Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 97, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

REGULARITY CRITERIA FOR 3D BOUSSINESQ EQUATIONS WITH ZERO THERMAL DIFFUSION

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Abstract. In this article, we consider the three-dimensional (3D) incompressible Boussinesq equations with zero thermal diffusion. We establish a regularity criterion for the local smooth solution in the framework of Besov spaces in terms of the velocity only.

1. Introduction

In this article, we consider the 3D Boussinesq equations with zero thermal diffusion,

$$\begin{gathered}
\partial_t u + (u\cdot\nabla)u - \mu\Delta u + \nabla P = \theta e_3, \quad x\in\mathbb{R}^3,\ t>0,\\
\partial_t\theta + (u\cdot\nabla)\theta = 0, \quad x\in\mathbb{R}^3,\ t>0,\\
\nabla\cdot u = 0, \quad x\in\mathbb{R}^3,\ t>0,\\
u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), \quad x\in\mathbb{R}^3,
\end{gathered}\tag{1.1}$$

where $\mu\geq 0$ is the viscosity, $u=u(x,t)\in\mathbb{R}^3$ is the velocity, $P=P(x,t)\in\mathbb{R}$ is the scalar pressure, $\theta=\theta(x,t)\in\mathbb{R}^3$ is the temperature, and $e_3=(0,0,1)^{\mathrm{T}}$. The Boussinesq equations are of relevance to study a number of models coming from atmospheric or oceanographic turbulence (see [20, 24]).

It is easy to check that in the case $\theta=0$, the system (1.1) reduces to the 3D classical Navier-Stokes equations. Although the local existence and uniqueness of smooth solutions for the system (1.1) with large initial data were easily obtained (see [6, 20]), whether the unique local smooth solution can exist globally is an outstanding challenging open problem. Therefore, it is important to study the mechanism of blowup and structure of possible singularities of smooth solutions to the system (1.1). For this reason, many researchers were devoted to finding sufficient conditions to ensure the smoothness of the solutions; see [10,9,11,26,25,27,28,29,31] and so forth. For many interesting results on the high dimensional Boussinesq equations with axisymmetric data, we refer the readers to [1, 14, 15, 22, 23]. We remark that the 2D Boussinesq equations also has recently attracted considerable attention, just name a few (see [4, 7, 5, 8, 12, 13, 16, 18, 17]).

2010 *Mathematics Subject Classification.* 35Q35, 35B65, 76W05.
Key words and phrases. 3D Boussinesq equations; Besov spaces; regularity criterion.

Submitted January 14, 2015. Published April 14, 2015.

The aim of this paper is to improve the previous regularity criterion results on the system (1.1). Since the concrete value of μ does not play a special role in our discussion, for simplicity, we set $\mu = 1$. Now we state the main results as follows

Theorem 1.1. *Assume that* $(u_0, \theta_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. *Let* (u, θ) *be a local smooth solution of the system* (1.1). *If the following condition holds*

$$\int_0^T \|u(t)\|^q_{\dot{B}^{\frac{3}{p}+\frac{2}{q}-1}_{p,\infty}}\,dt < \infty, \tag{1.2}$$

with $\frac{3}{p} + \frac{2}{q} \le 2$ *and* $(p, q) \neq (\infty, \infty)$ *for* $1 < p, q \le \infty$, *then the solution pair* (u, θ) *can be extended beyond time* T. *Here* $\dot{B}^s_{p,q}$ *stands for the homogeneous Besov space. In other words, if* $T < \infty$ *is the maximal existence time, then*

$$\int_0^T \|u(t)\|^q_{\dot{B}^{\frac{3}{p}+\frac{2}{q}-1}_{p,\infty}}\,dt = +\infty.$$

Remark 1.2. When the thermal diffusion $\Delta\theta$ was added in the second equation of the system (1.1), Xiang [27] obtained the same regularity result (1.2). As a result, Theorem 1.1 significantly improves the result (Theorem 1.1) in [27]. Moreover, we chose not to apply the Littlewood-Paley decomposition on the system itself, in contrast to the proof of [27].

