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# EXISTENCE AND NONEXISTENCE OF NONTRIVIAL SOLUTIONS FOR CHOQUARD TYPE EQUATIONS

TAO WANG

Abstract. In this article, we consider the nonlocal problem

$$-\Delta u + u = q(x)\Big(\int_{\mathbb{R}^N} \frac{q(y)|u(y)|^p}{|x-y|^{N-\alpha}} dy\Big)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in (0, N)$, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $q(x)$ is a given potential. Under suitable assumptions on $q(x)$, we prove the existence and nonexistence of nontrivial solutions.

## 1. Introduction

In this article, we are concerned with the equation

$$-\Delta u + u = q(x)\Big(\int_{\mathbb{R}^N} \frac{q(y)|u(y)|^p}{|x-y|^{N-\alpha}} dy\Big)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$, $q(x) \geq 0$ and $q(x)$ is continuous in $\mathbb{R}^N$.

It is well known that when $N = 3$, $\alpha = 2$, $p = 2$ and $q \equiv 1$, Equation (1.1) becomes the classical stationary Choquard equation

$$-\Delta u + u = (|u|^2 * \frac{1}{|x|})u, \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

It appeared at least as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest[26]. In 1976, Choquard used (1.2) to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [11]. In 1996, Penrose proposed (1.2) as a model of self-gravitating matter, in a program in which quantum state reduction is understood as a gravitational phenomenon [20], see also [10, 13, 25] for more details.

In 1977, using symmetric rearrangement inequalities, Lieb [11] showed the existence and uniqueness of the minimizer of (1.2) up to translations. Later, Lions [14] proved the existence of a sequence of radially symmetric solutions to (1.2) by dual variational methods. Further results for related problems can be founded in [1, 6, 15, 19, 24, 27, 28] and references therein. In recent years, the existence and properties of solutions for the generalized Choquard type equation (1.1) with $q \equiv 1$ have been considered by many authors. In 2010, Ma and Zhao [18] proved the positive solutions for the generalized Choquard equation (1.1) with $q \equiv 1$ must be

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radially symmetric and monotone decreasing about some point under appropriate assumptions on $p, \alpha, N$, see also [5, 8]. They also showed the positive solutions of (1.2) is uniquely determined, up to translations. Moroz and Van Schaftingen [21] obtained the existence, regularity, positivity and radial symmetry of ground state solution of (1.1) with $q \equiv 1$ for the optimal range of parameters. They also derived the sharp decay asymptotic of the ground state solution, see also [22]. Alves and Yang [2] studied the multiplicity and concentration behaviour of positive solutions for quasilinear Choquard equation

$$-\epsilon^p \Delta_p u + V(x)|u|^{p-2}u = \epsilon^{\mu-N}\Big(\int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x-y|^\mu} dy\Big) Q(x) f(u), \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $\Delta_p$ is the $p$-Laplacian operator, $1 < p < N$, $V$ and $Q$ are two continuous real functions on $\mathbb{R}^N$, $F(s)$ is the primate function of $f(s)$ and $\epsilon$ is a positive parameter, see [3, 7, 23] for more related problems.

Motivated by the above work, in this article, we consider (1.1) with the nonlinear potential $q$ on the right side of the equation. To be precise, using variational method, we investigate the existence and nonexistence of nontrivial solutions to (1.1) where nonlinear potential $q$ is radial or non-radial. To deduce our statements, we need the following assumptions:

- (H1) $\lim_{|x|\to\infty} q(x) = q_\infty$, where $q_\infty > 0$ is a positive number;
- (H2) $\lim_{|x|\to\infty} q(x) = 0$;
- (H3) $q$ is bounded in $\mathbb{R}^N$ and there exists $R_0 > 0$ such that $\min_{2R \le |x| \le 4R} q(x) \ge \max_{|x| \le R} q(x)$ for all $R \ge R_0$;
- (H4) $q$ is radial in $\mathbb{R}^N$ and $q(r) \le C(1 + r^l)$ with $0 \le l < \frac{(N-1)p-N-\alpha}{2}$, where $C > 0$ is a positive constant.

We remark (H1)–(H4) were introduced by Ding and Ni [9] with some modifications. Recall here that $u \in H^1(\mathbb{R}^N)$ is said to be a ground state solution to (1.1), if $u$ solves (1.1) and minimizes the energy functional associated with (1.1) among all possible nontrivial solutions. Now we are ready to state our main results.

