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EXISTENCE OF SOLUTIONS FOR FRACTIONAL p-KIRCHHOFF EQUATIONS WITH CRITICAL NONLINEARITIES

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Abstract. In this article, we show the existence of non-negative solutions of the fractional p-Kirchhoff problem

$$-M(\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^p K(x-y)dx\,dy)\mathcal{L}_K u = \lambda f(x,u) + |u|^{p^*-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where $\mathcal{L}_K$ is a p-fractional type non local operator with kernel K, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, M and f are continuous functions, and p^* is the fractional Sobolev exponent.

1. Introduction

In this work, we study the existence of solutions for the following p-Kirchhoff equation

$$\begin{gathered}-M\Big(\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^p K(x-y)dx\,dy\Big)\mathcal{L}_K u = \lambda f(x,u) + |u|^{p^*-2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\end{gathered} \tag{1.1}$$

where $p > 1$, $n > ps$ with $s \in (0,1)$, $p^* = \frac{np}{n-ps}$, λ is a positive parameter, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $M : \mathbb{R}^+ \to \mathbb{R}^+$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy some growth assumptions which will be stated later. Here the operator $\mathcal{L}_K$ is the p-fractional type non-local operator defined as follows:

$$\mathcal{L}_K u(x) = 2\int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}(u(x)-u(y))K(x-y)dy \quad \text{for all } x \in \mathbb{R}^n,$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a measurable function with the property that

there exists $\theta > 0$ and $s \in (0,1)$ such that $\theta|x|^{-(n+ps)} \le K(x) \le \theta^{-1}|x|^{-(n+ps)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$. (1.2)

It is immediate to observe that $mK \in L^1(\mathbb{R}^n)$ by setting $m(x) = \min\{|x|^p, 1\}$. A typical example for K is given by $K(x) = |x|^{-(n+ps)}$. In this case problem (1.1)

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becomes

$$M\Big(\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dx\,dy\Big)(-\Delta)_p^s u=\lambda f(x,u)+|u|^{p^*-2}u\quad\text{in }\Omega,\qquad u=0\quad\text{in }\mathbb{R}^n\setminus\Omega,\tag{1.3}$$

where $(-\Delta)_p^s$ is the fractional p-Laplace operator defined as

$$-2\int_{\mathbb{R}^n}\frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{n+p\alpha}}dy.$$

Problems (1.1) and (1.3) are variational in nature and the natural space to look for solutions is the fractional Sobolev space $W_0^{s,p}(\Omega)$ (see [9]). To study (1.1) and (1.3), it is important to encode the 'boundary condition' $u=0$ in $\mathbb{R}^n\setminus\Omega$ (which is different from the classical case of the Laplacian) in the weak formulation. Also that in the norm $\|u\|_{W^{s,p}(\mathbb{R}^n)}$, the interaction between Ω and $\mathbb{R}^n\setminus\Omega$ gives positive contribution. Inspired by [18, 19], we define the function space for p-case as

$$X=\Big\{u:\mathbb{R}^n\to\mathbb{R}: u\text{ is measurable, } u\big|_\Omega\in L^p(\Omega),\quad (u(x)-u(y))\sqrt[p]{K(x-y)}\in L^p(Q)\Big\},$$

where $Q:=\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega)$. The space X is endowed with a norm, defined as

$$\|u\|_X=\Big(\|u\|_{L^p(\Omega)}+\int_Q|u(x)-u(y)|^pK(x-y)dx\,dy\Big)^{1/p}.\tag{1.4}$$

It is immediate to observe that bounded and Lipschitz functions belong to X, thus X is not reduced to $\{0\}$. These spaces for the case $p=2$ are studied in [18, 19]. The function space X_0 denotes the closure of $C_0^\infty(\Omega)$ in X. By [11, Lemma 4], the space X_0 is a Banach space which can be endowed with the norm, defined as

$$\|u\|_{X_0}=\Big(\int_Q|u(x)-u(y)|^pK(x-y)dx\,dy\Big)^{1/p}.\tag{1.5}$$

Note that in (1.4) and (1.5), the integrals can be extended to all $\mathbb{R}^{2n}$, since $u=0$ a.e. in $\mathbb{R}^n\setminus\Omega$. In view of our problem, we assume that $M:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the following conditions:

- (M1) $M:\mathbb{R}^+\to\mathbb{R}^+$ is an increasing and continuous function.
- (M2) There exists $m_0>0$ such that $M(t)\ge m_0=M(0)$ for any $t\in\mathbb{R}^+$.

A typical example for M is given by $M(t)=m_0+tb$ with $b\ge 0$.

Also, we assume that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function that satisfies:

- (F1) $f(x,t)=0$ for any $x\in\Omega$, $t\le 0$ and $\lim_{t\to 0}\frac{f(x,t)}{t^{p-1}}=0$, uniformly in $x\in\Omega$;
- (F2) There exists $q\in(p,p^*)$ such that $\lim_{t\to\infty}\frac{f(x,t)}{t^{q-1}}=0$, uniformly in $x\in\Omega$;
- (F3) There exists $\sigma\in(p,p^*)$ such that for any $x\in\Omega$ and $t>0$,

$$0<\sigma F(x,t)=\sigma\int_0^t f(x,s)ds\le tf(x,t).$$

Definition 1.1. A function $u\in X_0$ is called weak solution of (1.1) if u satisfies

$$M(\|u\|_{X_0}^p)\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy=\lambda\int_\Omega f(x,u(x))\varphi(x)\,dx+\int_\Omega|u(x)|^{p^*-2}u(x)\varphi(x)dx\quad\forall\varphi\in X_0.\tag{1.6}$$

Thanks to our assumptions on Ω, M, f and K, all the integrals in (1.6) are well defined if u, $\varphi \in X_0$. We also point out that the odd part of function K gives no contribution to the integral of the left-hand side of (1.6). Therefore, it would be not restrictive to assume that K is even.

