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RELAXATION IN CONTROLLED SYSTEMS DESCRIBED BY FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONTROL CONDITIONS

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Abstract. A control system described by fractional evolution integro-differential equations and fractional integral nonlocal control conditions is investigated. This posed system is subjected to mixed multivalued control constraints whose values are nonconvex closed sets. Along with the original system, we consider the system in which the constraints on the controls are the closed convex hulls of the original constraints. More precisely, existence results for the mentioned nonlocal control systems are proved. Furthermore, we study relations between the solution sets of both two systems.

1. Introduction

We are interested with the following fractional nonlocal control abstract evolution systems

$$ {}^C D_t^{\alpha} x(t) + Ax(t) = f(t, x(t)) + \int_0^t g(t, s, x(s), B_1(s)u_1(s))ds, \tag{1.1} $$

$$ x(0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x(s), B_2(s)u_2(s))ds, \tag{1.2} $$

with the mixed nonconvex constraint on the controls

$$ u_1(t), u_2(t) \in U(t, x(t)) \quad \text{a.e. on } J, \tag{1.3} $$

where ${}^C D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha, 0 < \alpha < 1$ and $t \in J = [0, b]$. Let $-A$ be the infinitesimal generator of a strongly continuous semigroup $\{Q(t), t \geq 0\}$ in a separable reflexive Banach space X, the operators $B_1, B_2 : J \to \mathcal{L}(Y, X)$ are linear continuous from Y into X. We assume that $f : J \times X \to X, g : \Delta \times X^2 \to X$ and $h : C(J : X, X) \to X$ are given abstract functions to be specified later. It is also assumed that $U : J \times X \to 2^Y \backslash \{\emptyset\}$ is a multivalued map with closed values (not necessarily convex). Here, $\Delta = \{(t, s) : 0 \leq s \leq t \leq b\}, \Gamma$ is the classical gamma function and Y is a separable reflexive Banach space modeling the control space.

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Along with the constraint (1.3) on the controls, we also consider the constraint

$$u_1(t), u_2(t) \in \overline{\text{co}}U(t, x(t)) \quad \text{a.e. on } J \tag{1.4}$$

on the controls, where $\overline{\text{co}}$ stands for the closed convex hull of a set. We denote by $\mathcal{R}_U$, $\mathcal{T}r_U$ ($\mathcal{R}_{\overline{\text{co}}U}$, $\mathcal{T}r_{\overline{\text{co}}U}$) the sets of all solutions, all admissible trajectories of the control system (1.1)–(1.3) (the control system (1.1)–(1.2), (1.4), respectively).

The main results obtained in this paper are to show that: $\mathcal{T}r_{\overline{\text{co}}U}$ is a compact set in $C(J, X)$ and the relaxation property

$$\mathcal{T}r_{\overline{\text{co}}U} = \overline{\mathcal{T}r_U} \tag{1.5}$$

holds, where the bar stands for the closure in $C(J, X)$.

The applied sciences confirmed that fractional differential equations play an important role in many fields, including viscoelasticity, electrochemistry, control, porous media, electromagnetic and so on. Some works have done on the qualitative properties of solutions for these equations; see [2, 12, 15, 21] and the references therein. The existence of solutions for fractional semilinear differential or integrodifferential equations is one of the theoretical fields being investigated by many authors. There has been a significant development in nonlocal problems for fractional differential equations or inclusions (see for instance [3, 5, 6, 7, 8, 9, 20, 28, 29]).

Relaxation property, such as (1.5), has important ramifications in control theory, since it implies that every trajectory of the convexified (full) system can be approximated in $C(J, X)$ norm, with arbitrary degree of accuracy, by trajectories of the original system. There are many papers dealing with the verification of the relaxation property for various classes of control systems, for instance, Tolstonogov [22] of control systems of subdifferential type, Migórski [18, 19], Tolstonogov [23], Tolstonogov et al [24], Denkowski et al [10] (c.f. Section 7.4) of nonlinear evolution inclusions or equations.

In recent publications, X. Liu et al [16, 17] studied the relaxation properties in both control systems and nonconvex optimal control problems described by fractional differential equations. Debbouche and Torres [7] and [8] introduced the notions of fractional nonlocal condition and nonlocal control condition, respectively, and then investigated the approximate controllability question for both differential equations and inclusions.

Motivated by the above facts, we extend the results, with the same schemes of proof, of [16] for studying a relaxation property in control systems described by fractional integrodifferential equations, and under a comparison between [7] and [8], we also introduce a new concept called fractional integral nonlocal control condition, so that our new complex considered system appears in terms of two controls. The control systems established here are closed-loop systems (feedback control systems) while the ones considered in papers related to this work cited above were concerned with open-loop systems.

The article is organized as follows: In section 2, we introduce some preliminary results and give the assumptions on the data of our problems which will be used throughout the paper. Auxiliary results required to realize our investigation are addressed in section 3. Section 4 deals with the existence of solutions for the considered control systems. The main results are presented in section 5.

2. Preliminaries and assumptions

We start by recalling some well-known facts in fractional calculus, in particular we give the notions of fractional integral and derivative that can be found in [15,21].

Definition 2.1. The fractional integral of order $\alpha > 0$ of $\varpi \in L^1([a,b],\mathbb{R}^+)$ is

$$I_a^\alpha \varpi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \varpi(s) ds,$$

where Γ is the classical gamma function. If $a = 0$, we can write $I^\alpha \varpi(t) = (g_\alpha * \varpi)(t)$, where

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

and as usual, $*$ denotes the convolution of functions. Moreover, $\lim_{\alpha\to 0} g_\alpha(t) = \delta(t)$, with δ the delta Dirac function.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in \mathbb{N}$, is

$${}^L D^\alpha \varpi(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{\varpi(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,$$

where the function ϖ has absolutely continuous derivatives up to order $(n-1)$.

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is

$${}^C D^\alpha \varpi(t) = {}^L D^\alpha \Big(\varpi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \varpi^{(k)}(0) \Big), \quad t > 0,$$

where the function ϖ has absolutely continuous derivatives up to order $(n-1)$.

Remark 2.4. The following properties hold. Let $n-1 < \alpha < n$, $n \in \mathbb{N}$.

(i) If $\varpi \in C^n([0,\infty))$, then

$${}^C D^\alpha \varpi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\varpi^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} \varpi^{(n)}(t), \quad t > 0.$$

(ii) The Caputo derivative of a constant function is equal to zero.

(iii) The Riemann-Liouville derivative of a constant function is given by

$${}^L D_{a^+}^\alpha C = \frac{C}{\Gamma(1-\alpha)} (x-a)^{-\alpha}.$$

(v) If ϖ is an abstract function with values in X, then the previous integrals are taken in Bochner's sense.

According to previous definitions, it is suitable to rewrite problem (1.1)–(1.2) as the equivalent integral equation

$$\begin{aligned} x(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x(s), B_2(s) u_2(s)) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[-Ax(s) + f(s, x(s)) \\ &+ \int_0^s g(s,\eta,x(\eta), B_1(\eta) u_1(\eta)) d\eta \Big] ds, \end{aligned} \tag{2.1}$$

provided the integrals exist.

Let $J=[0,b]$ be the closed interval of the real line with the Lebesgue measure μ and the σ-algebra Σ of μ measurable sets. The norm of the space X (or Y) will be denoted by $\|\cdot\|_X$ (or $\|\cdot\|_Y$). We denote by $C(J,X)$ the space of all continuous functions from J into X with the supremum norm given by $\|x\|_C=\sup_{t\in J}\|x(t)\|_X$ for $x\in C(J,X)$. For any Banach space V, the symbol ω-V stands for V equipped with the weak topology $\sigma(V,V^*)$. The same notation will be used for subsets of V. In all other cases, we assume that V and its subsets are equipped with the strong (normed) topology.

We now proceed to some basic definitions and results from multivalued analysis. For more details on multivalued analysis, see the books [1, 14].

We use the following symbols: $P_f(Y)$ is the set of all nonempty closed subsets of Y, $P_{bf}(Y)$ is the set of all nonempty, closed and bounded subsets of Y.