**Theorem 1.1.** *Let $N \ge 3$, $\alpha \in (0, N)$, $p \in [2, \frac{N+\alpha}{N-2})$ and $q$ satisfies* (H1). *Then the following statements are true:*

- (i) *If $\lim_{|x|\to\infty} q(x) = \inf_{x\in\mathbb{R}^N} q(x)$, then* (1.1) *has a ground state solution in $H^1(\mathbb{R}^N)$.*
- (ii) *If $\lim_{|x|\to\infty} q(x) = \sup_{x\in\mathbb{R}^N} q(x)$ and $q$ is not constant, then* (1.1) *has no ground state solution in $H^1(\mathbb{R}^N)$.*

Note that if $\alpha \le N - 4$, then assumptions of Theorem 1.1 can not be satisfied.

**Theorem 1.2.** *Let $N \ge 3$, $\alpha \in (0, N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$. Then the following statements are true:*

- (i) *If $q$ satisfies (H2) and $q \not\equiv 0$. Then* (1.1) *has a ground state solution in $H^1(\mathbb{R}^N)$.*
- (ii) *Suppose $q$ satisfies (H3) and $q$ is not constant. Then* (1.1) *has no ground state solution in $H^1(\mathbb{R}^N)$.*

**Theorem 1.3.** *Let $N \ge 3$, $\alpha \in (0, N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$. If $q$ satisfies* (H4) *and $q \not\equiv 0$, then* (1.1) *has a nonnegative radial solution in $H^1_r(\mathbb{R}^N)$ defined in section 2.*

**Remark 1.4.** If the nontrivial radial solution $u$ obtained in Theorem 1.3 tends to zero exponentially fast at infinity, then according to the proof of symmetric criticality principle (see Theorem 1.28 in[29]), we conclude $u$ is a solution of (1.1) in $H^1(\mathbb{R}^N)$.

**Theorem 1.5.** *Let $N \geq 3$, $\alpha \in (0, N)$, $p \in (\frac{N+\alpha}{N-1}, \frac{N+\alpha}{N-2})$. Suppose that $q$ satisfies* (H4) *and $q(r) \to +\infty$ as $r \to \infty$. Then for large $k$, (4.1) has two nontrivial weak solutions in $H_0^1(B_k)$ defined in section 2, one of which is radial while the other is not.*

The remainder of this article is organized as follows. In section 2, we introduce some notation and give some important compactness lemmas, which play a key role for the the existence results. In sections 3 and 4, we prove our main results. Throughout the paper, we write $C > 0$ for different positive constant.

## 2. Preliminary results

We shall use the following notation:

• Let $N$ and $k$ be positive integers and $B_R(0)$ be an open ball of radius $R$ centered at the origin in $\mathbb{R}^N$.

• $H^1(\mathbb{R}^N)$ is the usual Sobolev space with the standard norm
$$\|u\| = \Big( \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx \Big)^{1/2}.$$
$H_r^1(\mathbb{R}^N)$ is the set of all radial functions in $H^1(\mathbb{R}^N)$.

• Let $C_c^\infty(B_k(0))$ be the set of infinitely differential functions with compact support in $B_k(0)$ and $H_0^1(B_k)$ be the closure of $C_c^\infty(B_k(0))$ in the norm defined by
$$\|u\|_{H_0^1(B_k)} = \Big( \int_{B_k(0)} (|\nabla u|^2 + |u|^2) dx \Big)^{1/2}.$$
$H_{0,r}^1(B_k)$ is the set of all radial functions in $H_0^1(B_k)$. We can identify $u \in H_0^1(B_k)$ with its extension to $\mathbb{R}^N$ obtained by setting $u = 0$ in $\mathbb{R}^N \backslash B_k$.

• Let $\Omega \subset \mathbb{R}^N$ be a domain. For $1 \leq s < \infty$, $L^s(\Omega)$ denotes the Lebesgue space with the norm
$$|u|_{L^s(\Omega)} = \Big( \int_\Omega |u|^s dx \Big)^{1/s}.$$
If $\Omega = \mathbb{R}^N$, we write $|u|_{L^s} = |u|_{L^s(\Omega)}$.

• The dual space of $H^1(\mathbb{R}^N)$ is denoted by $H^{-1}(\mathbb{R}^N)$.

• Let $\langle \cdot, \cdot \rangle$ be the duality pairing between $H^1(\mathbb{R}^N)$ and $H^{-1}(\mathbb{R}^N)$.