The fractional Laplacian $(-\Delta)^{\frac{s}{2}}_2$ operator has been a classical topic in Fourier analysis and nonlinear partial differential equations for a long time. Non-local operators, naturally arise in continuum mechanics, phase transition phenomena, population dynamics and game theory, see [5] and references therein. Fractional operators are also involved in financial mathematics, where Levy processes with jumps appear in modeling the asset prices (see [2].) In [12] author gave motivation for the study of fractional Kirchhoff equations occurring in vibrating strings. Here we study the p-fractional version of the problem studied in [12]. We follow and adopt the same approach as in [12] to obtain our results.

Recently, much interest has grown to the study of critical exponent problem for non-local equations. The Brezis-Nirenberg problem for the Kirchhoff type equations are studied in [1, 8, 10] and references therein. Also, there are many works on the study of critical problems in a non-local setting inspired by fractional Laplacian [7, 10, 12, 17, 18, 19, 22]. Variational problems involving p-fractional operator with sub-critical and sign changing nonlinearities are studied in [13, 14], using Nehari manifold and fibering maps.

In [12], authors considered the fractional Kirchhoff problem

$$\begin{gathered} -M\Big(\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy\Big)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{gathered} \tag{1.7}$$

with $K(x) \sim |x|^{-(n+2s)}$ and $f(x,u)$ having sub-critical growth. Using mountain pass Lemma and the study of compactness of Palais-Smale sequences, they established the existence of solutions of (1.7) for large λ. Inspired by the above articles, in this paper we will investigate the existence of a nontrivial solution for p-fractional Kirchhoff problem stated in (1.1). To the best of our knowledge, there are no works on p-Kirchhoff fractional equations. With this introduction, we state our main result.

Theorem 1.2. *Let $s \in (0,1)$, $p > 1$, $n > ps$ and Ω be a bounded open subset of $\mathbb{R}^n$. Assume that the functions $K(x)$, $M(t)$ and $f(x,t)$ satisfy conditions* (1.2), (M1)–(M2) *and* (F1)–(F3). *Then there exists $\lambda^* > 0$ such that problem* (1.1) *has a nontrivial solution u_λ for all $\lambda \geq \lambda^*$. Moreover, $\lim_{\lambda\to\infty} \|u_\lambda\|_{X_0} = 0$.*

2. Auxiliary problem and variational formulation

To prove Theorem 1.2, we first study an auxiliary truncated problem. Given σ as in (F3) and $a \in \mathbb{R}$ such that $m_0 < a < \frac{\sigma}{p} m_0$, by (M1) there exists $t_0 > 0$ such that $M(t_0) = a$. Now, by setting

$$M_a(t) := \begin{cases} M(t) & \text{if } 0 \le t \le t_0, \\ a & \text{if } t \ge t_0, \end{cases} \tag{2.1}$$

we introduce the auxiliary problem

$$\begin{gathered} -M_a(\|u\|^p_{X_0})\mathcal{L}_K u = \lambda f(x,u) + |u|^{p^*-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{gathered} \tag{2.2}$$

with f satisfying conditions (F1)–(F3) and λ being a positive parameter. By (M1), we also note that

$$M_a(t) \le a \quad \text{for all } t \ge 0. \tag{2.3}$$

We obtain the following result.

Theorem 2.1. *Assume that $K(x)$, $M(t)$ and $f(x,t)$ satisfies* (1.2), (M1)–(M2) *and* (F1)–(F3)*, respectively. Then there exists $\lambda_0 > 0$ such that problem* (2.2) *has a nontrivial weak solution, for all $\lambda \ge \lambda_0$ and for all $a \in \big(m_0, \frac{\sigma}{p} m_0\big)$.*

For the proof of Theorem 2.1, we observe that problem (2.2) has a variational structure. The Euler functional corresponding to (2.2) is $\mathcal{J}_{a,\lambda} : X_0 \to \mathbb{R}$ defined as follows

$$\mathcal{J}_{a,\lambda}(u) = \frac{1}{p}\widehat{M_a}(\|u\|_{X_0}^p) - \lambda \int_\Omega F(x,u(x))dx - \frac{1}{p^*}\int_\Omega |u(x)|^{p^*} dx,$$

where

$$\widehat{M_a}(t) = \int_0^t M_a(s)ds.$$

Then the functional $\mathcal{J}_{a,\lambda}$ is Fréchet differentiable on X_0 and for any $\varphi \in X_0$,

$$\begin{aligned} &\langle \mathcal{J}'_{a,\lambda}(u), \varphi\rangle \\ &= M_a(\|u\|_{X_0}^p) \int_Q |u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ &\quad - \lambda \int_\Omega f(x,u(x))\varphi(x)\,dx - \int_\Omega |u(x)|^{p^*-2}u(x)\varphi(x)dx\,. \end{aligned} \tag{2.4}$$

Now we prove that the functional $\mathcal{J}_{a,\lambda}$ has the geometric features required by the Mountain Pass Theorem.

Lemma 2.2. *Let $K(x)$, $M(t)$ and $f(x,t)$ be three functions satisfying* (1.2), (M1)–(M2) *and* (F1)–(F3)*, respectively. Then there exist two positive constants ρ and α such that*

$$\mathcal{J}_{a,\lambda}(u) \ge \alpha > 0, \tag{2.5}$$

for any $u \in X_0$ with $\|u\|_{X_0} = \rho$.