On $P_{bf}(Y)$, we have a metric known as the "Hausdorff metric" and defined by

$$H(A,B)=\max\big\{\sup_{a\in A}d(a,B),\ \sup_{b\in B}d(b,A)\big\},$$

where $d(x,C)$ is the distance from a point x to a set C. We say a multivalued map is H-continuous if it is continuous in the Hausdorff metric $H(\cdot,\cdot)$.

We say that a multi-valued map $F:J\to P_f(Y)$ is measurable if $F^{-1}(E)=\{t\in J:F(t)\cap E\neq\emptyset\}\in\Sigma$ for every closed set $E\subseteq Y$. If $F:J\times X\to P_f(Y)$, then the measurability of F means that $F^{-1}(E)\in\Sigma\otimes\mathcal{B}_X$, where $\Sigma\otimes\mathcal{B}_X$ is the σ-algebra of subsets in $J\times X$ generated by the sets $A\times B$, $A\in\Sigma$, $B\in\mathcal{B}_X$, and $\mathcal{B}_X$ is the σ-algebra of the Borel sets in X.

Suppose V, Z are two Hausdorff topological spaces and $F:V\to 2^Z\backslash\{\emptyset\}$. We say that F is lower semicontinuous in the sense of Vietoris (l.s.c. for short) at a point $x_0\in V$, if for any open set $W\subseteq Z$, $F(x_0)\cap W\neq\emptyset$, there is a neighborhood $O(x_0)$ of x_0 such that $F(x)\cap W\neq\emptyset$ for all $x\in O(x_0)$. F is said to be upper semicontinuous in the sense of Vietoris (u.s.c. for short) at a point $x_0\in V$, if for any open set $W\subseteq Z$, $F(x_0)\subseteq W$, there is a neighborhood $O(x_0)$ of x_0 such that $F(x)\subseteq W$ for all $x\in O(x_0)$. For more properties of l.s.c and u.s.c, readers may refer to the book [14].

Besides the standard norm on $L^q(J,Y)$ (here Y is a separable, reflexive Banach space), $1<q<\infty$, we also consider the so called weak norm

$$\|u_i(\cdot)\|_\omega=\sup_{0\le t_1\le t_2\le b}\Big\|\int_{t_1}^{t_2}u_i(s)ds\Big\|_Y,\ \text{for } u_i\in L^q(J,Y),\quad i=1,2. \tag{2.2}$$

The space $L^q(J,Y)$ furnished with this norm will be denoted by $L^q_\omega(J,Y)$. The following result establishes a relation between convergence in ω-$L^q(J,Y)$ and convergence in $L^q_\omega(J,Y)$.

Lemma 2.5 ([23]). *If sequences $\{u_{1,n}\}_{n\ge1},\{u_{2,n}\}_{n\ge1}\subseteq L^q(J,Y)$ are bounded and converge to u_1,u_2 in $L^q_\omega(J,Y)$, respectively, then they converge to u_1,u_2 in ω-$L^q(J,Y)$, respectively.*

We use the following assumptions on the data of our problems.

- (H1) The operator $-A$ generates a strongly continuous semigroup $Q(t)$, $t\ge0$ in X, and there exists a constant $M_0\ge1$ such that $\sup_{t\in[0,\infty)}\|Q(t)\|\le M_0$. For any $t>0$, $Q(t)$ is compact.
- (H2) The operators $B_i:J\to\mathcal{L}(Y,X), i=1,2$, are such that:

(1) the maps $t \to B_1(t)u_1$ and $t \to B_2(t)u_2$ are measurable for any $u_1, u_2 \in Y$;
(2) for a.e. $t \in J$,

$$\|B_1(t)\|_{\mathcal{L}(Y,X)} \le d_1, \|B_2(t)\|_{\mathcal{L}(Y,X)} \le d_2, \text{ with } d_1, d_2 > 0.$$

(H3) The function $f : J \times X \to X$ satisfies the following:
(1) $t \to f(t, x)$ is measurable for all $x \in X$;
(2) there exists a function $l_1 \in L^\infty(J, \mathbb{R}^+)$ such that for a.e. $t \in J$ and all x, $y \in X$,

$$\|f(t, x) - f(t, y)\|_X \le l_1(t)\|x - y\|_X;$$

(3) there exists a constant $0 < \beta_1 < \alpha$ such that for a.e. $t \in J$, and all $x \in X$,

$$\|f(t, x)\|_X \le a_1(t) + c_1\|x\|_X,$$

where $a_1 \in L^{1/\beta_1}(J, \mathbb{R}^+)$ and $c_1 > 0$.
(H4) The function $g : \Delta \times X^2 \to X$ satisfies the following:
(1) $t \to g(t, s, x, y)$ is measurable for all $x, y \in X$;
(2) there exists a function $l_2 \in L^\infty(\Delta, \mathbb{R}^+)$ such that for a.e. $(t, s) \in \Delta$ and all $x_1, x_2, y_1, y_2 \in X$,

$$\|g(t, s, x_1, y_1) - g(t, s, x_2, y_2)\|_X \le l_2(t, s)\{\|x_1 - x_2\|_X + \|y_1 - y_2\|_X\};$$

(3) there exists a constant $0 < \beta_2 < \alpha$ such that for a.e. $(t, s) \in \Delta$, and all $x, y \in X$,

$$\|g(t, s, x, y)\|_X \le a_2(t, s) + c_2\{\|x\|_X + \|y\|_X\},$$

where $a_2 \in L^{1/\beta_2}(\Delta, \mathbb{R}^+)$ and $c_2 > 0$.
(H5) The function $h : C(J : X, X) \to X$ satisfies the following:
(1) $t \to h(x, y)$ is measurable for all $x, y \in X$;
(2) there exists a function $l_3 \in L^\infty(\mathbb{R}^+)$ such that for all $x_1, x_2, y_1, y_2 \in X$,

$$\|h(x_1, y_1) - h(x_2, y_2)\|_X \le l_3\{\|x_1 - x_2\|_X + \|y_1 - y_2\|_X\};$$

(3) there exists a constant $0 < \beta_3 < \alpha$ such that for all $x, y \in X$,

$$\|h(x, y)\|_X \le a_3 + c_3\{\|x\|_X + \|y\|_X\},$$

where $a_3 \in L^{1/\beta_3}(\mathbb{R}^+)$ and $c_3 > 0$.
(H6) The multivalued map $U : J \times X \to P_f(Y)$ is such that:
(1) $t \to U(t, x)$ is measurable for all $x \in X$;
(2) there exists a function $l_4 \in L^\infty(J, \mathbb{R}^+)$ such that for a.e. $t \in J$ and all $x, y \in X$,

$$H(U(t, x), U(t, y)) \le l_4(t)\|x - y\|_X,$$

(3) there exists a constant $0 < \beta_4 < \alpha$ such that for a.e. $t \in J$, and all $x \in X$,

$$\|U(t, x)\|_Y = \sup\{\|v\|_Y : v \in U(t, x)\} \le a_4(t) + c_4\|x\|_X,$$

where $a_4 \in L^{1/\beta_4}(J, \mathbb{R}^+)$ and $c_4 > 0$.

Definition 2.6 ([4,28,29]). A triple of functions (x, u_1, u_2) is a mild solution of the control system (1.1)–(1.3), if $x \in C(J, X)$ and there exist $u_1, u_2 \in L^1(J, Y)$ such that $u_1(t), u_2(t) \in U(t, x(t))$ a.e. $t \in J$,

$$x(0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x(s), B_2(s)u_2(s))ds,$$

and the following integral equation is satisfied

$$\begin{aligned} x(t) &= S_\alpha(t)\Big[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x(s), B_2(s)u_2(s))ds\Big] \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\Big[f(s, x(s)) + \int_0^s g(s, \eta, x(\eta), B_1(\eta)u_1(\eta))d\eta\Big]ds, \end{aligned} \tag{2.3}$$

where

$$S_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) Q(t^\alpha \theta)d\theta, \quad T_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) Q(t^\alpha \theta)d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha\big(\theta^{-\frac{1}{\alpha}}\big) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha),$$

with ξ_α is a probability density function defined on $(0, \infty)$; that is, $\xi_\alpha(\theta) \geq 0, \theta \in (0, \infty)$ and $\int_0^\infty \xi_\alpha(\theta)d\theta = 1$.