From this, the energy functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated with (1.1) is defined by
$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{q(x)q(y)|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy.$$
Suppose $q$ is bounded in $\mathbb{R}^N$. Then the functional is well defined by Hardy-Littlewood-Sobolev inequality (see[12]), which states that if $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2}$ and $u \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, then
$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{q(x)q(y)|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} \, dx \, dy &\leq C(N, \alpha, s, t)|q|_{L^\infty}^2 |u^p|_{L^s} |u^p|_{L^t} \\ &= C(N, \alpha, s, t)|q|_{L^\infty}^2 |u|_{L^{sp}}^p |u|_{L^{tp}}^p < \infty. \end{aligned} \tag{2.1}$$

where $C$ depends only on $N, \alpha, s, t$, and $\frac{1}{s}+\frac{1}{t}+\frac{N-\alpha}{N}=2$. This also implies that $I$ is $C^1$ functional whose derivative is given by

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)|u(y)|^p|u(x)|^{p-2}u(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy$$

for all $v \in H^1(\mathbb{R}^N)$. It is easy to see the critical points of $I$ are solutions to (1.1) in the weak sense. We consider the Nehari manifold

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \langle I'(u), u\rangle = 0\}$$

and $c = \inf_{u\in\mathcal{N}} I(u)$.

In what follows, we consider the limit problem when $q$ satisfies (H1)

$$-\Delta u + u = \Big(\int_{\mathbb{R}^N} \frac{q_\infty^2|u(y)|^p}{|x-y|^{N-\alpha}}dy\Big)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{2.2}$$

The associated energy functional is

$$I_\infty(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q_\infty^2|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy,$$

and the corresponding Nehari manifold is

$$\mathcal{N}_\infty = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \langle I'_\infty(u), u\rangle = 0\}.$$

We define $c_\infty = \inf_{u\in\mathcal{N}_\infty} I_\infty(u)$.

For convenience, we introduce

$$\mathbb{D}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy, \quad J(u) = \frac{\|u\|^2}{\mathbb{D}^{1/p}(u)}.$$

The existence of ground state solution for (2.2) has been investigated in [21, Theorem 1].

**Lemma 2.1** ([21]). *Let $N \geq 3$, $\alpha \in (0, N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$. Then there exists a ground state solution $w \in H^1(\mathbb{R}^N)$ such that $w$ satisfies (2.2) weakly in $\mathbb{R}^N$ and $c_\infty = I_\infty(w)$.*

In the sequel, we shall establish a compactness lemma which plays an important role in our existence results. To achieve this, we need some basic lemmas.

**Lemma 2.2** ([21]). *Let $\Omega \subset \mathbb{R}^N$ be a domain, $s \in [1,\infty)$ and $(u_n)_{n\geq 1}$ be a bounded sequence in $L^r(\Omega)$. If $u_n \to u$ almost everywhere on $\Omega$ as $n \to \infty$, then for every $s \in [1, r]$,*

$$\lim_{n\to\infty} \int_\Omega \big||u_n|^s - |u_n - u|^s - |u|^s\big|^{r/s} = 0.$$

**Lemma 2.3** ([30]). *Let $\Omega \subset \mathbb{R}^N$ be a domain, $s \in (1,\infty)$ and $(u_n)_{n\geq 1}$ be a bounded sequence in $L^s(\Omega)$. If $u_n \to u$ almost everywhere on $\Omega$ as $n \to \infty$, then $u_n \rightharpoonup u$ weakly in $L^s(\Omega)$.*

According to Lemmas 2.2 and 2.3, we obtain the following three lemmas, whose proofs are similar as that of [16, Proposition A.1] and [17, Lemma 2.15] with some necessary modifications. For the sake of completeness, we prove them here. In addition, we remark that we can identify $u \in L^s(\Omega)$ with its extension to $\mathbb{R}^N$ obtained by setting $u = 0$ in $\mathbb{R}^N\backslash\Omega$, which ensures that we can use Hardy-Littlewood-Sobolev inequality to handle with the nonlocal problem.

**Lemma 2.4.** *Let $N \geq 3$, $\alpha \in (0, N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and suppose $q$ is bounded in $\mathbb{R}^{\mathbb{N}}$. If $(u_n)_{n\geq 1}$ is a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, and $u_n \to u$ almost everywhere on $\mathbb{R}^N$ as $n \to \infty$, then*

$$\lim_{n\to\infty} (\mathbb{D}(u_n) - \mathbb{D}(u_n - u)) = \mathbb{D}(u).$$

*Proof.* The proof can be split into three steps.