Proof. By (F1) and (F2), it follows that, for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|F(x,t)| \le \epsilon |t|^p + \delta |t|^q\,. \tag{2.6}$$

By (M2) and (2.6), we obtain

$$\mathcal{J}_{a,\lambda}(u) \ge \frac{m_0}{p}\|u\|_{X_0}^p - \epsilon\lambda \int_\Omega |u(x)|^p dx - \delta\lambda \int_\Omega |u(x)|^q dx - \frac{1}{p^*}\int_\Omega |u(x)|^{p^*}dx.$$

So, by fractional Sobolev inequality (see [9, Theorem 6.5]), there is a positive constant $C = C(\Omega)$ such that

$$\mathcal{J}_{a,\lambda}(u) \ge \big(\frac{m_0}{p} - \epsilon\lambda C\big)\|u\|_{X_0}^p - \delta\lambda C\|u\|_{X_0}^q - C\|u\|_{X_0}^{p^*}.$$

Therefore, by fixing ϵ such that $\frac{m_0}{p} - \epsilon\lambda C > 0$, since $p < q < p^*$, the result follows by choosing ρ sufficiently small. $\square$

Lemma 2.3. *Let $K(x)$, $M(t)$ and $f(x,t)$ be three functions satisfying* (1.2), (M1)–(M2) *and* (F1)–(F3)*, respectively. Then there exists $e \in X_0$ with $\mathcal{J}_{a,\lambda}(e) < 0$ and $\|e\|_{X_0} > \rho$.*

Proof. We fix $u_0 \in X_0$ such that $\|u_0\|_{X_0} = 1$ and $u_0 \geq 0$ a.e. in $\mathbb{R}^n$. For $t > 0$, by $(F3)$ and (2.3), we obtain

$$\mathcal{J}_{a,\lambda}(tu_0) \leq a\frac{t^p}{p} - c_1 t^\sigma \lambda \int_\Omega |u_0(x)|^\sigma dx + c_2|\Omega| - \frac{t^{p^*}}{p^*}\int_\Omega |u_0(x)|^{p^*} dx.$$

Since $\sigma > p$, passing to the limit as $t \to +\infty$, we obtain that $\mathcal{J}_{a,\lambda}(tu_0) \to -\infty$, so that the assertion follows by taking $e = t_* u_0$, with $t_* > 0$ large enough. □

Now, we prove that the Palais-Smale sequence is bounded.

Lemma 2.4. *Let $K(x)$, $M(t)$ and $f(x,t)$ be three functions satisfying* (1.2), (M1)–(M2) *and* (F1)–(F3)*, respectively. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in X_0 such that, for any $c \in (0,\infty)$,*

$$\mathcal{J}_{a,\lambda}(u_j) \to c, \quad \mathcal{J}'_{a,\lambda}(u_j) \to 0, \tag{2.7}$$

as $j \to +\infty$. Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

Proof. By (2.7), there exists $C > 0$ such that

$$|\mathcal{J}_{a,\lambda}(u_j)| \leq C, \quad \langle \mathcal{J}'_{a,\lambda}(u_j), u_j\rangle \leq C\|u_j\|_{X_0}, \tag{2.8}$$

for any $j \in \mathbb{N}$. Moreover, by (M2), (F3), and (2.3) it follows that

$$\begin{aligned} \mathcal{J}_{a,\lambda}(u_j) - \frac{1}{\sigma}\mathcal{J}'_{a,\lambda}(u_j)(u_j) &\geq \frac{1}{p}\widehat{M_a}(\|u_j\|^p_{X_0}) - \frac{1}{\sigma}M_a(\|u_j\|^p_{X_0})\|u_j\|^p_{X_0} \\ &\geq \big(\frac{1}{p}m_0 - \frac{1}{\sigma}a\big)\|u_j\|^p_{X_0}. \end{aligned} \tag{2.9}$$

On the other hand, from (2.8), we obtain

$$\mathcal{J}_{a,\lambda}(u_j) - \frac{1}{\sigma}\langle \mathcal{J}'_{a,\lambda}(u_j)(u_j)\rangle \leq C(1 + \|u_j\|_{X_0}). \tag{2.10}$$

Now, from (2.9) and (2.10) along with the assumption, $m_0 < a < \frac{\sigma}{p}m_0$, we obtain

$$\|u_j\|^p_{X_0} \leq C(1 + \|u_j\|_{X_0}), \tag{2.11}$$

which implies that sequence $\{u_j\}$ is bounded in X_0 □

Now, we define

$$c_{a,\lambda} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}_{a,\lambda}(\gamma(t)) > 0, \tag{2.12}$$

where

$$\Gamma := \{\gamma \in C([0,1],\, X_0) : \gamma(0) = 0,\ \mathcal{J}_{a,\lambda}(\gamma(1)) < 0\}.$$

The following result is needed to study the asymptotic behavior of the solution of problem (1.6).

Lemma 2.5. *Let $K(x)$, $M(t)$ and $f(x,t)$ be three functions satisfying* (1.2), (M1)–(M2) *and* (F1)–(F3)*. Then $\lim_{\lambda\to+\infty} c_{a,\lambda} = 0$.*

Proof. Let $e \in X_0$ be the function given by Lemma 2.3 and let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence such that $\lambda_j \to +\infty$. Since $\mathcal{J}_{a,\lambda}$ satisfies the Mountain Pass geometry, it follows that there exists $t_\lambda > 0$ such that $\mathcal{J}_{a,\lambda}(t_\lambda e) = \max_{t\geq 0} \mathcal{J}_{a,\lambda}(te)$. Hence, $\langle \mathcal{J}'_{a,\lambda}(t_\lambda e), e\rangle = 0$ and by (2.4), we obtain

$$t_\lambda^{p-1}\|e\|_{X_0}^p M_a(t_\lambda^p\|e\|_{X_0}^p) = \lambda \int_\Omega f(x, t_\lambda e(x))e(x)\,dx + t_\lambda^{p^*-1}\int_\Omega |e(x)|^{p^*}dx\,. \quad (2.13)$$

Now, by construction $e \geq 0$ a.e. in $\mathbb{R}^n$. So, by (F3), (2.3) and (2.13) it follows that

$$a\|e\|_{X_0}^p \geq t_\lambda^{p^*-p}\int_\Omega |e(x)|^{p^*}dx,$$

which implies that t_λ is bounded for any $\lambda > 0$. Thus, there exists $\beta \geq 0$ such that $t_{\lambda_j} \to \beta$ as $j \to +\infty$. So, by (2.3) and (2.13) there exists $D > 0$ such that

$$\lambda_j \int_\Omega f(x, t_{\lambda_j} e(x))e(x)\,dx + t_{\lambda_j}^{p^*-1}\int_\Omega |e(x)|^{p^*}dx = t_{\lambda_j}^{p-1}M_a(t_{\lambda_j}^p\|e\|_{X_0}^p) \leq D, \quad (2.14)$$

for any $j \in \mathbb{N}$. We claim that $\beta = 0$. Indeed, if $\beta > 0$ then by $(F1)$, $(F2)$, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(x,t)| \leq \epsilon|t|^{p-1} + q\delta|t|^{q-1} \quad \text{for all } t \in \mathbb{R},$$

and so, by the Dominated Convergence Theorem,

$$\int_\Omega f(x, t_{\lambda_j} e(x))e(x)\,dx \to \int_\Omega f(x, \beta e(x))e(x)\,dx \quad \text{as } j \to +\infty.$$