Remark 1.3. The system (1.1) has scaling property that if (u, θ, P) is a solution of the system (1.1), then for any $\lambda > 0$ the functions

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad \theta_\lambda(x,t) = \lambda^3\theta(\lambda x, \lambda^2 t), \quad P_\lambda(x,t) = \lambda^2 P(\lambda x, \lambda^2 t),$$

are also solutions of (1.1) with the corresponding initial data $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$ and $\theta_{0,\lambda}(x) = \lambda^3\theta_0(\lambda x)$. It is an obvious fact that the assumption (1.2) does belong to the invariant spaces.

The method may also be adapted with almost no change to the study of the following Bénard system:

$$\begin{gathered}
\partial_t u + (u\cdot\nabla)u - \mu\Delta u + \nabla P = \theta e_3, \quad x \in \mathbb{R}^3,\ t > 0,\\
\partial_t\theta + (u\cdot\nabla)\theta = u_3, \quad x \in \mathbb{R}^3,\ t > 0,\\
\nabla\cdot u = 0, \quad x \in \mathbb{R}^3,\ t > 0,\\
u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^3,
\end{gathered} \tag{1.3}$$

which describes convective motions in a heated incompressible fluid (see [2, Chap. 6]). Because of the similar structure to Boussinesq system (1.1), it is not difficult to show that Bénard system (1.3) admits the same conclusion as Theorem 1.1, namely, we have the following result.

Theorem 1.4. *Assume that* $(u_0, \theta_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. *Let* (u, θ) *be a local smooth solution of the system* (1.3). *If the following condition holds*

$$\int_0^T \|u(t)\|^q_{\dot{B}^{\frac{3}{p}+\frac{2}{q}-1}_{p,\infty}}\,dt < \infty, \tag{1.4}$$

with $\frac{3}{p} + \frac{2}{q} \le 2$ *and* $(p, q) \neq (\infty, \infty)$ *for* $1 < p, q \le \infty$, *then the solution pair* (u, θ) *can be extended beyond time* T. *In other words, if* $T < \infty$ *is the maximal existence* *time, then*

$$\int_0^T \|u(t)\|^q_{\dot{B}^{\frac{3}{p}+\frac{2}{q}-1}_{p,\infty}}\,dt = +\infty.$$

2. Proof of Theorem 1.1

As stated above that the local smooth solution was obtained, we only need to establish *a priori* estimates. Throughout the paper, C represents a real positive constant which may be different in each occurrence.

Proof of Theorem 1.1. Multiplying the second equation of (1.1) by $|\theta|^{p-2}\theta$ and and integrating the resulting equation over $\mathbb{R}^3$ yield that

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}, \quad \forall p \in [1,\infty]. \tag{2.1}$$

Testing $(1.1)_1$ and $(1.1)_2$ by u and θ, respectively, it gives

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2_{L^2} + \|\theta(t)\|^2_{L^2}) + \|\nabla u\|^2_{L^2} \le \|u\|_{L^2}\|\theta\|_{L^2}, \tag{2.2}$$

which together with (2.1) implies that

$$\|u(t)\|^2_{L^2} + \|\theta(t)\|^2_{L^2} + \int_0^t \|\nabla u(\tau)\|^2_{L^2}\,d\tau \le C < \infty. \tag{2.3}$$

Multiplying equation $(1.1)_1$ by Δu, integration by parts and taking the divergence free property into account, one concludes that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2} + \|\Delta u\|^2_{L^2} = -\int_{\mathbb{R}^3} \theta e_3 \cdot \Delta u\,dx + \int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot\Delta u\,dx. \tag{2.4}$$

Integrating by parts and using Young inequality, we obtain

$$-\int_{\mathbb{R}^3} \theta e_3 \cdot \Delta u\,dx \le \|\Delta u\|_{L^2}\|\theta\|_{L^2} \le \frac{1}{4}\|\Delta u\|^2_{L^2} + C\|\theta\|^2_{L^2}. \tag{2.5}$$