**Step 1.** For every $n$, we have

$$\begin{aligned}
&\mathbb{D}(u_n) - \mathbb{D}(u_n - u)\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)(|u_n(y)|^p - |u_n - u|^p(y))(|u_n(x)|^p - |u_n - u|^p(x))}{|x-y|^{N-\alpha}}\,dx\,dy\\
&\quad + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)(|u_n(y)|^p - |u_n - u|^p(y))|u_n - u|^p(x)}{|x-y|^{N-\alpha}}\,dx\,dy\\
&=: I_1 + 2I_2.
\end{aligned}$$

**Step 2.** By Lemmas 2.2 and 2.3, the following statements are true, as $n \to \infty$,

$$q(|u_n|^p - |u_n - u|^p) \to q|u|^p \quad \text{strongly in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N), \tag{2.3}$$

$$\begin{gathered}
\int_{\mathbb{R}^N} \frac{|u_n(y)|^p - |u_n - u|^p(y)}{|x-y|^{N-\alpha}}dy \to \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}dy\\
\text{strongly in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N),
\end{gathered} \tag{2.4}$$

$$q|u_n - u|^p \rightharpoonup 0 \quad \text{weakly in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N). \tag{2.5}$$

**Step 3.** By (2.1) and (2.3), we obtain

$$\begin{aligned}
&|I_1 - \mathbb{D}(u)|\\
&\leq \Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(q(x)q(y)(|u_n(y)|^p - |u_n - u|^p(y) - |u(y)|^p)(|u_n(x)|^p\\
&\quad - |u_n - u|^p(x))\Big)/|x-y|^{N-\alpha}\,dx\,dy\Big|\\
&\quad + \Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)|u(y)|^p(|u_n(x)|^p - |u_n - u|^p(x) - |u(x)|^p)}{|x-y|^{N-\alpha}}\,dx\,dy\Big|\\
&\leq C|q|_{L^\infty}\, \big|q|u_n|^p - q|u_n - u|^p - q|u|^p\big|_{L^{\frac{2N}{N+\alpha}}}\, \big||u_n|^p - |u_n - u|^p\big|_{L^{\frac{2N}{N+\alpha}}}\\
&\quad + C|q|_{L^\infty}\, \big|q|u_n|^p - q|u_n - u|^p - q|u|^p\big|_{L^{\frac{2N}{N+\alpha}}}\, |u|^p_{L^{\frac{2Np}{N+\alpha}}} \to 0,
\end{aligned} \tag{2.6}$$

as $n \to \infty$. By (2.1),(2.3), (2.4) and (2.5), we conclude that

$$\begin{aligned}
|I_2| &\leq \Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)(|u_n(y)|^p - |u_n - u|^p(y) - |u(y)|^p)|u_n - u|^p(x)}{|x-y|^{N-\alpha}}\,dx\,dy\Big|\\
&\quad + \Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)|u(y)|^p|u_n - u|^p(x)}{|x-y|^{N-\alpha}}\,dx\,dy\Big| \to 0,
\end{aligned}$$

as $n \to \infty$. Then the proof is complete. □

**Lemma 2.5.** *Let $N \geq 3$, $\alpha \in (0, N)$, $p \in [2, \frac{N+\alpha}{N-2})$ and suppose $q$ is bounded in $\mathbb{R}^{\mathbb{N}}$. If $(u_n)_{n\geq 1}$ is a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, and $u_n \to u$ almost everywhere on $\mathbb{R}^N$ as $n \to \infty$, then*

$$\lim_{n\to\infty} (\mathbb{D}'(u_n) - \mathbb{D}'(u_n - u)) = \mathbb{D}'(u) \quad \text{in } H^{-1}(\mathbb{R}^N).$$

*Proof.* The outline of the proof is as follows.

**Step 1.** By Lemmas 2.2 and 2.3, for any $v \in H^1(\mathbb{R}^N)$, we have

$$q|u_n|^{p-2}u_n - q|u_n-u|^{p-2}(u_n-u) \to q|u|^{p-2}u \quad \text{strongly in } L^{\frac{2Np}{(N+\alpha)(p-1)}}(\mathbb{R}^N). \tag{2.7}$$

