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AN EXTENSION OF THE LAX-MILGRAM THEOREM AND ITS APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, using an iterative technique, we introduce an extension of the Lax-Milgram theorem which can be used for proving the existence of solutions to boundary-value problems. Also, we apply of the obtained result to the fractional differential equation

$$
\begin{gathered}
{}_tD_T^{\alpha}{}_0D_t^{\alpha}u(t)+u(t)=\lambda f(t,u(t)) \quad t\in(0,T),\\
u(0)=u(T)=0,
\end{gathered}
$$

where ${}_tD_T^{\alpha}$ and ${}_0D_t^{\alpha}$ are the right and left Riemann-Liouville fractional derivative of order $\frac{1}{2}<\alpha\leq 1$ respectively, λ is a parameter and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Applying a regularity argument to this equation, we show that every weak solution is a classical solution.

1. Introduction

Fractional differential equations form a very important and significant part of mathematical analysis and its applications to real-world problems. On the other hand, the Lax-Milgram theorem is a very useful tool in the wide area of functional analysis such as the theory of operator equations in Banach spaces. It is also used in the studies of fractional differential equations, ordinary and partial differential equations (see [1, 3, 4] and the references therein). For example, Ervin and Roop [3] using the Lax-Milgram theorem, investigated the existence of solutions to the following fractional boundary value problem

$$
\begin{gathered}
-Da\big(p_0D_t^{-\beta}+q_tD_1^{-\beta}\big)Du(t)+b(t)Du(t)+c(t)u(t)=f(t),\\
u(0)=u(1)=0,
\end{gathered}
$$

where D represents a single spatial derivative, ${}_0D_t^{-\beta}$ and ${}_tD_1^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0\leq\beta<1$, respectively, $f,c\in C([0,1])$ and $b\in C^1([0,1])$, $a>0$ and $0\leq p,q\leq 1$ with $p+q=1$.

Recently, Jiao and Zhou [7], for the first time, showed that the critical point theory is an effective approach for studying the existence for the following fractional boundary value problem

$$
{}_tD_T^{\alpha}\big({}_0D_t^{\alpha}u(t)\big)=\nabla F(t,u(t)), \quad t\in(0,T),
$$

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$$u(0) = u(T) = 0,$$

and obtained the existence of at least one nontrivial solution.

We know that, we can only use the Lax-Milgram theorem to prove the existence of solutions to equations in the form

$$Lu = f(t),$$

where f is independent of u and L is an operator. In our investigations, we apply the iterative technique to generalize the Lax-Milgram theorem [2]. Moreover, we are going to study the solvability of the following fractional differential equation

$$\begin{gathered} {}_tD_T^\alpha {}_0D_t^\alpha u(t) + u(t) = \lambda f(t, u(t)) \quad t \in (0, T), \\ u(0) = u(T) = 0, \end{gathered} \tag{1.1}$$

where ${}_tD_T^\alpha$ and ${}_0D_t^\alpha$ are the right and left Riemann-Liouville fractional derivative of order $1/2 < \alpha \leq 1$ respectively, λ is a parameter and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

This article is organized as follows: Section 2 is devoted to our main results. In Section 3, we prove any weak solution of Problem (1.1) is a classical solution.

2. Main results

In this section, firstly, we recall some notation and theorems to obtain the results of this work. Let $(X, \|\cdot\|_X)$ be a real Banach space with dual space X^*. Denote by $B_r(x_0)$ the ball $B_r(x_0) = \{x \in X : \|x - x_0\|_X \leq r\}$.

In the following, we state the Lax-Milgram theorem.