Now, since $\lambda_j \to +\infty$, we obtain

$$\lim_{j\to+\infty} \lambda_j \int_\Omega f(x, t_{\lambda_j} e(x))e(x)\,dx + t_{\lambda_j}^{p^*-1}\int_\Omega |e(x)|^{p^*}dx = +\infty,$$

which contradicts (2.14). Thus, we have that $\beta = 0$. Now, we consider the following path $\gamma_*(t) = te$ for $t \in [0,1]$ which belongs to Γ. Using $(F3)$, we obtain

$$0 < c_{a,\lambda} \leq \max_{t\in[0,1]} \mathcal{J}_{a,\lambda}(\gamma_*(t)) \leq \mathcal{J}_{a,\lambda}(t_\lambda e) \leq \frac{1}{p}\widehat{M_a}(t_\lambda^p\|e\|_{X_0}^p). \quad (2.15)$$

By (M1) and the fact that $\beta = 0$, we obtain

$$\lim_{\lambda\to+\infty} \widehat{M_a}(t_\lambda^p\|e\|_{X_0}^p) = 0,$$

and so by (2.15), we conclude the proof. $\square$

Now we prove the following proposition, which will be useful in applying the concentration-compactness principle (see [16, Theorem 2]) to prove Lemma 3.1.

Proposition 2.6. *Let $\xi \in \mathbb{R}^n$, $\delta \in (0,1)$, $u \in L^{p^*}(\mathbb{R}^n)$. Let either $U \times V = B_\delta(\xi) \times \mathbb{R}^n$ or $U \times V = \mathbb{R}^n \times B_\delta(\xi)$. Then*

$$\lim_{\delta\to 0} \delta^{-p}\int_U \int_{V\cap\{|x-y|\leq\delta\}} |u(x)|^p|x-y|^{p-n-ps}\,dx\,dy = 0, \quad (2.16)$$

$$\lim_{\delta\to 0} \int_U \int_{V\cap\{|x-y|>\delta\}} |u(x)|^p|x-y|^{-n-ps}\,dx\,dy = 0. \quad (2.17)$$

Proof. We set $\zeta_\delta := \big(\int_{B_\delta(\xi)} |u(x)|^{p^*}\,dx\big)^{p/p^*}$ and we remark that

$$\lim_{\delta\to 0}\zeta_\delta = 0. \tag{2.18}$$

Also we observe that, using the Hölder's inequality with exponents $\frac{p^*}{p} = \frac{n}{n-ps}$ and $\frac{n}{ps}$, we obtain

$$\int_{B_\delta(\xi)} |u(x)|^p\,dx \le \Big(\int_{B_\delta(\xi)} |u(x)|^{p^*}\,dx\Big)^{p/p^*}\Big(\int_{B_\delta(\xi)} 1\,dx\Big)^{ps/n} \le C\zeta_\delta\delta^{ps}, \tag{2.19}$$

for some $C>0$ independent of δ (in what follows we will possibly change C from line to line). Moreover

$$(U\times V)\cap\{|x-y|\le\delta\} \subseteq B_{2\delta}(\xi)\times B_{2\delta}(\xi). \tag{2.20}$$

Indeed, if $(x,y)\in U\times V = B_\delta(\xi)\times\mathbb{R}^n$, with $|x-y|\le\delta$, we obtain $|\xi-y|\le |\xi-x|+|x-y|\le\delta+\delta$, and so we obtain (2.20). On the other hand, if $(x,y)\in U\times V=\mathbb{R}^n\times B_\delta(\xi)$ with $|x-y|\le\delta$, we obtain

$$|\xi-x|\le|\xi-y|+|y-x|\le\delta+\delta,$$

and this completes the proof of (2.20).

Now using the change of variable $z:=x-y$ and using (2.20), we obtain

$$\begin{aligned}
&\int_{x\in U}\int_{y\in V\cap\{|x-y|\le\delta\}} |u(x)|^p|x-y|^{p-n-ps}\,dx\,dy\\
&\le \int_{x\in B_{2\delta}(p)}\int_{y\in B_{2\delta}(p)\cap\{|x-y|\le\delta\}} |u(x)|^p|x-y|^{p-n-ps}\,dx\,dy\\
&\le \int_{x\in B_{2\delta}(\xi)}\int_{z\in B_\delta} |u(x)|^p|z|^{p-n-ps}\,dx\,dz\\
&\le C\delta^{p-ps}\int_{x\in B_{2\delta}(\xi)} |u(x)|^p\,dx.
\end{aligned}$$

Using this and (2.19), we obtain

$$\begin{aligned}
&\delta^{-p}\int_U\int_{V\cap\{|x-y|\le\delta\}} |u(x)|^p|x-y|^{p-n-ps}\,dx\,dy\\
&\le C\delta^{-ps}\int_{x\in B_{2\delta}(\xi)} |u(x)|^p\,dx \le C\zeta_\delta.
\end{aligned} \tag{2.21}$$

So, (2.21) and (2.18) imply (2.16). Now, we prove (2.17). For this, we fix an auxiliary parameter $K>2$ (such parameter will be taken arbitrarily large at the end, after taking $\delta\to 0$). We observe that

$$U\times V \subseteq \big(B_{K\delta}(\xi)\times\mathbb{R}^n\big)\cup\big((\mathbb{R}^n\setminus B_{K\delta}(\xi))\times B_\delta(\xi)\big). \tag{2.22}$$