**Step 2.** It is easy to check that

$$\begin{aligned}
&\langle \mathbb{D}'(u_n), v\rangle - \langle \mathbb{D}'(u_n-u), v\rangle \\
&= 2p\Big[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)|u_n(y)|^p|u_n(x)|^{p-2}u_n(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)|u_n-u|^p(y)|u_n-u|^{p-2}(x)(u_n-u)(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy\Big] \\
&=: 2pK.
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
K &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(q(x)q(y)(|u_n(y)|^p - |u_n-u|^p(y))(|u_n(x)|^{p-2}u_n(x)v(x) \\
&\quad - |u_n-u|^{p-2}(x)(u_n-u)(x)v(x))\Big)/|x-y|^{N-\alpha}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(q(x)q(y)(|u_n(y)|^p - |u_n-u|^p(y))|u_n-u|^{p-2}(x)(u_n-u)(x)v(x)\Big) \\
&\quad \div |x-y|^{N-\alpha}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(q(x)q(y)|u_n-u|^p(y)(|u_n(x)|^{p-2}u_n(x)v(x) \\
&\quad - |u_n-u|^{p-2}(x)(u_n-u)(x)v(x))\Big)/|x-y|^{N-\alpha}\,dx\,dy \\
&=: K_1 + K_2 + K_3.
\end{aligned}$$

**Step 3.** By direct calculations, from (2.1), (2.3) and (2.7) we deduce that for $n$ large enough,

$$\begin{aligned}
&|K_1 - \frac{1}{2p}\langle \mathbb{D}'(u), v\rangle| \\
&= \Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(q(x)q(y)(|u_n(y)|^p - |u_n-u|^p(y) - |u(y)|^p)(|u_n(x)|^{p-2}u_n(x)v(x) \\
&\quad - |u_n-u|^{p-2}(x)(u_n-u)(x)v(x))\Big)/|x-y|^{N-\alpha}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(q(x)q(y)|u(y)|^p(|u_n(x)|^{p-2}u_n(x)v(x) \\
&\quad - |u_n-u|^{p-2}(x)(u_n-u)(x)v(x)) - |u(x)|^{p-2}u(x)v(x)\Big)/|x-y|^{N-\alpha}\,dx\,dy| \\
&\le C|q|_{L^\infty}|q|u_n|^p - q|u_n-u|^p - q|u|^p|_{L^{\frac{2N}{N+\alpha}}}\big||u_n|^{p-2}u_n \\
&\quad - |u_n-u|^{p-2}(u_n-u)\big|_{L^{\frac{2Np}{(N+\alpha)(p-1)}}}\|v\| \\
&\quad + C|q|_{L^\infty}|u|^p_{L^{\frac{2Np}{N+\alpha}}}\big||q|u_n|^{p-2}u_n - q|u_n-u|^{p-2}(u_n-u) \\
&\quad - q|u|^{p-2}u\big|_{L^{\frac{2Np}{(N+\alpha)(p-1)}}}\|v\| = o(1)\|v\|.
\end{aligned}$$

Here and in the following part, we point out that $o(1) \to 0$ as $n \to \infty$.

Since $u \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, for any $\epsilon > 0$, there exists $R_1 > 0$ such that

$$|u|_{L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N \setminus B_{R_1}(0))} < \epsilon.$$

Fix $R_1 > 0$. Then there exists $R_2 > 0$ large enough such that

$$\begin{aligned} &\Big| \int_{\mathbb{R}^N \setminus B_{R_1+R_2}(0)} \int_{B_{R_1}(0)} \frac{q(x)q(y)|u(y)|^p |u_n - u|^{p-2}(x)(u_n - u)(x)v(x)}{|x-y|^{N-\alpha}} \, dx \, dy \Big| \\ &\le C{R_2}^{\alpha - N} \|v\| < \epsilon \|v\| \end{aligned} \tag{2.9}$$

Note that $|u_n - u|^p \to 0$ in $L_{\rm loc}^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. For $n$ large enough, we deduce from (2.1), (2.3) and (2.9) that

$$\begin{aligned} &|K_2| \\ &\le \Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big( q(x)q(y)(|u_n(y)|^p - |u_n - u|^p(y) - |u(y)|^p)|u_n - u|^{p-2}(x) \\ &\quad \times (u_n - u)(x)v(x) \Big) / |x-y|^{N-\alpha} \, dx \, dy \Big| \\ &\quad + \Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_{R_1}(0)} \frac{q(x)q(y)|u(y)|^p |u_n - u|^{p-2}(x)(u_n - u)(x)v(x)}{|x-y|^{N-\alpha}} \, dx \, dy \Big| \\ &\quad + \Big| \int_{\mathbb{R}^N \setminus B_{R_1+R_2}(0)} \int_{B_{R_1}(0)} \frac{q(x)q(y)|u(y)|^p |u_n - u|^{p-2}(x)(u_n - u)(x)v(x)}{|x-y|^{N-\alpha}} \, dx \, dy \Big| \\ &\quad + \Big| \int_{B_{R_1+R_2}(0)} \int_{B_{R_1}(0)} \frac{q(x)q(y)|u(y)|^p |u_n - u|^{p-2}(x)(u_n - u)(x)v(x)}{|x-y|^{N-\alpha}} \, dx \, dy \Big| \\ &= o(1)\|v\|. \end{aligned}$$