Theorem 2.1 ([2, Proposition 1.2.41]). *Let H be a complex Hilbert space and let $B : H \times H \to \mathbb{C}$ be a mapping with the following properties:*

- (i) *The mapping $x \mapsto B(x, y)$ is linear for any $y \in H$.*
- (ii) *$B(x, \alpha_1 y_1 + \alpha_2 y_2) = \bar{\alpha}_1 B(x, y_1) + \bar{\alpha}_2 B(x, y_2)$ for every $x, y_1, y_2 \in H$, $\alpha_1, \alpha_2 \in \mathbb{C}$.*
- (iii) *There is a constant c such that $|B(x, y)| \leq c\|x\|\,\|y\|$ for every $x, y \in H$.*

Then there is $A \in L(H)$, $\|A\|_{L(H)} \leq c$, such that

$$B(x, y) = (x, Ay), \quad x,\ y \in H.$$

Moreover, if

- (iv) *there is a positive constant d such that*

$$B(x, x) \geq d\|x\|^2 \quad \forall x \in H,$$

then A is invertible, $A^{-1} \in L(H)$ and $\|A^{-1}\|_{L(H)} \leq \frac{1}{d}$.

The main result of this section reads as follows.

Theorem 2.2. *Suppose that H is a Hilbert space, $B(u, v)$ is a continuous coercive bilinear form on H and $F : H \to H^*$ satisfying the following conditions:*

- (F1) *There exists a constant $N > 0$ such that*

$$\|F(u)\|_{H^*} \leq N \quad \forall u \in B_1(0),$$

where $B_1(0) = \{u \in H : \|u\|_H \leq 1\}$

- (F2) *If $\{u_k\}$ is a sequence in H such that $u_k \rightharpoonup u$ weakly in H, then the sequence $\{F(u_k)\}$ has a subsequence $\{F(u_{k_n})\}$ such that $F(u_{k_n}) \rightharpoonup F(u)$ weakly in H^*.*

Then, there exists a constant $L > 0$ such that for any $\lambda \in \mathbb{R}$ with $|\lambda| \le L$, there exists an element $\tilde{u} \in H$ such that

$$B(\tilde{u}, v) = \lambda\langle F(\tilde{u}), v\rangle \quad \forall v \in H.$$

Proof. Take any $u_0 \in H$ with $\|u_0\|_H \le 1$. From the Riesz representation theorem, there exists a unique element $G(u_0) \in H$ such that $\|G(u_0)\|_H = \|F(u_0)\|_{H^*}$ and

$$\langle F(u_0), v\rangle = (G(u_0), v) \quad \forall v \in H, \tag{2.1}$$

where $(\cdot, \cdot)$ denotes the inner product of H.

By the hypotheses of the theorem, there are two constants $a > 0$ and $b > 0$ such that for all $u, v \in H$, we have

$$|B(u, v)| \le a\|u\|_H\|v\|_H,$$
$$|B(u, u)| \ge b\|u\|_H^2.$$

Thus, the Lax-Milgram theorem (see [2]) yields the existence of a continuous and invertible linear operator A on H such that $\|A\|_{L(H)} \le a$, $\|A^{-1}\|_{L(H)} \le 1/b$ and

$$B(u, v) = (Au, v) \quad \forall u, v \in H. \tag{2.2}$$

Then, one can conclude the existence of a unique element $u_1 \in H$ such that $Au_1 = \lambda G(u_0)$. So, in view of (2.1) and (2.2), we have that

$$B(u_1, v) = (Au_1, v) = \lambda(G(u_0), v) = \lambda\langle F(u_0), v\rangle \forall v \in H$$

such that

$$\|u_1\|_H = |\lambda|\ \|A^{-1}G(u_0)\|_H \le \frac{N|\lambda|}{b}.$$

Set $L = b/N$. Hence if $|\lambda| \le L$, one can get

$$B(u_1, v) = \lambda\langle F(u_0), v\rangle \quad \forall v \in H,$$
$$\|u_1\|_H \le 1.$$

Similarly, there exists an element $u_2 \in H$ such that

$$B(u_2, v) = \lambda\langle F(u_1), v\rangle \quad \forall v \in H,$$
$$\|u_2\|_H \le 1.$$

So by induction, we have a sequence $\{u_n\}$ such that

$$\begin{gathered} B(u_n, v) = \lambda\langle F(u_{n-1}), v\rangle \quad \forall v \in H, \\ \|u_n\|_H \le 1. \end{gathered} \tag{2.3}$$

The reflexivity of H implies that there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n \rightharpoonup \tilde{u}$ weakly in H. Finally, in view of (F2), the desired conclusion follows from (2.3) and letting $n \to +\infty$. □

Now, by using Theorem 2.2, we prove the existence of one solution to Problem (1.1). To this end, we need the following preliminaries.