A similar definition can be introduced for the control system (1.1)–(1.2),(1.4).

Remark 2.7 ([29]). It is not difficult to verify that

$$\int_0^\infty \theta \xi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}.$$

Lemma 2.8 ([29]). *Let* (H1) *hold. Then the operators* S_α *and* T_α *have the following properties:*

(1) *For any fixed* $t \geq 0$, $S_\alpha(t)$ *and* $T_\alpha(t)$ *are linear and bounded operators, i.e., for any* $x \in X$,

$$\|S_\alpha(t)x\|_X \leq M_0 \|x\|_X, \quad \|T_\alpha(t)x\|_X \leq \frac{M_0}{\Gamma(\alpha)} \|x\|_X;$$

(2) $\{S_\alpha(t), t \geq 0\}$ *and* $\{T_\alpha(t), t \geq 0\}$ *are strongly continuous;*
(3) *For every* $t > 0$, $S_\alpha(t)$ *and* $T_\alpha(t)$ *are compact operators.*

The proof of the above lemma can be found in [29].

3. Auxiliary results

In this section, we shall give some auxiliary results needed in the proof of the main results. We begin with the a prior estimation of the trajectory of the control systems.

Lemma 3.1. *For any admissible trajectory* x *of control system* (1.1)–(1.2), (1.4); *i.e.,* $x \in \mathcal{T}r_{\overline{\text{co}}U}$, *there is a constant* L *such that*

$$\|x\|_C \leq L. \tag{3.1}$$

Proof. From Definition 2.6, we have for any $x \in \mathcal{T}r_{\overline{\text{co}}U}$, there exist $u_1(t), u_2(t) \in \overline{\text{co}}U(t, x(t))$ a.e. $t \in J$ and

$$x(t) =_{\alpha} (t)\Big[x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(x(s), B_2(s)u_2(s))ds\Big] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\Big[f(s, x(s)) + \int_0^s g(s, \eta, x(\eta), B_1(\eta)u_1(\eta))d\eta\Big]ds.$$

Then by Lemma 2.8, we obtain

$$\begin{aligned} \|x(t)\|_X \le M_0\Big[\|x_0\|_X + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|h(x(s), B_2(s)u_2(s))\|_X ds\Big] \\ + \frac{M_0}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\|f(s, x(s))\|_X \\ + \int_0^s \|g(s, \eta, x(\eta), B_1(\eta)u_1(\eta))\|_X d\eta\Big]ds. \end{aligned} \tag{3.2}$$

From (H3.2), (H3.3) and Hölder's inequality, we have

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1}\|f(s, x(s))\|_X ds \\ &\le \int_0^t (t-s)^{\alpha-1}\|f(s, x(s)) - f(s, 0)\|_X ds + \int_0^t (t-s)^{\alpha-1}\|f(s, 0)\|_X ds \\ &\le \int_0^t (t-s)^{\alpha-1} l_1(s)\|x(s)\|_X ds + \int_0^t (t-s)^{\alpha-1} a_1(s) ds \\ &\le \Big[\frac{1-\beta_1}{\alpha-\beta_1} b^{\frac{\alpha-\beta_1}{1-\beta_1}}\Big]^{1-\beta_1}\|a_1\|_{L^{1/\beta_1}(J,\mathbb{R}^+)} + \|l_1\|_{L^\infty(J,\mathbb{R}^+)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds. \end{aligned} \tag{3.3}$$

Also, we use (H4.2), (H4.3) and Hölder's inequality to obtain

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1}\int_0^s \|g(s, \eta, x(\eta), B_1(\eta)u_1(\eta))\|_X d\eta ds \\ &\le \int_0^t (t-s)^{\alpha-1}\int_0^s \|g(s, \eta, x(\eta), B_1(\eta)u_1(\eta)) - g(s, \eta, 0, 0)\|_X d\eta ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\int_0^s \|g(s, \eta, 0, 0)\|_X d\eta ds \\ &\le \int_0^t (t-s)^{\alpha-1}\int_0^s l_2(s, \eta)\{\|x(\eta)\|_X + \|B_1(\eta)u_1(\eta)\|_X\} d\eta ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\int_0^s a_2(s, \eta) d\eta ds \\ &\le \Big[\frac{1-\beta_2}{\alpha-\beta_2} b^{\frac{\alpha-\beta_2}{1-\beta_2}}\Big]^{1-\beta_2} b\|a_2\|_{L^{1/\beta_2}(\Delta,\mathbb{R}^+)} \\ &\quad + b\|l_2\|_{L^\infty(\Delta,\mathbb{R}^+)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds \\ &\quad + b\|l_2\|_{L^\infty(\Delta,\mathbb{R}^+)}\int_0^t (t-s)^{\alpha-1}\|B_1(s)u_1(s)\|_X ds, \end{aligned} \tag{3.4}$$

applying (H2.2), (H6.3) and Hölder's inequality for the above integral,

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1}\|B_1(s)u_1(s)\|_X ds \\ &\leq d_1 \int_0^t (t-s)^{\alpha-1}\big(a_4(s)+c_4\|x(s)\|_X\big)ds \\ &\leq d_1\Big[\frac{1-\beta_4}{\alpha-\beta_4}b^{\frac{\alpha-\beta_4}{1-\beta_4}}\Big]^{1-\beta_4}\|a_4\|_{L^{1/\beta_4}(J,\mathbb{R}^+)} + d_1c_4\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds. \end{aligned} \tag{3.5}$$

Again, assumption (H5.3) and Hölder inequality, give

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1}\|h(x(s),B_2(s)u_2(s))\|_X ds \\ &\leq \int_0^t (t-s)^{\alpha-1}[a_3+c_3\{\|x(t)\|_X+\|B_2(t)u_2(t))\|_X\}]ds \\ &\leq \Big[\frac{1-\beta_3}{\alpha-\beta_3}b^{\frac{\alpha-\beta_3}{1-\beta_3}}\Big]^{1-\beta_3}\|a_3\|_{L^{1/\beta_3}(\mathbb{R}^+)} + c_3\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds \\ &\quad + c_3\int_0^t (t-s)^{\alpha-1}\|B_2(t)u_2(t))\|_X ds, \end{aligned} \tag{3.6}$$

by applying (H2.2), (H6.3) and Hölder's inequality for the above integral,

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1}\|B_2(s)u_2(s)\|_X ds \\ &\leq d_2 \int_0^t (t-s)^{\alpha-1}\big(a_4(s)+c_4\|x(s)\|_X\big)ds \\ &\leq d_2\Big[\frac{1-\beta_4}{\alpha-\beta_4}b^{\frac{\alpha-\beta_4}{1-\beta_4}}\Big]^{1-\beta_4}\|a_4\|_{L^{1/\beta_4}(J,\mathbb{R}^+)} + d_2c_4\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds. \end{aligned} \tag{3.7}$$

Combining (3.3)–(3.7) with (3.2), we obtain

$$\begin{aligned} \|x(t)\|_X \leq M_0\Big[&\|x_0\|_X + \frac{1}{\Gamma(\alpha)}\Big[\frac{1-\beta_3}{\alpha-\beta_3}b^{\frac{\alpha-\beta_3}{1-\beta_3}}\Big]^{1-\beta_3}\|a_3\|_{L^{1/\beta_3}(\mathbb{R}^+)} \\ &+ c_3d_2\Big[\frac{1-\beta_4}{\alpha-\beta_4}b^{\frac{\alpha-\beta_4}{1-\beta_4}}\Big]^{1-\beta_4}\|a_4\|_{L^{1/\beta_4}(J,\mathbb{R}^+)} \\ &+ c_3(1+d_2c_4)\int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds\Big] \\ &+ \frac{M_0}{\Gamma(\alpha)}\Big[\Big[\frac{1-\beta_1}{\alpha-\beta_1}b^{\frac{\alpha-\beta_1}{1-\beta_1}}\Big]^{1-\beta_1}\|a_1\|_{L^{1/\beta_1}(J,\mathbb{R}^+)} \\ &+ \Big[\frac{1-\beta_2}{\alpha-\beta_2}b^{\frac{\alpha-\beta_2}{1-\beta_2}}\Big]^{1-\beta_2}b\|a_2\|_{L^{1/\beta_2}(\Delta,\mathbb{R}^+)} \\ &+ b\|l_2\|_{L^\infty(\Delta,\mathbb{R}^+)}d_1\Big[\frac{1-\beta_4}{\alpha-\beta_4}b^{\frac{\alpha-\beta_4}{1-\beta_4}}\Big]^{1-\beta_4}\|a_4\|_{L^{1/\beta_4}(J,\mathbb{R}^+)} \\ &+ \{\|l_1\|_{L^\infty(J,\mathbb{R}^+)} + b\|l_2\|_{L^\infty(\Delta,\mathbb{R}^+)} + d_1c_4b\|l_2\|_{L^\infty(\Delta,\mathbb{R}^+)}\} \\ &\times \int_0^t (t-s)^{\alpha-1}\|x(s)\|_X ds\Big]. \end{aligned}$$

