

where $p > 0$, $\frac{r}{p} > \frac{b-a}{\alpha-1}$ *and* $q : [a,b] \to \mathbb{R}$ *is a continuous function, then*

$$\int_a^b |q(s)|\, ds \ge \big(1 + \frac{p}{r}(b-a)\big)\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}. \tag{2.3}$$

Proof. Let $X = C[a,b]$ be the Banach space endowed with the norm

$$\|y\|_\infty = \max\{|y(t)| : a \le t \le b\}.$$

It follows from Lemma 2.1 that

$$u(t) = \int_a^b G(t,s)q(s)u(s)\, ds, \quad t \in [a,b].$$

We obtain

$$|u(t)| \le \|u\|_\infty \max|G(t,s)|_{a\le t,s\le b} \int_a^b |q(s)|\, ds.$$

Now, Lemma 2.2 yields

$$\|u\|_\infty \le \|u\|_\infty \frac{1}{\Gamma(\alpha)} \frac{\frac{r}{p}(b-a)^{\alpha-1}}{(\frac{r}{p}+b-a)} \int_a^b |q(s)|\, ds,$$

from which the inequality (2.3) follows. □

Similarly, using Lemma 2.1 and Lemma 2.3, we obtain the following result.

Theorem 2.5. *If there exists a nontrivial continuous solution of the fractional boundary-value problem*

$$\begin{gathered}({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha < 2,\\ pu(a) - ru'(a) = u(b) = 0,\end{gathered}$$

where $p > 0$, $0 \le \frac{r}{p} \le \frac{b-a}{\alpha-1}$ *and* $q : [a,b] \to \mathbb{R}$ *is a continuous function, then*

$$\int_a^b |q(s)|\,ds \ge \frac{\Gamma(\alpha)}{\max\{\mathcal{A}(\alpha, r/p), \mathcal{B}(\alpha, r/p)\}}\,. \tag{2.4}$$

2.4. **Particular cases.**

Case $r = 0$**.** In the case $r = 0$, from Theorem 2.5, taking $r = 0$ in (2.4), we obtain

$$\int_a^b |q(s)|\,ds \ge \frac{\Gamma(\alpha)}{\max\{\mathcal{A}(\alpha, 0), \mathcal{B}(\alpha, 0)\}}\,.$$

On the other hand, we have

$$\begin{gathered}\mathcal{A}(\alpha, 0) = \frac{2-\alpha}{(\alpha-1)^{\frac{\alpha-1}{\alpha-2}}}(b-a)^{\alpha-1},\\ \mathcal{B}(\alpha, 0) = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b-a)^{\alpha-1}\,.\end{gathered}$$

Using the inequality (see [5])

$$\frac{2-\alpha}{(\alpha-1)^{\frac{\alpha-1}{\alpha-2}}} \le \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}, \quad 1 < \alpha < 2,$$

we deuce that

$$\max\{\mathcal{A}(\alpha, 0), \mathcal{B}(\alpha, 0)\} = \mathcal{B}(\alpha, 0).$$

Thus we obtain the following result (see [5, Theorem 1]).

Corollary 2.6. *If there exists a nontrivial continuous solution of the fractional boundary-value problem*

$$\begin{gathered}({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha < 2,\\ u(a) = u(b) = 0,\end{gathered}$$

where $q : [a,b] \to \mathbb{R}$ *is a continuous function, then*

$$\int_a^b |q(s)|\,ds \ge \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}\,.$$

Case $\frac{r}{p} = \frac{b-a}{\alpha-1}$ **with** $\alpha \simeq 2$**.** In the case $\frac{r}{p} = \frac{b-a}{\alpha-1}$, from Theorem 2.5, taking $\frac{r}{p} = \frac{b-a}{\alpha-1}$ in (2.4), we obtain

$$\int_a^b |q(s)|\,ds \ge \frac{\Gamma(\alpha)}{\max\{\mathcal{A}\big(\alpha, \frac{b-a}{\alpha-1}\big), \mathcal{B}\big(\alpha, \frac{b-a}{\alpha-1}\big)\}}\,.$$