Definition 2.3 ([9, 8])**.** Let ϕ be a function defined on $[0, T]$. Then, the left and right Riemann-Liouville fractional integrals of order $0 < \alpha < 1$ on the interval $[0, T]$ are respectively defined by

$${}_0I_t^\alpha\phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\phi(\xi)d\xi,$$

$$
{}_tI_T^\alpha\phi(t)=\frac{1}{\Gamma(\alpha)}\int_t^T(\xi-t)^{\alpha-1}\phi(\xi)d\xi.
$$

The left and right Riemann-Liouville fractional derivatives of order $0<\alpha<1$ on the interval $[0,T]$ are respectively defined by

$$
\begin{gathered}
{}_0D_t^\alpha\phi(t)=\frac{d}{dt}\Big({}_0I_t^{1-\alpha}\phi(t)\Big),\\
{}_tD_T^\alpha\phi(t)=-\frac{d}{dt}\Big({}_tI_T^{1-\alpha}\phi(t)\Big).
\end{gathered}
$$

Taking $p=2$ in Definition 3.1, [6, Propositions 3.1 and 3.3], we deduce the following definition and theorems.

Definition 2.4 ([6]). Let $0<\alpha\le 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty([0,T])$ with respect to the norm

$$
\|u\|_{E_0^\alpha}=\Big(\|u\|_{L^2(0,T)}^2+\|{}_0D_t^\alpha u\|_{L^2(0,T)}^2\Big)^{1/2}.
$$

Theorem 2.5 ([6]). *Let $0<\alpha\le 1$. The fractional derivative space E_0^α is a reflexive and separable Banach space.*

Remark 2.6. In fact, the space E_0^α is a separable Hilbert space with the inner product

$$
(u,v)_{E_0^\alpha}=\int_0^T\Big({}_0D_t^\alpha u(t)\ {}_0D_t^\alpha v(t)+u(t)v(t)\Big)dt.
$$

Theorem 2.7 ([6]). *Assume that $\alpha>\frac12$ and the sequence $\{u_k\}$ converges weakly to u in E_0^α, then $u_k\to u$ in $C([0,T])$.*

Remark 2.8 ([6]). We have

$$
\|u\|_\infty\le\frac{T^{\alpha-\frac12}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u\|_{E_0^\alpha}\quad\forall u\in E_0^\alpha.
$$

Definition 2.9. A function $u\in E_0^\alpha$ is a weak solution of Problem (1.1), provided that

$$
\int_0^T\Big({}_0D_t^\alpha u(t){}_0D_t^\alpha v(t)+u(t)v(t)\Big)dt=\lambda\int_0^T f(t,u(t))v(t)dt. \tag{2.4}
$$

for any $v\in E_0^\alpha$.

Set

$$
\Delta=\max\Big\{f(t,s):(t,s)\in[0,T]\times\Big[\frac{-T^{\alpha-\frac12}}{\Gamma(\alpha)(2\alpha-1)^{1/2}},\frac{T^{\alpha-\frac12}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\Big]\Big\}.
$$

Theorem 2.10. *Suppose that $\frac12<\alpha\le 1$ and $f\in C([0,T]\times\mathbb{R},\mathbb{R})$, then for any $|\lambda|\le\frac{1}{\Delta T^{1/2}}$, Problem (1.1) has at least one weak solution.*