From the above inequality, using the well-known singular-version Gronwall's inequality (see [11, Theorem 3.1]), we can deduce that the inequality (3.1) is satisfied, i.e., there exists a constant $L > 0$ such that $\|x\|_C \le L$. □

Let $\mathrm{pr}_L : X \to X$ be the L-radial retraction; i.e.,

$$\mathrm{pr}_L(x) = \begin{cases} x, & \|x\|_X \le L, \\ \frac{Lx}{\|x\|_X}, & \|x\|_X > L. \end{cases}$$

This map is Lipschitz continuous. We define $U_1(t,x) = U(t, \mathrm{pr}_L\, x)$. Evidently, U_1 satisfies (H6.1) and (H6.2). Moreover, by the properties of pr_L, we have, for a.e. $t \in J$, all $x \in X$ and all $u_1, u_2 \in U_1(t,x)$ such that

$$\sup\{\|u_1\|_Y, \|u_2\|_Y\} \le a_4(t) + c_4 L \text{ and } \sup\{\|u_1\|_Y, \|u_2\|_Y\} \le a_4(t) + c_4\|x\|_X.$$

Hence, Lemma 3.1 is still valid with $U(t,x)$ substituted by $U_1(t,x)$. Consequently, without loss of generality, we assume that, for a.e. $t \in J$ and all $x \in X$,

$$\sup\{\|v\|_Y : v \in U(t,x)\} \le \varphi(t) = a_4(t) + c_4 L, \quad \text{with } \varphi \in L^{1/\beta_4}(J, \mathbb{R}^+). \tag{3.8}$$

Let φ be defined by (3.8), we put

$$\begin{aligned} Y_\varphi = \big\{(u_1,u_2) : u_1 \in L^{1/\beta_4}(J,Y) : \|u_1(t)\|_Y \le \varphi(t) \text{ a.e. } t \in J, \\ u_2 \in L^{1/\beta_4}(J,Y) : \|u_2(t)\|_Y \le \varphi(t) \text{ a.e. } t \in J\big\}. \end{aligned} \tag{3.9}$$

$$\begin{aligned} X_\varphi = \big\{(K_1,K_2,K_3) : K_1 &\in L^{1/\beta_1}(J,X) : \|K_1\|_X \le a_1(t) + c_1 L \text{ a.e. } t \in J, \\ K_2 &\in L^{1/\beta_2}(\Delta,X) : \|K_2\|_X \le a_2(t,s) + c_2\{L + d_1\varphi\} \text{ a.e. } t,s \in J, \\ K_3 &\in L^{1/\beta_3}(J,X) : \|K_3\|_X \le a_3 + c_3\{L + d_2\varphi\} \text{ a.e. } t \in J.\big\} \end{aligned} \tag{3.10}$$

According to (H2)–(H5), for any $x \in C(J,X)$ and $u_1, u_2 \in L^{1/\beta_4}(J,Y)$, the functions f, g and h are elements of the spaces $L^{1/\beta_1}(J,X), L^{1/\beta_2}(J,X)$ and $L^{1/\beta_3}(J,X)$, respectively. Hence, we can consider operators $\mathcal{A}_1, \mathcal{A}_2 : C(J,X) \times L^{1/\beta_4}(J,Y) \to L^{1/\beta_4}(J,X)$ defined by

$$\begin{aligned} \mathcal{A}_1(x,u_1)(t) &= f(t,x(t)) + \int_0^t g(t,s,x(s),B_1(s)u_1(s))ds, \\ \mathcal{A}_2(x,u_2)(t) &= \int_0^t (t-s)^{\alpha-1} h(x(s),B_2(t)u_2(s))ds. \end{aligned} \tag{3.11}$$

Lemma 3.2. *The maps $\mathcal{A}_1(x,u_1)$ and $\mathcal{A}_2(x,u_2)$ are sequentially continuous from $C(J,X) \times \omega\text{-}L^{1/\beta_4}(J,Y)$ to $\omega\text{-}L^{1/\beta_4}(J,X)$.*

Proof. Suppose that $x_n \to x$ in $C(J,X), u_{1,n} \to u_1$ in $\omega\text{-}L^{1/\beta_4}(J,Y)$ and $u_{2,n} \to u_2$ in $\omega\text{-}L^{1/\beta_4}(J,Y)$. Let $f \in L^{1/(1-\beta_1)}(J,X^*), g \in L^{1/(1-\beta_2)}(\Delta,X^*)$ and $h \in L^{1/(1-\beta_3)}(X^*)$ be fixed. Now we may assume that $\|x_n\|_C \le M$ for some constant $M > 0$ and $n \ge 1$. Then from (H2)–(H5), we can have the following facts

$$f(t,x_n(t)) \to f(t,x(t)) \text{ in } X \text{ a.e. } t \in J,\ \|f(t,x_n(t))\|_X \le a_1(t) + c_1 M, \tag{3.12}$$

$$\begin{aligned} g(t,s,x_n(t),\cdot) \to g(t,s,x(t),\cdot) \text{ in } X \text{ a.e. } (t,s) \in \Delta, \\ \|g(t,s,x_n(t),\cdot)\|_X \le a_2(t,s) + c_2 M, \end{aligned} \tag{3.13}$$

$$h(x_n(t),\cdot) \to h(x(t),\cdot) \quad \text{in } X,\ \|h(x_n(t),\cdot)\|_X \le a_3 + c_3 M, \tag{3.14}$$

$$\int_J \langle B_1^*(t)g(t), u_{1,n}(t)\rangle dt \to \int_J \langle B_1^*(t)g(t), u_1(t)\rangle dt, \tag{3.15}$$

$$\int_J \langle B_2^*(t)h(t), u_{2,n}(t)\rangle dt \to \int_J \langle B_2^*(t)h(t), u_2(t)\rangle dt, \tag{3.16}$$

where $B_1^*(t)$ and $B_2^*(t)$ are the operators adjoint to $B_1(t)$ and $B_2(t)$, respectively. From (3.12)–(3.14), using Lebesgue's dominated convergence theorem, we obtain

$$f(t, x_n(t)) \to f(t, x(t)) \quad \text{in } L^{1/\beta_1}(J, X), \tag{3.17}$$

$$g(t, s, x_n(t), \cdot) \to g(t, s, x(t), \cdot) \quad \text{in } L^{1/\beta_2}(\Delta, X), \tag{3.18}$$

$$h(x_n(t), \cdot) \to h(x(t), \cdot) \quad \text{in } L^{1/\beta_3}(X). \tag{3.19}$$

Since

$$\langle g(t), B_1(t)u_1(t)\rangle = \langle B_1^*(t)g(t), u_1(t)\rangle \text{ and } \langle h(t), B_2(t)u_2(t)\rangle = \langle B_2^*(t)h(t), u_2(t)\rangle$$

for some arbitrary $g \in L^{1/(1-\beta_2)}(\Delta, X^*)$ and $h \in L^{1/(1-\beta_3)}(X^*)$, by (3.15) and (3.16), we deduce that

$$B_1(t)u_{1,n}(t) \to B_1(t)u_1(t), \quad B_2(t)u_{2,n}(t) \to B_2(t)u_2(t) \quad \text{in } \omega\text{-}L^{1/\beta_4}(J, X).$$

Together with (3.17)–(3.19) imply

$$\mathcal{A}_1(x_n, u_{1,n}) \to \mathcal{A}_1(x, u_1), \quad \mathcal{A}_2(x_n, u_{2,n}) \to \mathcal{A}_2(x, u_2) \quad \text{in } \omega\text{-}L^{1/\beta_4}(J, X).$$

□

Now we consider the nonlocal auxiliary problem

$$\begin{gathered} {}^C D_t^\alpha x(t) = -Ax(t) + f(t), \quad t \in J = [0, b], \\ x(0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x(s)) ds, \end{gathered} \tag{3.20}$$

where $f(t)$ and $h(x(t))$ reduce $f(t, x(t)) + \int_0^t g(t, s, x(s), \cdot)ds$ and $h(x(s), \cdot)$, respectively.