An easy computation gives us

$$\mathcal{A}\big(\alpha, \frac{b-a}{\alpha-1}\big) = \frac{(b-a)^{\alpha-1}}{\alpha}\big(\frac{2-\alpha}{\alpha^{\frac{1}{\alpha-2}}} - 1\big),$$

$$\mathcal{B}\big(\alpha, \frac{b-a}{\alpha-1}\big) = \frac{(b-a)^{\alpha-1}}{\alpha}\,.$$

Thus we have

$$\mathcal{A}\big(\alpha, \frac{b-a}{\alpha-1}\big) - \mathcal{B}\big(\alpha, \frac{b-a}{\alpha-1}\big) = \frac{(b-a)^{\alpha-1}}{\alpha}\big(\frac{2-\alpha}{\alpha^{\frac{1}{\alpha-2}}} - 2\big)\,.$$

On the other hand,

$$\lim_{\alpha\to 2^-} \frac{2-\alpha}{\alpha^{\frac{1}{\alpha-2}}} = +\infty.$$

Then there exists $\delta > 0$ such that

$$2-\delta < \alpha < 2 \Rightarrow \frac{2-\alpha}{\alpha^{\frac{1}{\alpha-2}}} > 2.$$

Thus for $2-\delta < \alpha < 2$, we have

$$\max\big\{\mathcal{A}\big(\alpha, \frac{b-a}{\alpha-1}\big), \mathcal{B}\big(\alpha, \frac{b-a}{\alpha-1}\big)\big\} = \mathcal{A}\big(\alpha, \frac{b-a}{\alpha-1}\big).$$

Hence we have the following result.

Corollary 2.7. *There exists $\delta > 0$ such that if there exists a nontrivial continuous solution of the fractional boundary-value problem*

$$({}^C_a D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2-\delta < \alpha < 2,$$
$$pu(a) - ru'(a) = u(b) = 0,$$

where $\frac{r}{p} = \frac{b-a}{\alpha-1}$ and $q : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\,ds \ge \frac{\Gamma(\alpha)\alpha^{\frac{\alpha-1}{\alpha-2}}}{(b-a)^{\alpha-1}(2-\alpha-\alpha^{\frac{1}{\alpha-2}})}\,.$$

Case $p \simeq 0$. Letting $p \to 0^+$ in the inequality (2.3), from Theorem 2.4 we obtain the following result.

Corollary 2.8. *If there exists a nontrivial continuous solution of the fractional boundary-value problem*

$$\begin{aligned} &({}^C_a D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha < 2,\\ &u'(a) = u(b) = 0, \end{aligned}$$

where $q : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\,ds \ge \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\,. \tag{2.5}$$

Taking $\alpha = 2$ in the inequality (2.5), we obtain the following result.

Corollary 2.9. *If there exists a nontrivial continuous solution of the boundary-value problem*

$$\begin{aligned} &u''(t) + q(t)u(t) = 0, \quad a < t < b,\\ &u'(a) = u(b) = 0, \end{aligned}$$

where $q : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|\,ds \ge \frac{1}{b-a}\,.$$

3. Applications

In this section, we present some applications of our main results.

3.1. Real zeros of certain Mittag-Leffler functions.

Let $\alpha, \beta > 0$ be fixed. The complex function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0, \beta > 0, z \in \mathbb{C}$$

is analytic in the whole complex plane; it will be referred to [10, 12] as the Mittag-Leffler function with parameters (α, β).

Next, using the above Lyapunov-type inequalities, we give intervals where linear combinations of some Mittag-Leffler functions have no real zeros.

Theorem 3.1. *Let $1 < \alpha < 2$. The Mittag-Leffler function $E_{\alpha,1}(x)$ has no real zeros for*

$$x \in (-\Gamma(\alpha), 0].$$

Proof. Let $(a, b) = (0, 1)$, and consider the fractional Sturm-Liouville eigenvalue problem

$$({}_0^C D^\alpha u)(t) + \lambda u(t) = 0, \quad 0 < t < 1,$$
$$u'(0) = u(1) = 0.$$