Proof. First, we define $B(u,v)=\int_0^T\Big({}_0D_t^\alpha u(t){}_0D_t^\alpha v(t)+u(t)v(t)\Big)dt$. Since,

$$
\begin{gathered}
|B(u,v)|\le\|{}_0D_t^\alpha u\|_2\|{}_0D_t^\alpha v\|_2+\|u\|_2\|v\|_2\le 2\|u\|_{E_0^\alpha}^2\|v\|_{E_0^\alpha}^2,\\
|B(u,u)|\ge\|u\|_{E_0^\alpha}^2,
\end{gathered}
$$

it follows that B is a continuous coercive bilinear form on E_0^α with $a=2$ and $b=1$.

We now define

$$F : E_0^\alpha \to (E_0^\alpha)^*,$$
$$\langle F(u), v\rangle = \int_0^T f(t,u(t))v(t)dt.$$

Assume $u \in E_0^\alpha$ with $\|u\|_{E_0^\alpha} \le 1$. Then, in view of Remark 2.8, one has

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u\|_{E_0^\alpha} \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}. \tag{2.5}$$

and we have $|u(t)| \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}$ for any $t \in [0,T]$. So, we can conclude that $|f(t,u(t))| \le \Delta$ for any $t \in [0,T]$.

For any $v \in E_0^\alpha$ with $\|v\|_{E_0^\alpha} = 1$, by the Hölder inequality, we have

$$|\langle F(u), v\rangle| = \Big|\int_0^T f(t,u(t))v(t)dt\Big| \le \Big(\int_0^T |f(t,u)|^2 dt\Big)^{1/2}\|v\|_{L^2(0,T)} \le \Delta T^{1/2}.$$

Taking $N = \Delta T^{1/2}$, Condition (F1) holds.

Suppose $\{u_k\}$ is a sequence in E_0^α such that $u_k \rightharpoonup u$ weakly in E_0^α. Then, Theorem 2.7 yields that for any $t \in [0,T]$

$$u_k(t) \to u(t) \quad \forall t \in [0,T].$$

By using it and that f is continuous, we have

$$f(t,u_k(t)) \to f(t,u(t)) \quad \text{as } k \to \infty, \quad \forall t \in [0,T]. \tag{2.6}$$

On the other hand, $\{u_k\}$ is a bounded subset of E_0^α (see [10, Theorem 3.18]). In other words, there exists a constant $K > 0$ such that $\|u_k\|_{E_0^\alpha} \le K$ for any $k \in \mathbb{N}$. From (2.5), we have $\|u_k\|_\infty \le \frac{KT^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}$ for any $k \in \mathbb{N}$. Therefore, we have that there exists a constant $\Delta_0 > 0$ such that

$$|f(t,u_k(t))| < \Delta_0 \quad \text{a.e. on } [0,T],\ k = 1,2,3,\dots. \tag{2.7}$$

From (2.7), (2.6) and the Lebesgue's dominated theorem, we conclude that

$$\int_0^T \Big|f(t,u_k) - f(t,u)\Big|^2 dt \to 0.$$

For any $v \in E_0^\alpha$ with $\|v\|_{E_0^\alpha} = 1$, we have

$$|\langle F(u_k) - F(u), v\rangle| = \Big|\int_0^T \Big(f(t,u_k(t)) - f(t,u(t)\Big)v(t)dt\Big| \le \|f(t,u_k) - f(t,u)\|_2 \to 0,$$

which yields that F satisfies (F2). Then by Theorem 2.2, we obtain the desired conclusion. $\square$

3. Regularity

The main result of this section reads as follows.

Theorem 3.1. *Under the assumptions of Theorem 2.10, every weak solution of Problem* (1.1) *is a classical solution.*

To prove the above theorem, we need the following lemmas and definitions.