Indeed, if $U\times V = B_\delta(\xi)\times\mathbb{R}^n$, then of course $U\times V\subseteq B_{K\delta}(\xi)\times\mathbb{R}^n$, hence (2.22) is obvious. If instead $(x,y)\in U\times V=\mathbb{R}^n\times B_\delta(\xi)$, we distinguish two cases: if $x\in B_{K\delta}(\xi)$ then $(x,y)\in B_{K\delta}(\xi)\times\mathbb{R}^n$; if $x\in\mathbb{R}^n\setminus B_{K\delta}(\xi)$, then

$$(x,y)\in(\mathbb{R}^n\setminus B_{K\delta}(\xi))\times V = (\mathbb{R}^n\setminus B_{K\delta}(\xi))\times B_\delta(\xi).$$

This completes the proof of (2.22). Now, we compute

$$\begin{aligned}
&\int_{x\in B_{K\delta}(\xi)}\int_{y\in\mathbb{R}^n\cap\{|x-y|>\delta\}}|u(x)|^p|x-y|^{-n-ps}\,dx\,dy\\
&=\int_{x\in B_{K\delta}(\xi)}\int_{z\in\mathbb{R}^n\setminus B_\delta}|u(x)|^p|z|^{-n-ps}\,dx\,dz\\
&=C\delta^{-ps}\int_{x\in B_{K\delta}(\xi)}|u(x)|^p\,dx\le C\zeta_{K\delta},
\end{aligned}\tag{2.23}$$

where (2.19) has been used again in the last step. Now, we observe that if $x\in\mathbb{R}^n\setminus B_{K\delta}(\xi)$ and $y\in B_\delta(\xi)$ then

$$\begin{aligned}
|x-y|\ge|x-\xi|-|y-\xi|&=\frac{|x-\xi|}{2}+\frac{|x-\xi|}{2}-|y-p|\\
&\ge\frac{|x-\xi|}{2}+\frac{K\delta}{2}-\delta\ge\frac{|x-\xi|}{2}.
\end{aligned}$$

As a consequence we infer that

$$\begin{aligned}
&\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(\xi)}\int_{y\in B_\delta(\xi)}|u(x)|^p|x-y|^{-n-ps}\,dx\,dy\\
&\le C\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(\xi)}\int_{y\in B_\delta(\xi)}|u(x)|^p|x-\xi|^{-n-ps}\,dx\,dy\\
&=C\delta^n\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(p)}|u(x)|^p|x-\xi|^{-n-ps}\,dx.
\end{aligned}$$

Now using the Hölder's inequality with exponents $\frac{p^*}{p}=\frac{n}{n-ps}$ and $\frac{n}{ps}$, we obtain

$$\begin{aligned}
&\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(\xi)}\int_{y\in B_\delta(\xi)}|u(x)|^p|x-y|^{-n-ps}\,dx\,dy\\
&\le C\delta^n\Big(\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(\xi)}|u(x)|^{p^*}\,dx\Big)^{p/p^*}\Big(\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(\xi)}|x-\xi|^{-(n+ps)n/ps}\,dx\Big)^{ps/n}\\
&\le C\delta^n\|u\|^p_{L^{p^*}(\mathbb{R}^n)}\Big(\int_{K\delta}^{+\infty}\rho^{-((n+ps)n/ps)+(n-1)}d\rho\Big)^{ps/n}\\
&=C\delta^n\|u\|^p_{L^{p^*}(\mathbb{R}^n)}\big((K\delta)^{-n^2/ps}\big)^{ps/n}\\
&=CK^{-n}\|u\|^p_{L^{p^*}(\mathbb{R}^n)}.
\end{aligned}\tag{2.24}$$

By collecting the results in (2.22), (2.23) and (2.24), we obtain

$$\begin{aligned}
&\int_U\int_{V\cap\{|x-y|>\delta\}}|u(x)|^p|x-y|^{-n-ps}\,dx\,dy\\
&\le\int_{x\in B_{K\delta}(\xi)}\int_{y\in\mathbb{R}^n\cap\{|x-y|>\delta\}}|u(x)|^p|x-y|^{-n-ps}\,dx\,dy\\
&\quad+\int_{x\in\mathbb{R}^n\setminus B_{K\delta}(\xi)}\int_{y\in B_\delta(\xi)}|u(x)|^p|x-y|^{-n-ps}\,dx\,dy\\
&\le C\zeta_{K\delta}+CK^{-n}\|u\|^p_{L^{p^*}(\mathbb{R}^n)}.
\end{aligned}$$

From this, we first take $\delta\to0$ and then $K\to+\infty$ to obtain (2.17) (using again (2.18)). $\square$

3. Proofs of Theorems 1.2 and 2.1

We need the following lemma in which we study the local Palais-Smale sequences and show the Palais-Smale condition, $(PS)_c$ in short, below the first critical level.

Lemma 3.1. *There exists $\lambda_0 > 0$ such that $\mathcal{J}_{a,\lambda}$ satisfies $(PS)_{c_{a,\lambda}}$ for all $\lambda > \lambda_0$, where $c_{a,\lambda}$ is defined in* (2.12).

Proof. Let $\{u_j\}$ be a Palais-Smale sequence in X_0 at level $c_{a,\lambda}$ i.e. $\{u_j\}$ satisfies (2.7). By lemma 2.4, $\{u_j\}$ is bounded in X_0 and so upto subsequence $\{u_j\}$ converges weakly to u in X_0, strongly in L^q for all $1 \leq q < p^*$ and point wise to u almost everywhere in Ω. Also there exists $h \in L^p(\Omega)$ such that $|u_j(x)| \leq h(x)$ a.e. in Ω. Also $\{\|u_j\|_{X_0}\}$ as a real sequence converges to α (say). Since M_a is continuous, $M_a(\|u_j\|^p_{X_0}) \to M_a(\alpha^p)$. Now we claim that

$$\|u_j\|^p_{X_0} \to \|u\|^p_{X_0} \quad \text{as } j \to +\infty, \tag{3.1}$$

Once the claim is proved, we can invoke Brezis-Leib lemma to prove that u_j converges to u strongly in X_0. We know that $\{u_j\}$ is also bounded in $W^{s,p}_0(\Omega)$. So we may assume that there exists two positive measures μ and ν on $\mathbb{R}^n$ such that