By [3], we know that the eigenvalues $\lambda \in \mathbb{R}$ of the above problem satisfy

$$\lambda > 0 \quad \text{and} \quad E_{\alpha,1}(-\lambda) = 0.$$

The corresponding eigenfunctions are

$$u(t) = AE_{\alpha,1}(-\lambda t^\alpha), \quad t \in [0, 1].$$

By Corollary 2.8, if a real eigenvalue λ exists; i.e., $E_{\alpha,1}(-\lambda) = 0$, then $\lambda \geq \Gamma(\alpha)$, which concludes the proof. □

Theorem 3.2. *Let $1 < \alpha < 2$, $p > 0$, $\frac{r}{p} > \frac{1}{\alpha-1}$. The linear combination of Mittag-Leffler functions given by*

$$pE_{\alpha,2}(x) + qrE_{\alpha,1}(x)$$

has no real zeros for

$$x \in (-(1 + \frac{p}{r})\Gamma(\alpha), 0].$$

Proof. Let $(a, b) = (0, 1)$, and consider the following fractional Sturm-Liouville eigenvalue problem

$$({}_0^C D^\alpha u)(t) + \lambda u(t) = 0, \quad 0 < t < 1,$$
$$pu(0) - ru'(0) = u(1) = 0.$$

By [3], we know that the eigenvalues $\lambda \in \mathbb{R}$ of the above problem satisfies

$$\lambda > 0 \quad \text{and} \quad pE_{\alpha,2}(-\lambda) + qrE_{\alpha,1}(-\lambda) = 0.$$

The corresponding eigenfunctions are

$$u(t) = A\big(E_{\alpha,1}(-\lambda t^\alpha) + \frac{p}{r} t E_{\alpha,2}(-\lambda t^\alpha)\big), \quad t \in [0, 1].$$

By Theorem 2.4, if a real eigenvalue λ exists, then $\lambda \geq (1+\frac{p}{r})\Gamma(\alpha)$, which concludes the proof. □

3.2. **Applications to fractional boundary-value problems.** In this section, we apply the results on the Liapunov-type inequalities obtained previoulsy to study the nonexistence of solutions for certain fractional boundary-value problems. Consider the fractional boundary-value problem

$$({}_0^C D^\alpha u)(t) + q(t)u(t) = 0, \quad 0 < t < 1,\ 3/2 < \alpha < 2, \tag{3.1}$$

with the boundary conditions

$$u(0) - 2u'(0) = u(1) = 0, \tag{3.2}$$

where $q : [a, b] \to \mathbb{R}$ is a continuous function. We have the following result.

Theorem 3.3. *Assume that*

$$\int_0^1 |q(s)|\, ds < \frac{3}{2}\Gamma(\alpha). \tag{3.3}$$

Then(3.1)-(3.2) *has no nontrivial solution.*

Proof. Assume the contrary, i.e. (3.1)-(3.2) has a nontrivial solution $u(t)$. By Theorem 2.4 with $(p, r) = (1, 2)$, we obtain

$$\int_0^1 |q(s)|\, ds \geq \frac{3}{2}\Gamma(\alpha),$$

which contradicts assumption (3.3). □

Consider now the fractional boundary-value problem

$$({}_0^C D^\alpha u)(t) + q(t)u(t) = 0, \quad 0 < t < 1,\ 1 < \alpha < 2, \tag{3.4}$$

with the boundary conditions

$$2u(0) - u'(0) = u(1) = 0, \tag{3.5}$$

where $q : [a, b] \to \mathbb{R}$ is a continuous function. We have the following result.

Theorem 3.4. *Assume that*

$$\int_0^1 |q(s)|\, ds < \frac{\Gamma(\alpha)}{\max\{\mathcal{A}(\alpha, 1/2), \mathcal{B}(\alpha, 1/2)\}}\,. \tag{3.6}$$

Then (3.4)-(3.5) *has no nontrivial solution.*

Proof. Assume the contrary; i.e., (3.4)-(3.5) has a nontrivial solution $u(t)$. By Theorem 2.5 with $(p, r) = (2, 1)$, we obtain

$$\int_0^1 |q(s)|\, ds \geq \frac{\Gamma(\alpha)}{\max\{\mathcal{A}(\alpha, 1/2), \mathcal{B}(\alpha, 1/2)\}},$$

which contradicts assumption (3.6). □

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