Definition 3.2 ([5]). Let $u \in L^2(0,T)$, $v, w \in L^2(0,T)$ and

$$\int_0^T u(t)_t D_T^\alpha \varphi(t)dt = \int_0^T v(t)\varphi(t)dt \quad \forall \varphi \in C_0^\infty(0,T),$$
$$\int_0^T u(t)_0 D_t^\alpha \varphi(t)dt = \int_0^T w(t)\varphi(t)dt \quad \forall \varphi \in C_0^\infty(0,T).$$

The functions v and w given above will be called the weak left and the weak right fractional derivative of order $\alpha \in (0,1]$ of u respectively. Here, we denote them by ${}_0\overline{D}_t^\alpha u(t)$ and ${}_t\overline{D}_T^\alpha u(t)$ respectively.

In view of Definition 2.4, $u \in E_0^\alpha$ means that u is the limit of a Cauchy sequence $\{u_n\} \subset C_0^\infty(0,T)$. In other words, $u_n \to u$ in $L^2(0,T)$ and there exists an element $w \in L^2(0,T)$ such that ${}_0D_t^\alpha u_n \to w$ in $L^2(0,T)$. Then for any $\varphi \in C_0^\infty(0,T)$, we have

$$\begin{aligned}\int_0^T w(t)\varphi(t)dt &= \lim_{n\to\infty}\int_0^T {}_0D_t^\alpha u_n(t)\varphi(t)dt \\ &= \lim_{n\to\infty}\int_0^T u_n(t)_t D_T^\alpha \varphi(t)dt \\ &= \int_0^T u(t)_t D_T^\alpha \varphi(t)dt.\end{aligned}$$

So, $w = {}_0\overline{D}_t^\alpha u$ however it is not clear whether ${}_0D_t^\alpha u(t)$ exists in the usual sense, for any $t \in [0,T]$ or not (see [1, p. 202] for the case $\alpha = 1$).

Remark 3.3 ([3, Lemma 2.7]). Let $u \in E_0^\alpha$, then for any $v \in E_0^\alpha$, we have

$$\begin{aligned}\int_0^T u(t)_t D_T^\alpha v(t)dt &= \int_0^T {}_0I_t^\alpha {}_0D_t^\alpha u(t)_t D_T^\alpha v(t)dt \\ &= \int_0^T {}_0D_t^\alpha u(t)_t I_T^\alpha {}_t D_T^\alpha v(t)dt \\ &= \int_0^T {}_0D_t^\alpha u(t)\ v(t)dt.\end{aligned}$$

Since $C_0^\infty(0,T) \subset E_0^\alpha$, we conclude ${}_0D_t^\alpha u(t) = {}_0\overline{D}_t^\alpha u(t)$ a.e. on $[0,T]$.

Lemma 3.4. *Let $u \in E_0^\alpha$, then ${}_0\overline{D}_t^\alpha u$ is almost everywhere equal to the weak derivative of ${}_0I_t^{1-\alpha}u$ in the $H^1(0,T)$ sense. In other words*

$$ {}_0\overline{D}_t^\alpha u(t) = \overline{D}({}_0I_t^{1-\alpha}u(t)) \quad \textit{a.e. on } [0,T].$$

Proof. For any $\varphi \in C_0^\infty(0,T)$, we have (see [6, Remark 3.1] and [8, Theorem 2.1])

$$\begin{aligned}\int_0^T {}_0\overline{D}_t^\alpha u(t)\varphi(t)dt &= \int_0^T u(t)_t D_T^\alpha \varphi(t)dt \\ &= \int_0^T u(t)_t^C D_T^\alpha \varphi(t)dt \\ &= -\int_0^T u(t)_t I_T^{1-\alpha}(D\varphi(t))dt \\ &= -\int_0^T {}_0I_t^{1-\alpha}u(t)D\varphi(t)dt.\end{aligned}$$

□

Lemma 3.5. *Let $0 < \alpha \le 1$.*

(i) *If $\frac{1}{2} < \alpha \le 1$ and $u \in L^2(0,T)$, then ${}_0I_0^\alpha u(0) = 0$.*
(ii) *If $u \in C([0,T])$, then ${}_0I_t^\alpha u \in C([0,T])$.*
(iii) *If $u \in C^1([0,T])$ and $u(0) = 0$, then ${}_0I_t^\alpha u \in C^1([0,T])$.*

Proof. It is easy to see that

$$|{}_0I_t^\alpha u(t)| \le \frac{t^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u\|_2,$$

which completes the proof of (i).