$$|(-\Delta)^s_p u_j|^p dx \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_j|^{p^*} \rightharpoonup \nu, \tag{3.2}$$

in the sense of measure. Moreover, (see, [16]), we have a countable index set J, positive constants $\{\nu_j\}_{j\in J}$ and $\{\mu_j\}_{j\in J}$ such that

$$\nu = |u|^{p^*} dx + \sum_{i\in J} \nu_i \delta_{x_i}, \tag{3.3}$$

$$\mu \geq |(-\Delta)^s_p u|^p dx + \sum_{i\in J} \mu_i \delta_{x_i}, \quad \nu_i \leq S\mu_i^{p^*/p}, \tag{3.4}$$

where S is the best constant of the embedding $W^{s,p}_0(\Omega)$ into $L^{p^*}(\Omega)$. Our goal is to show that J is empty. Suppose not, then there exists $i \in J$. For this x_i, define $\phi^i_\delta(x) = \phi(\frac{x-x_i}{\delta}), x \in \mathbb{R}^n$ and $\phi \in C^\infty_0(\mathbb{R}^n, [0,1])$ such that $\phi = 1$ in $B(0,1)$ and $\phi = 0$ in $\mathbb{R}^n \setminus B(0,2)$. Since $\{\phi^i_\delta u_j\}$ is bounded in X_0, we have $\mathcal{J}'_{a,\lambda}(u_j)(\phi^i_\delta u_j) \to 0$ as $j \to +\infty$. That is,

$$\begin{aligned}
&M_a(\|u_j\|^p_{X_0}) \int_{\mathbb{R}^{2n}} u_j(x)|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y)) \\
&\quad \times \big(\phi^i_\delta(x) - \phi^i_\delta(y)\big) K(x-y)\,dx\,dy \\
&= -M_a(\|u_j\|^p_{X_0}) \int_{\mathbb{R}^{2n}} \phi^i_\delta(y)|u_j(x) - u_j(y)|^p K(x-y)\,dx\,dy \\
&\quad + \lambda \int_\Omega f(x, u_j(x))\phi^i_\delta(x) u_j(x) dx + \int_\Omega |u_j(x)|^{p^*} \phi^i_\delta(x) dx + o_j(1),
\end{aligned} \tag{3.5}$$

as $j \to \infty$. Now using Hölder's inequality and the fact that $\{u_j\}$ is bounded in X_0, we obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^{2n}} u_j(x)|(u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))\big(\phi^i_\delta(x) - \phi^i_\delta(y)\big)K(x-y)\,dx\,dy\Big| \\
&\leq C\Big(\int_{\mathbb{R}^{2n}} |u_j(x)|^p |\phi^i_\delta(x) - \phi^i_\delta(y)|^p K(x-y)\,dx\,dy\Big)^{1/p}.
\end{aligned}$$

Now we claim that

$$\lim_{\delta\to 0}\Big[\lim_{j\to+\infty}\Big(\int_{\mathbb{R}^{2n}}|u_j(x)|^p|\phi_\delta^i(x)-\phi_\delta^i(y)|^pK(x-y)\,dx\,dy\Big)\Big]=0. \tag{3.6}$$

Using the Lipschitz regularity of ϕ_δ^i, we have, for some $L\geq 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}}|u_j(x)|^p|\phi_\delta^i(x)-\phi_\delta^i(y)|^pK(x-y)\,dx\,dy\\
&\leq \frac{1}{\theta}\int_{\mathbb{R}^{2n}}|u_j(x)|^p|\phi_\delta^i(x)-\phi_\delta^i(y)|^p|x-y|^{-n-ps}\,dx\,dy\\
&\leq \frac{L^p\delta^{-p}}{\theta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\cap\{|x-y|\leq\delta\}}|u_j(x)|^p|x-y|^{p-n-ps}\,dx\,dy\\
&\quad+\frac{2^p}{\theta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\cap\{|x-y|>\delta\}}|u_j(x)|^p|x-y|^{-n-ps}\,dx\,dy\\
&\leq C\frac{(L^p\delta^{-p}+2^p)}{\theta}\int_{\mathbb{R}^n}|h(x)|^p\,dx\,dy<+\infty,
\end{aligned} \tag{3.7}$$

with $C=C(n,s,\delta)>0$. So, by dominated convergence theorem

$$\begin{aligned}
&\lim_{j\to+\infty}\int_{\mathbb{R}^{2n}}|u_j(x)|^p|\phi_\delta^i(x)-\phi_\delta^i(y)|^pK(x-y)\,dx\,dy\\
&=\int_{\mathbb{R}^{2n}}|u(x)|^p|\phi_\delta^i(x)-\phi_\delta^i(y)|^pK(x-y)\,dx\,dy.
\end{aligned}$$

Now, following the calculations in (3.7), we obtain

$$\begin{aligned}
&\int_{U\times V}|u(x)|^p|\phi_\delta^i(x)-\phi_\delta^i(y)|^pK(x-y)\,dx\,dy\\
&\leq \frac{L^p}{\theta}\delta^{-p}\int_U\int_{V\cap\{|x-y|\leq\delta\}}|u(x)|^p|x-y|^{p-n-ps}\,dx\,dy\\
&\quad+\frac{2^p}{\theta}\int_U\int_{V\cap\{|x-y|>\delta\}}|u(x)|^p|x-y|^{-n-ps}\,dx\,dy,
\end{aligned} \tag{3.8}$$

where U and V are two generic subsets of $\mathbb{R}^n$. Next we claim that

$$\int_{\mathbb{R}^{2n}}|u(x)|^p|\phi_\delta^i(x)-\phi_\delta^i(y)|^pK(x-y)\,dx\,dy\to 0,\quad \text{as } \delta\to 0.$$

When $U=V=\mathbb{R}^n\setminus B(x_i,\delta)$ claim follows. When $U\times V=B(x_i,\delta)\times\mathbb{R}^n$ and $U\times V=\mathbb{R}^n\times B(x_i,\delta)$, we can use Proposition 2.6 to prove the claim. Thus

$$\lim_{\delta\to 0}\int_{\mathbb{R}^{2n}}|u(x)|^p|\phi_\delta^i(x)-\phi_\delta^i(y)|^pK(x-y)\,dx\,dy=0. \tag{3.9}$$