It is clear that, for every $f \in L^{1/\beta}(J, X), h \in L^{1/\beta}(J : X, X), 0 < \beta < \alpha$, the problem (3.20) has a unique mild solution $H(f, h) \in C(J, X)$ which is given by

$$H(f, h)(t) = S_\alpha(t)\Big[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x(s))ds\Big] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s) ds.$$

The following lemma concerns with the property of the solution map H which is crucial in our investigation.

Lemma 3.3. *The solution map $H : X_\varphi \to C(J, X)$ is continuous from ω-X_φ into $C(J, X)$.*

Proof. We already know that H is linear and continuous from $L^{1/\beta}(J, X)$ to $C(J, X)$, hence H is also continuous from ω-$L^{1/\beta}(J, X)$ to ω-$C(J, X)$.

Let $C \in P_b(L^{1/\beta}(J, X))$ and suppose that for any $f, h \in C$, $\|f\|_{L^{1/\beta}(J,X)} \le K_1$ and $\|h\|_{L^{1/\beta}(J:X,X)} \le K_2$ ($K_1, K_2 > 0$ are constants). Next we will show that H is completely continuous.

Step 1: From Lemma 3.1, we have that the map $\|H(f, h)(t)\|_X$ is uniformly bounded.

Step 2: H is equicontinuous on C. Let $0 \le t_1 < t_2 \le b$. For any $f, h \in C$, we obtain

$$\|H(f, h)(t_2) - H(f, h)(t_1)\|_X$$

$$
\begin{aligned}
&\le \|\frac{1}{\Gamma(\alpha)}S_\alpha(t_2)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}h(x(s))ds\|_X\\
&\quad+\|\frac{1}{\Gamma(\alpha)}S_\alpha(t_2)\int_0^{t_1}\Big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big]h(x(s))ds\|_X\\
&\quad+\|\frac{1}{\Gamma(\alpha)}\Big[S_\alpha(t_2)-S_\alpha(t_1)\Big]\int_0^{t_1}(t_1-s)^{\alpha-1}h(x(s))ds\|_X\\
&\quad+\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)f(s)ds\|_X\\
&\quad+\|\int_0^{t_1}\Big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big]T_\alpha(t_2-s)f(s)ds\|_X\\
&\quad+\|\int_0^{t_1}(t_1-s)^{\alpha-1}\Big[T_\alpha(t_2-s)-T_\alpha(t_1-s)\Big]f(s)ds\|_X\\
&=: I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

By using analogous arguments as in Lemma 3.1, we have

$$
I_1\le\frac{M_0}{\Gamma(\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}K_2(t_2-t_1)^{\alpha-\beta},
$$

$$
\begin{aligned}
I_2&\le\frac{M_0}{\Gamma(\alpha)}\Big(\int_0^{t_1}\big((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big)^{1/(1-\beta)}ds\Big)^{1-\beta}K_2\\
&\le\frac{M_0}{\Gamma(\alpha)}\Big(\int_0^{t_1}\big((t_1-s)^{\frac{\alpha-1}{1-\beta}}-(t_2-s)^{\frac{\alpha-1}{1-\beta}}\big)ds\Big)^{1-\beta}K_2\\
&=\frac{M_0}{\Gamma(\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}\Big(t_1^{\frac{\alpha-\beta}{1-\beta}}-t_2^{\frac{\alpha-\beta}{1-\beta}}+(t_2-t_1)^{\frac{\alpha-\beta}{1-\beta}}\Big)^{1-\beta}K_2\\
&\le\frac{2M_0}{\Gamma(\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}\big(t_2-t_1\big)^{\alpha-\beta}K_2,
\end{aligned}
$$

$$
I_4\le\frac{M_0}{\Gamma(\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}K_1(t_2-t_1)^{\alpha-\beta},
$$

$$
\begin{aligned}
I_5&\le\frac{M_0}{\Gamma(\alpha)}\Big(\int_0^{t_1}\big((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big)^{1/(1-\beta)}ds\Big)^{1-\beta}K_1\\
&\le\frac{M_0}{\Gamma(\alpha)}\Big(\int_0^{t_1}\big((t_1-s)^{\frac{\alpha-1}{1-\beta}}-(t_2-s)^{\frac{\alpha-1}{1-\beta}}\big)ds\Big)^{1-\beta}K_1\\
&=\frac{M_0}{\Gamma(\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}\Big(t_1^{\frac{\alpha-\beta}{1-\beta}}-t_2^{\frac{\alpha-\beta}{1-\beta}}+(t_2-t_1)^{\frac{\alpha-\beta}{1-\beta}}\Big)^{1-\beta}K_1\\
&\le\frac{2M_0}{\Gamma(\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}\big(t_2-t_1\big)^{\alpha-\beta}K_1.
\end{aligned}
$$

For $t_1=0$, $0<t_2\le b$, it is easy to see that $I_3=I_6=0$. For $t_1>0$ and $\epsilon>0$ be small enough, we have

$$
\begin{aligned}
I_3&\le\|\Big(S_\alpha(t_2)-S_\alpha(t_1)\Big)\frac{1}{\Gamma(\alpha)}\int_0^{t_1-\epsilon}(t_1-s)^{\alpha-1}h(x(s))ds\|_X\\
&\quad+\|\Big(S_\alpha(t_2)-S_\alpha(t_1)\Big)\frac{1}{\Gamma(\alpha)}\int_{t_1-\epsilon}^{t_1}(t_1-s)^{\alpha-1}h(x(s))ds\|_X
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup \|S_\alpha(t_2) - S_\alpha(t_1)\| \frac{1}{\Gamma(\alpha)} \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} \big(t_1^{\frac{\alpha-\beta}{1-\beta}} - \epsilon^{\frac{\alpha-\beta}{1-\beta}}\big)^{1-\beta} K_2 \\
&\quad + \frac{2M_0}{\Gamma(\alpha)} \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} \epsilon^{\alpha-\beta} K_2,
\end{aligned}
$$

$$
\begin{aligned}
I_6 &\leq \| \int_0^{t_1-\epsilon} (t_1-s)^{\alpha-1} \Big(T_\alpha(t_2-s) - T_\alpha(t_1-s)\Big) f(s) ds \|_X \\
&\quad + \| \int_{t_1-\epsilon}^{t_1} (t_1-s)^{\alpha-1} \Big(T_\alpha(t_2-s) - T_\alpha(t_1-s)\Big) f(s) ds \|_X \\
&\leq \sup_{s\in[0,t_1-\epsilon]} \|T_\alpha(t_2-s) - T_\alpha(t_1-s)\| \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} \big(t_1^{\frac{\alpha-\beta}{1-\beta}} - \epsilon^{\frac{\alpha-\beta}{1-\beta}}\big)^{1-\beta} K_1 \\
&\quad + \frac{2M_0}{\Gamma(\alpha)} \Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta} \epsilon^{\alpha-\beta} K_1.
\end{aligned}
$$

Combining the estimations for $I_1, I_2, I_3, I_4, I_5, I_6$ and letting $t_2 \to t_1$, and $\epsilon \to 0$ in I_3, I_6, we conclude that H is equicontinuous.