Let $t_0 \in [0,T]$ and $\{t_n\}$ be a sequence in $[0,T]$ such that $t_n \to t_0$. We take

$$M = \max_{0\le t\le T} |u(t)|.$$

From $u(t_n - s) \to u(t_0 - s)$ a.e. on $[0,T]$,

$$s^{\alpha-1}|u(t_n - s) - u(t_0 - s)| \le 2Ms^{\alpha-1},$$

and the Lebesgue's dominated theorem, we conclude that

$$\begin{aligned}
&\Big|\int_0^{t_n} s^{\alpha-1}u(t_n - s)ds - \int_0^{t_0} s^{\alpha-1}u(t_0 - s)ds\Big| \\
&\le \int_0^{t_n} s^{\alpha-1}|u(t_n - s) - u(t_0 - s)|ds + \int_{t_0}^{t_n} |s^{\alpha-1}u(t_0 - s)|ds \\
&\le \int_0^{\xi} s^{\alpha-1}|u(t_n - s) - u(t_0 - s)|ds + \int_{t_0}^{t_n} |s^{\alpha-1}u(t_0 - s)|ds \to 0,
\end{aligned}$$

where $\xi = \max\{t_0, t_1, t_2, \dots\}$. This concludes the proof of (ii).

Suppose that

$$K = \max_{0\le t\le T} |u'(t)|,$$

then $|\frac{u(t_n-s)-u(t_0-s)}{t_n-t_0}| \le K$. Hence by the Lebesgue's dominated theorem, we conclude that

$$\begin{aligned}
&\Gamma(\alpha)\Big(\frac{{}_0I_{t_n}^\alpha u(t_n) - {}_0I_{t_0}^\alpha u(t_0)}{t_n - t_0}\Big) \\
&= \int_0^{t_0} s^{\alpha-1}\frac{u(t_n - s) - u(t_0 - s)}{t_n - t_0}ds + \int_{t_0}^{t_n} s^{\alpha-1}\frac{u(t_n - s) - u(t_0 - s)}{t_n - t_0}ds \\
&\quad + \int_{t_0}^{t_n} s^{\alpha-1}\frac{u(t_0 - s) - u(0)}{t_0 - s}\,\frac{t_0 - s}{t_n - t_0}ds \to \int_0^{t_0} s^{\alpha-1}u'(t_0 - s)ds.
\end{aligned}$$

Thus $\Big({}_0I_t^\alpha u(t)\Big)'(t_0) = {}_0I_{t_0}^\alpha u'(t_0)$. From this and part (ii), we can conclude (iii). □

Lemma 3.6. *Suppose that for some $u \in L^2(0,T)$, ${}_0\overline{D}_t^\alpha u$ exists and is almost everywhere equal to a function in $C([0,T])$. Then:*

(i) *u is almost everywhere equal to a function $\tilde{u} \in C([0,T])$.*
(ii) *${}_0D_t^\alpha u(t)$ exists for any $t \in [0,T]$ and ${}_0D_t^\alpha u \in C([0,T])$.*

Proof. Lemma 3.4 implies that $\overline{D}({}_0I_t^{1-\alpha}u)$ is almost everywhere equal to a function in $C([0,T])$. Therefore, ${}_0I_t^{1-\alpha}u$ is almost everywhere equal to a function in $C^1([0,T])$ (see [1, p. 204]). Thus by Lemma 3.5, $\int_0^t u(s)ds = {}_0I_t^{\alpha}{}_0I_t^{1-\alpha}u(t) \in C^1([0,T])$. Take $\tilde{u}(t) = D(\int_0^t u(s)ds)$ (see [1, Lemma 8.2]), this completes the proof of (i).