Hence

$$\begin{aligned}
&M_a(\|u_j\|_{X_0}^p)\int_{\mathbb{R}^{2n}}u_j(x)|(u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))\\
&\times\big(\phi_\delta^i(x)-\phi_\delta^i(y)\big)K(x-y)\,dx\,dy\to 0,
\end{aligned} \tag{3.10}$$

as $\delta \to 0$ and $j \to \infty$. Now, using Hölder's inequality,

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^n} \frac{u_j(x)-u_j(y)}{|x-y|^{n+ps}}\Big|^p \\
&\le 2^{p-1}\Big[|u_j(y)|^p\Big|\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+ps}}\Big|^p + \Big|\int_\Omega \frac{u_j(x)-u_j(y)}{|x-y|^{n+ps}}dx\Big|^p\Big] \\
&\le C_1|u_j(y)|^p + C_2\int_\Omega |u_j(x)-u_j(y)|^p K(x-y)dx,
\end{aligned}
\tag{3.11}
$$

where $C_1 = 2^{p-1}|\int_{\mathbb{R}^n\setminus\Omega}\frac{dx}{|x-y|^{n+ps}}|^p$ and $C_2 = 2^{p-1}/\theta$. Now using equations (3.11) and (3.2), we obtain

$$
\begin{aligned}
&\liminf_{j\to+\infty}\int_{\mathbb{R}^n}\phi_\delta^i(y)\int_\Omega |u_j(x)-u_j(y)|^p K(x-y)\,dx\,dy \\
&\ge C_3\frac{1}{c(n,s)}\liminf_{j\to+\infty}\int_{\mathbb{R}^n}\phi_\delta^i(y)c(n,s)\Big|\int_{\mathbb{R}^n}\frac{u_j(x)-u_j(y)}{|x-y|^{n+ps}}\,dx\Big|^p dy \\
&\quad - C_4\liminf_{j\to+\infty}\int_{\mathbb{R}^n}\phi_\delta^i(y)|u_j(y)|^p dy \\
&\ge C_3\frac{1}{c(n,s)}\int_{\mathbb{R}^n}\phi_\delta^i(y)d\mu - C_4\int_{B(x_i,\delta)}|u(y)|^p dy,
\end{aligned}
\tag{3.12}
$$

where $C_3 = 1/C_2$ and $C_4 = C_1/C_2$. Moreover, for a given $\epsilon > 0$ there exist $C_\epsilon > 0$ such that

$$
|f(x,t)| \le \epsilon|t|^{p-1} + C_\epsilon|t|^{q-1}\,. \tag{3.13}
$$

So, using Vitali's convergence theorem, we obtain

$$
\int_{B(x_i,\delta)} f(x,u_j(x))u_j(x)\phi_\delta^i(x)dx \to \int_{B(x_i,\delta)} f(x,u(x))u(x)\phi_\delta^i(x)dx, \tag{3.14}
$$

as $j \to +\infty$. We also observe that the integral goes to 0 as $\delta \to 0$. So, using (3.10), (3.12), (3.14) and (3.2) in (3.5), we obtain

$$
\begin{aligned}
&\int_\Omega \phi_\delta^i(x)d\nu + \lambda\int_{B(x_i,\delta)} f(x,u(x))u(x)\phi_\delta^i(x)dx \\
&\ge M_a(\alpha^p)C\Big(\int_\Omega \phi_\delta^i(y)d\mu - \int_{B(x_i,\delta)}|u(y)|^p dy\Big) + o_\delta(1).
\end{aligned}
$$

Now, by taking $\delta \to 0$, we conclude that $\nu_i \ge M_a(\alpha^p)C\mu_i \ge m_0C\mu_i$. Then by (3.4), we obtain

$$
\nu_i \ge \frac{(m_0C)^{n/ps}}{S^{(n-ps)/ps}}, \tag{3.15}
$$

for any $i \in J$, where $C = \frac{C_3}{c(n,s)}$, independent of λ. We will prove that (3.15) is not possible.
Consider

$$
\lim_{j\to+\infty}\Big(\mathcal{J}_{a,\lambda}(u_j) - \frac{1}{\sigma}\mathcal{J}'_{a,\lambda}(u_j)(u_j)\Big) = c_{a,\lambda}. \tag{3.16}
$$

Since, $m_0 < a < \frac{\sigma}{p} m_0$, we obtain

$$
\begin{aligned}
&\mathcal{J}_{a,\lambda}(u_j) - \frac{1}{\sigma}\mathcal{J}'_{a,\lambda}(u_j)(u_j)\\
&\geq \frac{1}{p}\widehat{M_a}(\|u_j\|_{X_0}^p) - \frac{1}{\sigma}M_a(\|u_j\|_{X_0}^p)\|u_j\|_{X_0}^p + (\frac{1}{\sigma} - \frac{1}{p^*})\int_\Omega |u_j(x)|^{p^*}dx\\
&\geq \frac{1}{p}m_0\|u_j\|_{X_0}^p - \frac{1}{\sigma}a\|u_j\|_{X_0}^p + (\frac{1}{\sigma} - \frac{1}{p^*})\int_\Omega |u_j(x)|^{p^*}dx \qquad (3.17)\\
&\geq (\frac{1}{p}m_0 - \frac{1}{\sigma}a)\|u_j\|_{X_0}^p + (\frac{1}{\sigma} - \frac{1}{p^*})\int_\Omega |u_j(x)|^{p^*}dx\\
&\geq (\frac{1}{\sigma} - \frac{1}{p^*})\int_\Omega \phi_\delta^i(x)|u_j(x)|^{p^*}dx.
\end{aligned}
$$

So, as $j \to 0$,

$$
c_{a,\lambda} \geq (\frac{1}{\sigma} - \frac{1}{p^*})\int_\Omega \phi_\delta^i(x)d\nu
$$