Similarly, $K_3 = o(1)\|v\|$ for $n$ large enough. This completes the proof. □

**Lemma 2.6.** *Let $N \ge 3$, $\alpha \in (0, N)$, $p \in [2, \frac{N+\alpha}{N-2})$ and suppose $q$ is bounded in $\mathbb{R}^{\rm N}$. If $(u_n)_{n\ge1}$ is a sequence such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then $\langle \mathbb{D}'(u_n), v\rangle \to \langle \mathbb{D}'(u), v\rangle$ for all $v \in H^1(\mathbb{R}^N)$.*

*Proof.* Since $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $(u_n)_{n\ge1}$ is bounded in $H^1(\mathbb{R}^N)$. Going if necessary to a subsequence, we assume $u_n \to u$ a.e. on $\mathbb{R}^N$. For any $v \in H^1(\mathbb{R}^N)$, it is easy to verify that

$$|u_n|^{p-2}u_n v \to |u|^{p-2}uv \quad \text{strongly in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N). \tag{2.10}$$

**Step 1.** A direct calculation yields

$$\begin{aligned} \langle \mathbb{D}'(u_n), v\rangle - \langle \mathbb{D}'(u), v\rangle &= 2p\Big[\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{q(x)q(y)|u_n(y)|^p |u_n(x)|^{p-2}u_n(x)v(x)}{|x-y|^{N-\alpha}} \, dx \, dy \\ &\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{q(x)q(y)|u(y)|^p |u(x)|^{p-2}u(x)v(x)}{|x-y|^{N-\alpha}} \, dx \, dy\Big] \\ &=: 2pT. \end{aligned}$$

and

$$\begin{aligned} T &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{q(x)q(y)(|u_n(y)|^p - |u(y)|^p)|u_n(x)|^{p-2}u_n(x)v(x)}{|x-y|^{N-\alpha}} \, dx \, dy \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{q(x)q(y)|u(y)|^p (|u_n(x)|^{p-2}u_n(x)v(x) - |u(x)|^{p-2}u(x)v(x))}{|x-y|^{N-\alpha}} \, dx \, dy \end{aligned}$$

$$= T_1 + T_2 \tag{2.11}$$

**Step 2.** Since $v \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, for any $\epsilon > 0$, there exists $R_1 > 0$ such that $|v|_{L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N\setminus B_{R_1}(0))} < \epsilon$. Fix $R_1 > 0$. Then there exists $R_2 > 0$ large enough such that

$$\begin{aligned}
&\Big|\int_{B_{R_1}(0)}\int_{\mathbb{R}^N\setminus B_{R_1+R_2}(0)} \frac{q(x)q(y)(|u_n(y)|^p - |u(y)|^p)|u_n(x)|^{p-2}u_n(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy\Big| \\
&\le C{R_2}^{\alpha-N} < \epsilon.
\end{aligned} \tag{2.12}$$

Note that $|u_n|^p \to |u|^p$ in $L^{\frac{2N}{N+\alpha}}_{\rm loc}(\mathbb{R}^N)$. Letting $n \to \infty$ and then $\epsilon \to 0$, we conclude from (2.1), (2.11) and (2.12) that

$$\begin{aligned}
&|T_1| \\
&\le \Big|\int_{B_{R_1}(0)}\int_{B_{R_1+R_2}(0)} \frac{q(x)q(y)(|u_n(y)|^p - |u(y)|^p)|u_n(x)|^{p-2}u_n(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy\Big| \\
&\quad + \Big|\int_{B_{R_1}(0)}\int_{\mathbb{R}^N\setminus B_{R_1+R_2}(0)} \frac{q(x)q(y)(|u_n(y)|^p - |u(y)|^p)|u_n(x)|^{p-2}u_n(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy\Big| \\
&\quad + \Big|\int_{\mathbb{R}^N\setminus B_{R_1}(0)}\int_{\mathbb{R}^N} \frac{q(x)q(y)(|u_n(y)|^p - |u(y)|^p)|u_n(x)|^{p-2}u_n(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy\Big| \to 0.
\end{aligned}$$

On the other hand, it follows from (2.10) that $T_2 \to 0$ as $n \to \infty$. This completes the proof. □

Now, we are ready to prove the compactness lemma, following exactly the same lines as the proof of [29, Proposition 8.4].