To bound the remainder term, we split it into the following two cases:

Case 1: $2 < q \le \infty$. The following bilinear estimate (see [30])

$$\|ff\|_{\dot{B}^s_{2,2}} \le C\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\|f\|_{\dot{B}^{s+\alpha}_{2,2}}, \quad \text{for any } s>0,\ \alpha>0$$

and Young inequality allow us to show that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot\Delta u\,dx \\
&\le \int_{\mathbb{R}^3} \nabla\cdot(u\otimes u)\cdot\Delta u\,dx \\
&\le C\|u\otimes u\|_{\dot{H}^1}\|\Delta u\|_{L^2} \\
&\le C(\|u\|_{\dot{B}^{-\beta}_{\infty,\infty}}\|u\|_{\dot{B}^{1+\beta}_{2,2}})\|\Delta u\|_{L^2} \quad (0<\beta\le 1) \\
&\le C\|u\|_{\dot{B}^{\frac{3}{p}-\beta}_{p,\infty}}(\|\nabla u\|^{1-\beta}_{L^2}\|\Delta u\|^{\beta}_{L^2})\|\Delta u\|_{L^2} \\
&\le \frac{1}{4}\|\Delta u\|^2_{L^2} + C\|u\|^{\frac{2}{1-\beta}}_{\dot{B}^{\frac{3}{p}-\beta}_{p,\infty}}\|\nabla u\|^2_{L^2} \\
&= \frac{1}{4}\|\Delta u\|^2_{L^2} + C\|u\|^{q}_{\dot{B}^{\frac{3}{p}+\frac{2}{q}-1}_{p,\infty}}\|\nabla u\|^2_{L^2} \quad \Big(q = \frac{2}{1-\beta} \in (2,\infty]\Big),
\end{aligned} \tag{2.6}$$

where we have used

$$\|u\|_{\dot{B}^{1+\beta}_{2,2}} \approx \|u\|_{\dot{H}^{1+\beta}} \le C\|\nabla u\|_{L^2}^{1-\beta}\|\Delta u\|_{L^2}^{\beta}, \quad \text{for } 0 \le \beta \le 1;$$
$$\|u\|_{\dot{B}^{-\beta}_{\infty,\infty}} \le C\|u\|_{\dot{B}^{\frac{3}{p}-\beta}_{p,\infty}}, \quad \text{for } 1 \le p \le \infty.$$

Case 2: $1 < q \le 2$**.** Now we recall the following interpolation inequality due to Meyer-Gerard-Oru [21] (see also [3, Theorem 2.42])

$$\|f\|_{L^m} \le C\|\Lambda^s f\|_{L^2}^{\frac{2}{m}}\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{m-2}{m}}, \tag{2.7}$$

for any $f \in \dot{H}^s \cap \dot{B}^{-\alpha}_{\infty,\infty}$, $s = \alpha(\frac{m}{2}-1) > 0$ and $2 < m < \infty$. Hence, by using (2.7) with $m = 4$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot\Delta u\,dx \\
&= -\int_{\mathbb{R}^3} (\partial_k u\cdot\nabla u)\cdot\partial_k u\,dx \quad (\nabla\cdot u = 0)\\
&\le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}\|\Lambda^\alpha\nabla u\|_{L^2}\|\nabla u\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \quad (0<\alpha\le 1)\\
&\le C\|\nabla u\|_{L^2}(\|\nabla u\|_{L^2}^{1-\alpha}\|\Delta u\|_{L^2}^{\alpha})\|\nabla u\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\\
&\le C\|\nabla u\|_{L^2}^{2-\alpha}\|\Delta u\|_{L^2}^{\alpha}\|u\|_{\dot{B}^{1-\alpha+\frac{3}{p}}_{p,\infty}}\\
&\le \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\|u\|_{\dot{B}^{1-\alpha+\frac{3}{p}}_{p,\infty}}^{\frac{2}{2-\alpha}}\|\nabla u\|_{L^2}^2\\
&= \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\|u\|_{\dot{B}^{\frac{3}{p}+\frac{2}{q}-1}_{p,\infty}}^{q}\|\nabla u\|_{L^2}^2 \quad \Big(q = \frac{2}{2-\alpha} \in (1,2]\Big),
\end{aligned} \tag{2.8}$$