Now, taking $\delta \to 0$,

$$
c_{a,\lambda} \geq (\frac{1}{\sigma} - \frac{1}{p^*})\frac{(m_0C)^{\frac{n}{ps}}}{S^{\frac{n-ps}{ps}}} > 0,
$$

for all λ, but from Lemma 2.5, there exists $\lambda_0 > 0$ such that

$$
c_{a,\lambda} < (\frac{1}{\sigma} - \frac{1}{p^*})\frac{(m_0C)^{\frac{n}{ps}}}{S^{\frac{n-ps}{ps}}}
$$

for all $\lambda > \lambda_0$, which is a contradiction. Therefore $\nu_i = 0$ or all $i \in J$. Hence J is empty. Which implies $u_j \to u$ in $L^{p^*}(\Omega)$. So, by (2.7) taking $\phi = u_j$ and using dominated convergence theorem,

$$
\lim_{j\to+\infty} M_a(\|u_j\|_{X_0}^p)\|u_j\|_{X_0}^p = \lambda\int_\Omega f(x,u(x))u(x)dx + \int_\Omega |u(x)|^{2^*}dx. \qquad (3.18)
$$

Now, we take $\phi = u$ in (2.7) and recalling that $M_a(\|u_j\|_{X_0}^p) \to M_a(\alpha^p)$, we obtain

$$
M_a(\alpha^p)\|u_j\|_{X_0}^p = \lambda\int_\Omega f(x,u(x))u(x)\,dx - \int_\Omega |u(x)|^{2^*}dx\,. \qquad (3.19)
$$

So, combining (3.18) and (3.19), we obtain

$$
M_a(\|u_j\|_{X_0}^p)\|u_j\|_{X_0}^p \to M_a(\alpha^p)\|u\|_{X_0}^p, \quad \text{as } j \to +\infty.
$$

So, using this result, we have

$$
\begin{aligned}
M_a(\|u_j\|_{X_0}^p)(\|u_j\|_{X_0}^p - \|u\|_{X_0}^p) &= M_a(\|u_j\|_{X_0}^p)\|u_j\|_{X_0}^p - M_a(\alpha^p)\|u\|_{X_0}^p\\
&\quad - M_a(\|u_j\|_{X_0}^p)\|u\|_{X_0}^p + M_a(\alpha^p)\|u\|_{X_0}^p,
\end{aligned}
$$

which leads to

$$
M_a(\|u_j\|_{X_0}^p)(\|u_j\|_{X_0}^p - \|u\|_{X_0}^p) \to 0.
$$

Also

$$
m_0(\|u_j\|_{X_0}^p - \|u\|_{X_0}^p) \leq M_a(\|u_j\|_{X_0}^p)(\|u_j\|_{X_0}^p - \|u\|_{X_0}^p), \qquad (3.20)
$$

which implies $\|u_j\|_{X_0}^p \to \|u\|_{X_0}^p$ and the claim is proved. Hence $u_j \to u$ strongly in X_0. ☐

Proof of Theorem 2.1. Using Lemma 3.1 and by Mountain Pass Theorem, we obtain a critical point u for the functional $\mathcal{J}_{a,\lambda}$ at the level $c_{a,\lambda}$. Since $\mathcal{J}_{a,\lambda}(u) = c_{a,\lambda} > 0 = \mathcal{J}_{a,\lambda}(0)$, we conclude that $u \not\equiv 0$. ☐

Proof of Theorem 1.2. Now to conclude the proof of Theorem 1.2 we claim that

$$\text{there exists } \lambda^* \geq \lambda_0 \text{ such that } \|u_\lambda\|_{X_0} \leq t_0 \text{ for all } \lambda \geq \lambda^*, \tag{3.21}$$

where t_0 is defined in (2.1). Suppose not, then there exists a sequence $\{\lambda_j\}$ in $\mathbb{R}$ such that $\|u_{\lambda_j}\|_{X_0} \geq t_0$. Since u_{λ_j} is a critical point of the functional $\mathcal{J}_{a,\lambda_j}$ which implies

$$\begin{aligned} c_{a,\lambda_j} &\geq \frac{1}{p}\widehat{M_a}(\|u_{\lambda_j}\|^p_{X_0}) - \frac{1}{\sigma} M_a(\|u_{\lambda_j}\|^p_{X_0})\|u_{\lambda_j}\|^p_{X_0} \\ &\geq \big(\frac{1}{p}m_0 - \frac{1}{\sigma}a\big)\|u_{\lambda_j}\|^p_{X_0} \\ &\geq \big(\frac{1}{p}m_0 - \frac{1}{\sigma}a\big)t_0^p, \end{aligned}$$

which contradicts Lemma 2.5, since $m_0 < a < \frac{\sigma}{p}m_0$. Hence u_λ is a solution of problem (M_λ). Since $c_{a,\lambda} \to 0$ as $\lambda \to 0$, implies $\|u_\lambda\|_{X_0} \to 0$ as $\lambda \to \infty$.

Now, we claim that u_λ is non-negative in $\mathbb{R}^n$. Take $v = u^- \in X_0$, in (2.4), where $u^- = \max(-u, 0)$. Then

$$\begin{aligned} &M(\|u_\lambda\|^p_{X_0}) \int_Q |u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y))K(x-y)\,dx\,dy \\ &= \int_\Omega f(x, u_\lambda)u_\lambda^-(x)\,dx + \int_\Omega |u_\lambda^-(x)|^{p^*}\,dx. \end{aligned}$$

Now, using

$$(u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y)) \leq -|u_\lambda^-(x) - u_\lambda^-(y)|^2$$

and $f(x, u_\lambda(x))u_\lambda^-(x) = 0$ for a.e. $x \in \mathbb{R}^n$ we obtain

$$0 \leq -M(\|u_\lambda\|^p_{X_0}) \int_Q |u_\lambda^-(x) - u_\lambda^-(y)|^p K(x-y) - \int_\Omega |u_\lambda^-(x)|^{p^*}\,dx \leq -m_0\|u_\lambda^-\|^p_{X_0}.$$

Thus $\|u_\lambda^-\|_{X_0} = 0$ and hence $u_\lambda > 0$. ☐

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