**Definition 2.7.** We say that $(u_n)_{n\ge1} \subset H^1(\mathbb{R}^N)$ is $(PS)_C$ sequence of $I$, if $(u_n)_{n\ge1}$ satisfies

$$I(u_n) \to C, \quad I'(u_n) \to 0. \tag{2.13}$$

**Lemma 2.8.** *Let $N \ge 3$, $\alpha \in (0, N)$ and $2 \le p < \frac{N+\alpha}{N-2}$. Suppose $q$ satisfies (H1) and $(u_n)_{n\ge1} \subset H^1(\mathbb{R}^N)$ is a $(PS)_C$ sequence of $I$. Then, replacing $(u_n)_{n\ge1}$ if necessary by a subsequence, there exists a solution $v_0 \in H^1(\mathbb{R}^N)$ of (1.1), $\{v_1, v_2, \cdots, v_k\} \subset H^1(\mathbb{R}^N)$ of solutions of (2.2), and $k$ sequences $(y_n^j)_{n\ge1}$, $1 \le j \le k$ satisfying*

$$\begin{gathered}
|y_n^j| \to \infty, \quad |y_n^j - y_n^{j'}| \to \infty, \quad j \ne j',\ n \to \infty, \\
\|u_n - v_0 - \sum_{j=1}^k v_j(\cdot - y_n^j)\| \to 0, \\
\|u_n\|^2 \to \sum_{j=0}^k \|v_j\|^2, \\
I(v_0) + \sum_{j=1}^k I_\infty(v_j) = C.
\end{gathered}$$

*Proof.* The proof can be split into three steps.

**Step 1.** Since $I(u_n) \to C$ and $I'(u_n) \to 0$, then for $n$ large enough, we have

$$\begin{aligned} C + 1 + \|u_n\| &\geq I(u_n) - \frac{1}{2p}\langle I'(u_n), u_n\rangle \\ &= (\frac{1}{2} - \frac{1}{2p}) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) \\ &= (\frac{1}{2} - \frac{1}{2p})\|u_n\|^2, \end{aligned}$$

which yields that $\|u_n\|$ is bounded.

**Step 2.** We assume that $u_n \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$ and $u_n \to v_0$ a.e. on $\mathbb{R}^N$. Then we claim that $I'(v_0) = 0$ and $u_n^1 := u_n - v_0$ such that

$$\begin{gathered} \|u_n^1\|^2 = \|u_n\|^2 - \|v_0\|^2 + o(1), \\ I_\infty(u_n^1) \to C - I(v_0), \\ I'_\infty(u_n^1) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N). \end{gathered}$$

Indeed, applying Lemma 2.6, we have $I'(v_0) = 0$. Since $\lim_{|x|\to\infty} q(x) = q_\infty$ and $u_n^1 \to 0$ in $L^{\frac{2Np}{N+\alpha}}_{\rm loc}(\mathbb{R}^N)$, then we derive from (2.1) that for $n$ large enough,

$$I_\infty(u_n^1) = I(u_n^1) + o(1). \tag{2.14}$$

On the other hand, $\|u_n^1\|^2 = \|u_n\|^2 - \|v_0\|^2 + o(1)$. Then it follows from Lemma 2.4 and (2.14) that

$$I_\infty(u_n^1) = I(u_n) - I(v_0) + o(1) = C - I(v_0) + o(1).$$

Since $I'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$, it follows from Lemma 2.5 that for $n$ large enough,

$$\begin{aligned} I'_\infty(u_n^1) &= I'(u_n^1) + o(1) \\ &= I'(u_n) - I'(v_0) + o(1) = o(1). \end{aligned} \tag{2.15}$$

Therefore, the claim holds.

**Step 3.** Let

$$\delta := \limsup_{n\to\infty} \Big( \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n^1|^2 dx \Big).$$

If $\delta = 0$, by Lemma 1.21 in [29], we have $u_n^1 \to 0$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. Moreover, $I'_\infty(u_n^1) \to 0$, then it follows from (2.1) that, for $n$ large enough,

$$\|u_n^1\|^2 = \langle I'_\infty(u_n^1), u_n^1\rangle + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{q_\infty^2 |u_n^1(y)|^p |u_n^1(x)|^p}{|x-y|^{N-\alpha}}\, dx\, dy = o(1),$$

and then we complete our proof. If $\delta > 0$, then there exists a sequence $(y_n^1)_{n\geq 1}$ such that $\int_{B_1(y_n^1)} |u_n^1|^2 > \frac{\delta}{2}$. Let $v_n^1 := u_n^1(\cdot + y_n^1)$. Then $v_n^1 \rightharpoonup v_1$ weakly in $H^1(\mathbb{R}^N)$ and $v_n^1 \to v_1$ a.e. on $\mathbb{R}^N$. Applying the compactness of the embedding $H_0^1(B_1(0)) \hookrightarrow L^2(B_1(0))$ and $\int_{B_1(0)} |v_n^1|^2 > \frac{\delta}{2}$, we have $\int_{B_1(0)} |v_1|^2 \geq \frac{\delta}{2}$ and $v_1 \neq 0$. Because $u_n^1 \rightharpoonup 0$ a.e. on $H^1(\mathbb{R}^N)$, so $(y_n^1)_{n\geq 1}$ must be unbounded.