where the following fact has been applied

$$\|\nabla u\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \le C\|u\|_{\dot{B}^{1-\alpha+\frac{3}{p}}_{p,\infty}}.$$

Substituting (2.5) and (2.6) (or (2.8)) into (2.4), we arrive at

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \le C\|\theta\|_{L^2}^2 + C\|u\|_{\dot{B}^{\frac{3}{p}+\frac{2}{q}-1}_{p,\infty}}^{q}\|\nabla u\|_{L^2}^2. \tag{2.9}$$

It thus follows from the Gronwall inequality that

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\Delta u(\tau)\|_{L^2}^2\,d\tau\\
&\le (\|\nabla u_0\|_{L^2}^2+1)\exp\Big[C\int_0^T \big(\|\theta(\tau)\|_{L^2}^2 + \|u(\tau)\|_{\dot{B}^{\frac{3}{p}+\frac{2}{q}-1}_{p,\infty}}^{q}\big)\,d\tau\Big] - 1\\
&\le C < \infty.
\end{aligned} \tag{2.10}$$

The Hölder inequality and Gagliardo-Nirenberg inequality lead to

$$\begin{aligned}
\|u\cdot\nabla u\|_{L^{3+\delta}} &\le C\|u\|_{L^{\frac{18+6\delta}{3-\delta}}}\|\nabla u\|_{L^6}\\
&\le C\|u\|_{L^2}^{\frac{3}{6+2\delta}}\|\Delta u\|_{L^2}^{\frac{3+2\delta}{6+2\delta}}\|\Delta u\|_{L^2}\\
&\le C\|u\|_{L^2}^{\frac{3}{6+2\delta}}\|\Delta u\|_{L^2}^{\frac{9+4\delta}{6+2\delta}},
\end{aligned} \tag{2.11}$$

where $0 < \delta < 3$. It follows from the bounds (2.1), (2.3) and (2.10) that

$$u \cdot \nabla u \in L^{\frac{2(6+2\delta)}{9+4\delta}}\big(0, T; L^{3+\delta}(\mathbb{R}^3)\big), \tag{2.12}$$

$$\theta \in L^{\frac{2(6+2\delta)}{9+4\delta}}\big(0, T; L^{3+\delta}(\mathbb{R}^3)\big). \tag{2.13}$$

Recall the first equation of (1.1), namely

$$\partial_t u - \Delta u + \nabla P = f := -(u \cdot \nabla)u + \theta e_3. \tag{2.14}$$

As a consequence of (2.12) and (2.13), it leads to

$$f \in L^{\frac{2(6+2\delta)}{9+4\delta}}\big(0, T; L^{3+\delta}(\mathbb{R}^3)\big). \tag{2.15}$$

According to the divergence-free condition, we can rewrite equation (2.14) as

$$\partial_t u - \Delta u = (I + \mathcal{R}_i \mathcal{R}_j) f, \tag{2.16}$$

where the singular operator $\mathcal{R}_i$ is the classical Riesz operator, more precisely

$$\mathcal{R}_i = \frac{\partial_{x_i}}{\sqrt{-\Delta}}.$$

Now we recall the following Maximal $L^q_t L^p_x$ regularity for the heat kernel (see [19])

Proposition 2.1. *The operator A defined by*

$$Af(x,t) := \int_0^t e^{(t-s)\Delta} \Delta f(s,x)\, ds$$

is bounded from $L^p(0,T; L^q(\mathbb{R}^n))$ to $L^p(0,T; L^q(\mathbb{R}^n))$ for very $(p,q) \in (1,\infty) \times (1,\infty)$ and $T \in (0,\infty]$.