Step 3: The set $\Pi(t) = \{H(f,h)(t) : f, h \in C\}$ is relatively compact in X. Clearly, $\Pi(0) = \{0\}$ is compact, and hence, it is only necessary to consider $t > 0$. For each $g \in (0,t)$, $t \in (0,b]$, $f, h \in C$, and $\delta > 0$ being arbitrary, we define

$$
\Pi_{g,\delta}(t) = \{H_{g,\delta}(f,h)(t) : f, h \in C\},
$$

where

$$
\begin{aligned}
&H_{g,\delta}(f,h)(t) \\
&= \int_\delta^\infty \xi_\alpha(\theta) Q(t^\alpha\theta) \Big[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t-g} (t-s)^{\alpha-1} h(x(s)) ds\Big] d\theta \\
&\quad + \alpha \int_0^{t-g} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^\alpha\theta) f(s) d\theta ds \\
&= Q(g^\alpha\delta) \int_\delta^\infty \xi_\alpha(\theta) Q(t^\alpha\theta - g^\alpha\delta) \left[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t-g} (t-s)^{\alpha-1} h(x(s)) ds\right] d\theta \\
&\quad + \alpha Q(g^\alpha\delta) \int_0^{t-g} \int_\delta^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) Q\big((t-s)^\alpha\theta - g^\alpha\delta\big) f(s) d\theta ds \\
&:= Q(g^\alpha\delta) y(t,g).
\end{aligned}
$$

Because $Q(g^\alpha\delta)$ is compact and $y(t,g)$ is bounded, we obtain that the set $\Pi_{g,\delta}(t)$ is relatively compact in X for any $g \in (0,t)$ and $\delta > 0$. Moreover, we have

$$
\begin{aligned}
&\|H(f,h)(t) - H_{g,\delta}(f,h)(t)\|_X \\
&= \Big\| \int_0^\delta \xi_\alpha(\theta) Q(t^\alpha\theta) \left[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x(s)) ds\right] d\theta \\
&\quad + \int_\delta^\infty \xi_\alpha(\theta) Q(t^\alpha\theta) \left[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x(s)) ds\right] d\theta \\
&\quad - \int_\delta^\infty \xi_\alpha(\theta) Q(t^\alpha\theta) \left[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t-g} (t-s)^{\alpha-1} h(x(s)) ds\right] d\theta \\
&+ \alpha \int_0^t \int_0^\delta \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^\alpha\theta) f(s) d\theta ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s)d\theta ds\\
&-\int_0^{t-g}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s)d\theta ds\Big\|_X\\
&\le \|\int_0^\delta \xi_\alpha(\theta)Q(t^\alpha\theta)\Big[x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(x(s))ds\Big]d\theta\|_X\\
&\quad+\|\int_\delta^\infty \xi_\alpha(\theta)Q(t^\alpha\theta)\left[\frac{1}{\Gamma(\alpha)}\int_{t-g}^t(t-s)^{\alpha-1}h(x(s))ds\right]d\theta\|_X\\
&\quad+\alpha\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s)d\theta ds\|_X\\
&\quad+\alpha\|\int_{t-g}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s)d\theta ds\|_X\\
&\le M_0\int_0^\delta \xi_\alpha(\theta)d\theta\Big[\|x_0\|_X+\frac{1}{\Gamma(\alpha)}\Big(\int_0^t(t-s)^{\frac{\alpha-1}{1-\beta}}ds\Big)^{1-\beta}\|h(x)\|_{L^{1/\beta}(J:X,X)}\Big]\\
&\quad+\frac{M_0}{\Gamma(\alpha)}\int_\delta^\infty \xi_\alpha(\theta)d\theta\Big[\Big(\int_{t-g}^t(t-s)^{\frac{\alpha-1}{1-\beta}}ds\Big)^{1-\beta}\|h(x)\|_{L^{1/\beta}(J:X,X)}\Big]\\
&\quad+M_0\alpha\Big(\int_0^t(t-s)^{\frac{\alpha-1}{1-\beta}}ds\Big)^{1-\beta}\|f\|_{L^{1/\beta}(J,X)}\int_0^\delta\theta\xi_\alpha(\theta)d\theta\\
&\quad+M_0\alpha\Big(\int_{t-g}^t(t-s)^{\frac{\alpha-1}{1-\beta}}ds\Big)^{1-\beta}\|f\|_{L^{1/\beta}(J,X)}\int_\delta^\infty\theta\xi_\alpha(\theta)d\theta\\
&\le M_0\Big\{\int_0^\delta \xi_\alpha(\theta)d\theta\Big[\|x_0\|_X+\frac{K_2}{\Gamma(\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}b^{\alpha-\beta}\Big]+\Big[\frac{K_2}{\Gamma(\alpha)}\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}g^{\alpha-\beta}\Big]\Big\}\\
&\quad+M_0K_1\alpha\Big[\frac{1-\beta}{\alpha-\beta}\Big]^{1-\beta}\Big(b^{\alpha-\beta}\int_0^\delta\theta\xi_\alpha(\theta)d\theta+\frac{1}{\Gamma(1+\alpha)}g^{\alpha-\beta}\Big).
\end{aligned}
$$

By Definition 2.6 and Remark 2.7, we deduce that the Right hand side of the above inequality tends to zero as $g\to 0$ and $\delta\to 0$. Therefore, there are relatively compact sets arbitrarily close to the set $\Pi(t)$, $t>0$. Hence the set $\Pi(t)$, $t>0$ is also relatively compact in X.

Since X_φ is a convex compact metrizable subset of ω-$L^{1/\beta}(J,X)$, it suffices to prove the sequential continuity of the map H. Now let $\{f_n\}_{n\ge1},\{h_n\}_{n\ge1}\subseteq X_\varphi$ such that

$$
f_n\to f \text{ and } h_n\to h \text{ in } \omega\text{-}L^{1/\beta}(J,X),\ f,h\in X_\varphi.
$$

By the properties of the operator H, we have $H(f_n,h_n)\to H(f,h)$ in ω-$C(J,X)$. Since $\{f_n\}_{n\ge1}$ and $\{h_n\}_{n\ge1}$ are bounded, there are subsequences $\{f_{n_k}\}_{k\ge1}$ and $\{h_{n_k}\}_{k\ge1}$ of $\{f_n\}_{n\ge1}$ and $\{f_n\}_{n\ge1}$, respectively, such that $H(f_{n_k},h_{n_k})\to z$ in $C(J,X)$ for some $z\in C(J,X)$. From the facts that

$$
H(f_n,h_n)\to H(f,h) \text{ in } \omega\text{-}C(J,X), \text{ and } H(f_{n_k},h_{n_k})\to z \text{ in } C(J,X),
$$

we obtain that $z=H(f,h)$ and $H(f_n,h_n)\to H(f,h)$ in $C(J,X)$. □

4. Existence results for control systems

In this section, we shall prove the existence of solutions for the control systems (1.1)–(1.3) and (1.1)–(1.2), (1.4).

Let $\Lambda = H(X_\varphi)$. From Lemma 3.3, we have Λ is a compact subset of $C(J,X)$. It follows from (3.8) and (3.10) that $\mathcal{T}r_U \subseteq \mathcal{T}r_{\overline{co}U} \subseteq \Lambda$. Let $\overline{U}: C(J,X) \to 2^{L^{1/\beta}(J,Y)}$ be defined by

$$\overline{U}(x) = \{\theta : J \to Y \text{ measurable} : \theta(t) \in U(t, x(t)) \text{ a.e.}\},\ x \in C(J,X). \tag{4.1}$$

Theorem 4.1. *The set $\mathcal{R}_U$ is nonempty and the set $\mathcal{R}_{\overline{co}U}$ is a compact subset of the space $C(J,X) \times \omega\text{-}L^{1/\beta}(J,Y)$.*

Proof. By the hypotheses (H6.1) and (H6.2), we have that for any measurable function $x : J \to X$, the map $t \to U(t, x(t))$ is measurable and has closed values [14, Proposition 2.7.9]. Therefore it has measurable selectors [13]. So the operator $\overline{U}$ is well defined and its values are closed decomposable subsets of $L^{1/\beta}(J,Y)$. We claim that $x \to \overline{U}(x)$ is l.s.c. Let $x_* \in C(J,X)$, $\theta_* \in \overline{U}(x_*)$ and let $\{x_n\}_{n\geq 1} \subseteq C(J,X)$ be a sequence converging to x_*. It follows from [30, Lemma 3.2] that there is a sequence $\theta_n \in \overline{U}(x_n)$ such that

$$\|\theta_*(t) - \theta_n(t)\|_Y \leq d_Y(\theta_*(t), U(t, x_n(t))) + \frac{1}{n}, \quad \text{a.e. } t \in J. \tag{4.2}$$