Lemma 3.5 implies that ${}_0I_t^{1-\alpha}\tilde{u}(t) \in C([0,T])$. Since

$$ {}_0I_t^{1-\alpha}u(t) = {}_0I_t^{1-\alpha}\tilde{u}(t) \quad \forall t \in [0,T], \tag{3.1} $$

we can conclude ${}_0I_t^{1-\alpha}u(t) \in C([0,T])$. By using it and the fact that ${}_0I_t^{1-\alpha}u$ is almost everywhere equal to a function in $C^1([0,T])$, we can conclude ${}_0I_t^{1-\alpha}u(t) \in C^1([0,T])$. The desired conclusion can be obtained from ${}_0D_t^{\alpha}u(t) = D\big({}_0I_t^{1-\alpha}u(t)\big)$. □

Quite similar to Lemma 3.5 and Lemma 3.6, we have the following lemmas.

Lemma 3.7. *Let $1/2 < \alpha \leq 1$.*

(i) *If $u \in L^2(0,T)$, then ${}_TI_T^{\alpha}u(T) = 0$.*
(ii) *If $u \in C([0,T])$, then ${}_tI_T^{\alpha}u \in C([0,T])$.*
(iii) *If $u \in C^1([0,T])$ and $u(T) = 0$, then ${}_tI_T^{\alpha}u \in C^1([0,T])$.*

Lemma 3.8. *Suppose that $u \in L^2(0,T)$, ${}_tD_T^{\alpha}u$ exists and is almost everywhere equal to a function in $C([0,T])$. Then:*

(i) *u is almost everywhere equal to a function $\tilde{u} \in C([0,T])$.*
(ii) *${}_tD_T^{\alpha}u(t)$ exists for any $t \in [0,T]$ and ${}_tD_T^{\alpha}u \in C([0,T])$.*

Proof of Theorem 3.1. Suppose that u is the weak solution of Problem (1.1). Set $g(t) = \lambda f(t,u(t)) - u(t)$. By Definition of weak solution, one has

$$ \int_0^T {}_0D_t^{\alpha}u(t){}_0D_t^{\alpha}v(t)dt = \int_0^T g(t)v(t)dt \quad \forall v \in E_0^{\alpha}. $$

Thus from definition 3.2, we have $g(t) = {}_t\overline{D}_T^{\alpha}{}_0D_t^{\alpha}u(t)$ and from Remark 3.3, we can conclude $g(t) = {}_t\overline{D}_T^{\alpha}{}_0\overline{D}_t^{\alpha}u(t)$.

From Theorem 2.7, $g \in C([0,T])$. Then Lemma 3.8 implies that ${}_tD_T^{\alpha}{}_0\overline{D}_t^{\alpha}u(t)$ exists for any $t \in [0,T]$, ${}_tD_T^{\alpha}{}_0\overline{D}_t^{\alpha}u(t) \in C([0,T])$ and ${}_0\overline{D}_t^{\alpha}u$ is almost everywhere equal to an element of C([0,T]). Then from Lemma 3.6(ii), we have ${}_0D_t^{\alpha}u$ exists for any $t \in [0,T]$ and Remark 3.3 yields ${}_0\overline{D}_t^{\alpha}u = {}_0D_t^{\alpha}u$ a.e. on $[0,T]$. Hence, we conclude that ${}_tD_T^{\alpha}{}_0D_t^{\alpha}u(t)$ exists for any $t \in [0,T]$.

Since g and ${}_tD_T^{\alpha}{}_0D_t^{\alpha}u$ are almost everywhere equal and they are continuous, we have

$$ {}_tD_T^{\alpha}{}_0D_t^{\alpha}u(t) = g(t) \quad \forall t \in [0,T], $$

which completes the proof. □

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