Applying operator Δ to (2.16), we have that the velocity Δu can be solved by the Duhamel's Principle,

$$\Delta u(x,t) = e^{t\Delta} \Delta u_0(x) + \int_0^t e^{(t-s)\Delta} \Delta (I + \mathcal{R}_i \mathcal{R}_j) f(x,s)\, ds. \tag{2.17}$$

By Proposition 2.1, one concludes from (2.17) that

$$\begin{aligned}
&\|\Delta u\|_{L_T^{\frac{12+4\delta}{9+4\delta}} L_x^{3+\delta}} \\
&\le \|e^{t\Delta}\Delta u_0\|_{L_T^{\frac{12+4\delta}{9+4\delta}} L_x^{3+\delta}} + \Big\| \int_0^t e^{(t-s)\Delta} \Delta (I + \mathcal{R}_i \mathcal{R}_j) f(x,s)\, ds \Big\|_{L_T^{\frac{12+4\delta}{9+4\delta}} L_x^{3+\delta}} \\
&\le C \|H(t,x)\|_{L_T^{\frac{12+4\delta}{9+4\delta}} L_x^{1}} \|\Delta u_0\|_{L_x^{3+\delta}} + C \big\| (I + \mathcal{R}_i \mathcal{R}_j) f(x,s) \big\|_{L_T^{\frac{12+4\delta}{9+4\delta}} L_x^{3+\delta}} \\
&\le C(T) \|u_0\|_{H^3} + C \|f\|_{L_T^{\frac{12+4\delta}{9+4\delta}} L_x^{3+\delta}} \\
&\le C < \infty,
\end{aligned} \tag{2.18}$$

where we have used the boundedness of the Calderon-Zygmund operator between the L^p $(1 < p < \infty)$ space and $H^3(\mathbb{R}^3) \hookrightarrow L^{3+\delta}(\mathbb{R}^3)$ for $0 < \delta < 3$.

Now we deduce that from the bounds (2.3), (2.10) and (2.18) that

$$u \in L^{\frac{12+4\delta}{9+4\delta}}\big(0, T; W^{2,3+\delta}(\mathbb{R}^3)\big). \tag{2.19}$$

Thus, we have

$$\nabla u \in L^1\big(0, T; L^\infty(\mathbb{R}^3)\big). \tag{2.20}$$

The above key estimate (2.20) as well as the local well-posedness result ensures implies that the local smooth solution pair (u, θ) can be extended beyond time T. This completes the proof. □

3. Proof of Theorem 1.4

Proof of Theorem 1.4. The proof is largely the same as Theorem 1.1 with only some modifications, thus we only say some words.

Testing $(1.3)_1$ and $(1.3)_2$ by u and θ, respectively, adding them up, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 \leq 2\|u\|_{L^2}\|\theta\|_{L^2}, \tag{3.1}$$

which together with Gronwall inequality yields

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \leq C < \infty. \tag{3.2}$$

The Sobolev interpolation together with (3.2) gives

$$u \in L^{\frac{4p}{3(p-2)}}\big(0, T; L^p(\mathbb{R}^3)\big), \quad 2 \leq p \leq 6. \tag{3.3}$$

Recalling the second equation of (1.3)

$$\partial_t \theta + (u \cdot \nabla)\theta = u_3,$$

it is easy to see that

$$\theta \in L^{\frac{4p}{3(p-2)}}\big(0, T; L^p(\mathbb{R}^3)\big), \quad 2 \leq p \leq 6. \tag{3.4}$$

Thus,

$$\theta \in L^{\frac{2(6+2\delta)}{9+4\delta}}\big(0, T; L^{3+\delta}(\mathbb{R}^3)\big),$$

where δ is the stated in previous section. Thus, we can obtain the desired result immediately with only some modifications correspondingly. □

Acknowledgements. The author would like to thank an anonymous referee for the careful reading and helpful suggestions.

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