Since the map $y \to U(t,y)$ is H-continuous a.e. $t \in J$ (by (H6.2)), then for a.e. $t \in J$, the map $y \to U(t,y)$ is l.s.c. [14, Proposition 1.2.66]. Hence by Proposition 1.2.26 in [14], the function $y \to d_Y(\theta_*(t), U(t,y))$ is u.s.c. for a.e. $t \in J$. It follows from (4.2) that, for a.e. $t \in J$,

$$\begin{aligned} \lim_{n\to\infty} \|\theta_*(t) - \theta_n(t)\|_Y &\leq \limsup_{n\to\infty} d_Y(\theta_*(t), U(t, x_n(t))) \\ &\leq d_Y(\theta_*(t), U(t, x_*(t))) = 0. \end{aligned}$$

This, together with (3.8), implies that $\theta_n \to \theta_*$ in $L^{1/\beta}(J,Y)$. Therefore the map $x \to \overline{U}(x)$ is l.s.c. By [25, Proposition 2.2] (also see [14, Theorem 2.8.7]), there exists a continuous function $m : \Lambda \to L^{1/\beta}(J,Y)$ such that

$$m(x) \in \overline{U}(x), \quad \text{for all } x \in \Lambda. \tag{4.3}$$

Consider the map $\mathcal{P} : L^{1/\beta}(J,X) \to L^{1/\beta}(J,Y)$ defined by $\mathcal{P}(f,h) = m(H(f,h))$. Due to Lemma 3.3 and the continuity of m, the map $\mathcal{P}$ is continuous from $\omega\text{-}X_\varphi$ into $L^{1/\beta}(J,Y)$. Then by Lemma 3.2, we deduce that the maps $f \to \mathcal{A}_1(H(f), \mathcal{P}(f))$ and $h \to \mathcal{A}_2(H(h), \mathcal{P}(h))$ are continuous from $\omega\text{-}X_\varphi$ into $\omega\text{-}L^{1/\beta}(J,X)$. For short, we denote $g \to \mathcal{A}(H(g), \mathcal{P}(g))$, where $g = (f,h)$. It follows from (3.8), (3.10) and (3.11) that $\mathcal{A}(H(g), \mathcal{P}(g)) \in X_\varphi$ for every $g \in X_\varphi$. Therefore, the map $g \to \mathcal{A}(H(g), \mathcal{P}(g))$ is continuous from $\omega\text{-}X_\varphi$ into $\omega\text{-}X_\varphi$. Since $\omega\text{-}X_\varphi$ is a convex metrizable compact set in $\omega\text{-}L^{1/\beta}(J,X)$, Schauder's fixed point theorem implies that this map has a fixed point $g_* \in X_\varphi$; i.e., $g_* = \mathcal{A}(H(g_*), \mathcal{P}(g_*))$. Let $(u_{1,*}, u_{2,*}) = \mathcal{P}(g_*)$ and $x_* = H(g_*)$, then we have $(u_{1,*}, u_{2,*}) = m(x_*)$, $f_* = \mathcal{A}_1(x_*, u_{1,*})$ and $h_* = \mathcal{A}_2(x_*, u_{2,*})$. That is to say we have

$$\begin{aligned} &x_*(t) \\ &= H(g_*)(t) \\ &= S_\alpha(t)\Big[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x_*(s), B_2(s) u_{2,*}(s)) ds\Big] \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \Big[f(s, x_*(s)) + \int_0^s g(s, \eta, x_*(\eta), B_1(\eta) u_{1,*}(\eta)) d\eta\Big] ds, \end{aligned}$$

$$u_{1,*}, u_{2,*} \in U(t, x_*(t)) \quad \text{a.e. } t \in J.$$

Which imply that $(x_*(\cdot), u_{1,*}(\cdot), u_{2,*}(\cdot))$ is a solution of the control system (1.1)–(1.3). Hence $\mathcal{R}_U$ is nonempty.

It is easy to see that $\mathcal{R}_{\overline{\text{co}}U} \subseteq \Lambda \times Y_\varphi$. Since Λ is compact in $C(J, X)$ and Y_φ is metrizable convex compact in $\omega\text{-}L^{1/\beta}(J, Y)$, we have that $\mathcal{R}_{\overline{\text{co}}U}$ is relatively compact in $C(J, X) \times \omega\text{-}L^{1/\beta}(J, Y)$. Hence to complete the proof of this theorem, it is sufficient to prove that $\mathcal{R}_{\overline{\text{co}}U}$ is sequentially closed in $C(J, X) \times \omega\text{-}L^{1/\beta}(J, Y)$.

Let $\{(x_n(\cdot), u_{1,n}(\cdot), u_{2,n}(\cdot))\}_{n\geq 1} \subseteq \mathcal{R}_{\overline{\text{co}}U}$ be a sequence converging to the a function $(x(\cdot), u_1(\cdot), u_2(\cdot))$ in the space $C(J, X) \times \omega\text{-}L^{1/\beta}(J, Y)$. Denote

$$\begin{gathered}
f_n(t) = f(t, x_n(t)) + \int_0^t g(t, s, x_n(s), B_1(s)u_{1,n}(s))ds, \\
f(t) = f(t, x(t)) + \int_0^t g(t, s, x(s), B_1(s)u_1(s))ds, \\
h_n(t) = \int_0^t (t-s)^{\alpha-1} h(x_n(s), B_2(t)u_{2,n}(s))ds, \\
h(t) = \int_0^t (t-s)^{\alpha-1} h(x(s), B_2(t)u_2(s))ds
\end{gathered}$$

According to Lemma 3.2, $f_n \to f, h_n \to h$ in $\omega\text{-}L^{1/\beta}(J, X)$. Since $f_n, h_n \in X_\varphi$ and $x_n = H(f_n, h_n)$, $n \geq 1$, Lemma 3.3 implies that

$$x = H(f, h).$$

Hence, to prove that $(x(\cdot), u_1(\cdot), u_2(\cdot)) \in \mathcal{R}_{\overline{\text{co}}U}$, we only need to verify that $u_1(t)$ and $u_2(t)$ belong to $\overline{\text{co}}U(t, x(t))$ a.e. $t \in J$.

Since $u_{1,n} \to u_1, u_{2,n} \to u_2$ in $\omega\text{-}L^{1/\beta}(J, Y)$, by Mazur's theorem, we have

$$u_1(t), u_2(t) \in \cap_{n=1}^{\infty} \overline{\text{co}}\big(\cup_{k=n}^{\infty} u_k(t)\big), \quad \text{for a.e. } t \in J. \tag{4.4}$$

By (H6.2) and the fact that $H(\overline{\text{co}}A, \overline{\text{co}}B) \leq h(A, B)$ for sets A, B, the map $x \to \overline{\text{co}}U(t, x)$ is H-continuous. Then from [14, Proposition 1.2.86], the map $x \to \overline{\text{co}}U(t, x)$ has property Q. Therefore we have

$$\cap_{n=1}^{\infty} \overline{\text{co}}\Big(\cup_{k=n}^{\infty} \overline{\text{co}}U(t, x_k(t))\Big) \subseteq \overline{\text{co}}U(t, x(t)), \quad \text{for a.e. } t \in J. \tag{4.5}$$

By (4.4) and (4.5), we obtain that $u_1(t), u_2(t) \in \overline{\text{co}}U(t, x(t))$ a.e. $t \in J$. This means that $\mathcal{R}_{\overline{\text{co}}U}$ is compact in $C(J, X) \times \omega\text{-}L^{1/\beta}(J, Y)$. □

5. Main results

Now we are in a position to state and prove the main results of this work.

Theorem 5.1. *For any $(x_*(\cdot), u_{1,*}(\cdot), u_{2,*}(\cdot)) \in \mathcal{R}_{\overline{\text{co}}U}$, we have that there exists a sequence $(x_n(\cdot), u_{1,n}(\cdot), u_{2,n}(\cdot)) \in \mathcal{R}_U$, $n \geq 1$, such that*

$$x_n \to x_* \quad \text{in } C(J, X), \tag{5.1}$$

$$u_{1,n} \to u_{1,*};\ u_{2,n} \to u_{2,*} \quad \text{in } L_\omega^{1/\beta}(J, Y) \text{ and in } \omega\text{-}L^{1/\beta}(J, Y). \tag{5.2}$$

Moreover, we have

$$\overline{\mathcal{T}r_U} = \mathcal{T}r_{\overline{\text{co}}U}, \tag{5.3}$$

where the bar stands for the closure in the space $C(J, X)$.

Proof. Let $(x_*(\cdot), u_{1,*}(\cdot), u_{2,*}(\cdot)) \in \mathcal{R}_{\overline{\text{co}}U}$, then $u_{1,*}(t), u_{2,*}(t) \in \overline{\text{co}}U(t, x_*(t))$ a.e. $t \in J$. It follows from (H6.1), (H6.2) and (3.8) that the map $t \to U(t, x_*(t))$ is measurable and integrally bounded. Hence by using [26, Theorem 2.2], we have that, for any $n \geq 1$, there exist measurable selections $v_{1,n}(t)$ and $v_{1,n}(t)$ of the multivalued map $t \to U(t, x_*(t))$ such that

$$\sup_{0 \leq t_1 \leq t_2 \leq b} \Big\| \int_{t_1}^{t_2} (u_{i,*}(s) - v_{i,n}(s)) ds \Big\|_Y \leq \frac{1}{n}, \quad i = 1, 2. \tag{5.4}$$

For each fixed $n \geq 1$, by (H6.2), we have that, for any $x \in X$ and a.e. $t \in J$, there exist $v_i \in U(t, x), i = 1, 2$, such that

$$\|v_{i,n}(t) - v_i\|_Y < k_i(t)\|x_*(t) - x\|_X + \frac{1}{n}, \quad i = 1, 2. \tag{5.5}$$

Let a map $\Upsilon_n : J \times X \to 2^Y$ be defined by

$$\Upsilon_n(t, x) = \{v_i \in Y : v_i, i = 1, 2, \text{ satisfy inequality (5.5)}\}. \tag{5.6}$$

It follows from (5.5) that $\Upsilon_n(t, x)$ is well defined for a.e. on J and all $x \in X$, and its values are open sets. Using [27, Corollary 2.1] (since we can assume without loss of generality that $U(t, x)$ is $\Sigma \otimes \mathcal{B}_X$ measurable, see [14, Proposition 2.7.9]), we obtain that, for any $\epsilon > 0$, there is a compact set $J_\epsilon \subseteq J$ with $\mu(J \backslash J_\epsilon) \leq \epsilon$, such that the restriction of $U(t, x)$ to $J_\epsilon \times X$ is l.s.c and the restrictions of $v_{1,n}(t), v_{2,n}(t), k_1(t)$ and $k_2(t)$ to J_ϵ are continuous. So (5.5) and (5.6) imply that the graph of the restriction of $\Upsilon_n(t, x)$ to $J_\epsilon \times X$ is an open set in $J_\epsilon \times X \times Y$. Let a map $\Upsilon : J \times X \to 2^Y$ be defined by

$$\Upsilon(t, x) = \Upsilon_n(t, x) \cap U(t, x). \tag{5.7}$$

It is obvious that, for a.e. $t \in J$ and all $x \in X$, $\Upsilon(t, x) \neq \emptyset$. Due to the arguments above and Proposition 1.2.47 in [14], we know that the restriction of $\Upsilon(t, x)$ to $J_\epsilon \times X$ is l.s.c. and so does $\overline{\Upsilon}(t, x) = \overline{\Upsilon(t, x)}$, here the bar stands for the closure of a set in Y.

Now we consider the system (1.1), (1.2) with the constraint on the controls

$$u_1(t), u_2(t) \in \overline{\Upsilon}(t, x(t)) \quad \text{a.e. on } J. \tag{5.8}$$

Since $\overline{\Upsilon}(t, x) \subseteq U(t, x)$, then a priori estimate Lemma 3.1 also holds in this situation. Repeating the proof of Theorem 4.1, we obtain that there is a solution $(x_n(\cdot), u_{1,n}(\cdot), u_{2,n}(\cdot))$ of the control system (1.1),(1.2), (5.8). The definition of $\overline{\Upsilon}$ implies that $(x_n(\cdot), u_{1,n}(\cdot), u_{2,n}(\cdot)) \in \mathcal{R}_U$ and

$$\|v_{i,n}(t) - u_{i,n}(t)\|_Y \leq k_i(t)\|x_*(t) - x_n(t)\|_X + \frac{1}{n}, \quad i = 1, 2. \tag{5.9}$$

Since $(x_n(\cdot), u_{1,n}(\cdot), u_{2,n}(\cdot)) \in \mathcal{R}_U$, $n \geq 1$, and $(x_*(\cdot), u_{1,*}(\cdot), u_{2,*}(\cdot)) \in \mathcal{R}_{\overline{\text{co}}U}$, we have

$$\begin{aligned} x_*(t) &= S_\alpha(t)\Big[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(x_*(s), B_2(s) u_{2,*}(s)) ds\Big] \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\Big[f(s, x_*(s)) \\ &\quad + \int_0^s g(s, \eta, x_*(\eta), B_1(\eta) u_{1,*}(\eta)) d\eta\Big] ds, \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} x_n(t) = S_\alpha(t)\Big[x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(x_n(s), B_2(s)u_{2,n}(s))ds\Big] \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\Big[f(s, x_n(s)) \\ + \int_0^s g(s,\eta,x_n(\eta), B_1(\eta)u_{1,n}(\eta))d\eta\Big]ds. \end{aligned} \tag{5.11}$$

Theorem 4.1 and $\{(x_n(\cdot), u_{1,n}(\cdot)), u_{1,n}(\cdot))\}_{n\geq 1} \subseteq \mathcal{R}_U \subseteq \mathcal{R}_{\overline{\text{co}}U}$ imply that we can assume, possibly up to a subsequence, that the sequence $(x_n(\cdot), u_{1,n}(\cdot), u_{2,n}(\cdot)) \to (\overline{x}(\cdot), \overline{u}_1(\cdot), \overline{u}_2(\cdot)) \in \mathcal{R}_{\overline{\text{co}}U}$ in $C(J,X)\times\omega\text{-}L^{1/\beta}(J,Y)$. Subtracting (5.11) from (5.10), using (H2.2), (H3.2),(H4.2), (H5.2) and (5.9), and according to previous estimations of our sufficient set of conditions, it is easy to get

$$\|x_*(t) - \overline{x}(t)\|_X \leq \tau \int_0^t (t-s)^{\alpha-1}\|x_*(s) - \overline{x}(s)\|_X ds,$$

where τ is a positive constant. Then by [11, Theorem 3.1], we obtain $x_* = \overline{x}$; i.e., we have $x_n \to x_*$ in $C(J,X)$. Hence from (5.9), we have $(v_{1,n} - u_{1,n}) \to 0, (v_{2,n} - u_{2,n}) \to 0$ in $L^{1/\beta}(J,Y)$. Therefore, $u_{1,n} = u_{1,n} - v_{1,n} + v_{1,n} \to u_{1,*}, u_{2,n} = u_{2,n} - v_{2,n} + v_{2,n} \to u_{2,*}$ in $\omega\text{-}L^{1/\beta}(J,Y)$ and $L^{1/\beta}_\omega(J,Y)$, i.e., (5.1) and (5.2) hold.

Since it is clear that $\mathcal{T}r_U \subseteq \mathcal{T}r_{\overline{\text{co}}U}$ and $\mathcal{T}r_{\overline{\text{co}}U}$ is compact in $C(J,X)$ by Theorem 4.1, then from the proof of the first part of this theorem, we have

$$\overline{\mathcal{T}r_U} = \mathcal{T}r_{\overline{\text{co}}U},$$

where the bar stands for the closure in $C(J,X)$. □

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