Suppose $|y_n^1| \to \infty$ as $n \to \infty$. We claim $I'_\infty(v_1) = 0$ and $u_n^2 := u_n^1 - v_1(\cdot - y_n^1)$ such that

$$\begin{gathered} \|u_n^2\|^2 = \|u_n\|^2 - \|v_0\|^2 - \|v_1\|^2 + o(1), \\ I_\infty(u_n^2) \to C - I(v_0) - I_\infty(v_1), \end{gathered}$$

$$I'_\infty(u_n^2) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).$$

Indeed, since $v_n^1 \rightharpoonup v_1$ weakly in $H^1(\mathbb{R}^N)$, we have

$$\|u_n^2\|^2 = \|u_n^1(\cdot + y_n^1) - v_1\|^2 = \|u_n^1\|^2 - \|v_1\|^2 + o(1).$$

Applying Lemma 2.4 and similar arguments as Step 2, we prove the claim. In the sequel, we iterate the above procedure, and then construct sequences $(v_j)$ and $(y_n^j)$ such that $|y_n^j| \to \infty$ and $|y_n^i - y_n^j| \to \infty$ for $i \neq j$, $n \to \infty$. Since $I(u_n) \to C$ and $I_\infty(v_j) \geq c_\infty$ for every nontrivial critical point $v_j$ of $I_\infty$, then the iteration must terminate at some finite number of steps, which completes the whole proof. □

## 3. Non-radial case

In this section, we first give some properties of the Nehari manifold $\mathcal{N}$ and the relationship between $c$ and $c_\infty$. Some of similar results can be found in [21] and [16]. Here we give the complete proof.

**Lemma 3.1.** *Let $N \geq 3$, $\alpha \in (0, N)$ and $2 \leq p < \frac{N+\alpha}{N-2}$. If $q(x)$ satisfies (H1), then the following statements are true:*

(i) *$\mathcal{N}$ is nonempty. Moreover, for every $u \in H^1(\mathbb{R}^N)$ with $\mathbb{D}(u) > 0$, there exists a unique $t_u \in (0, \infty)$ such that $t_u u \in \mathcal{N}$ and*

$$t_u = \Big(\frac{\|u\|^2}{\mathbb{D}(u)}\Big)^{\frac{1}{2p-2}}. \tag{3.1}$$

*Furthermore, $I(t_u u) = \sup_{t>0} I(tu) = (\frac{1}{2} - \frac{1}{2p}) J^{\frac{p}{p-1}}(u)$.*

(ii) $c = \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in H^1(\mathbb{R}^n)\setminus\{0\}} \sup_{t>0} I(tu)$.

(iii) $c > 0$.

(iv) *$\mathcal{N}$ is a $\mathcal{C}^2$-submanifold of $H^1(\mathbb{R}^N)$.*

(v) $c \leq c_\infty$.

*Proof.* (i) First, it follows from (H1) that there exists $R > 0$ large enough such that $q(x) > \frac{1}{2} q_\infty$ for $|x| > R$, and then we can find $u \in H^1(\mathbb{R}^N)$ such that $\mathbb{D}(u) > 0$. In addition, for $t > 0$, we have

$$\frac{d}{dt} I(tu) = \langle I'(tu), u\rangle = t\|u\|^2 - t^{2p-1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{q(x)q(y)|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\, dx\, dy.$$

Then there exists a unique $t_u$ such that $\langle I'(t_u u), u\rangle = 0$, which yields that $t_u u \in \mathcal{N}$. Since the map $t \mapsto I(tu)$ is increasing for $0 < t < t_u$ and decreasing for $t > t_u$, we have $I(t_u u) = \sup_{t>0} I(tu)$. Furthermore, it follows from direct calculation that

$$\begin{aligned} I(t_u u) &= \frac{1}{2}\Big(\frac{\|u\|^2}{\mathbb{D}(u)}\Big)^{\frac{2}{2p-2}} \|u\|^2 - \frac{1}{2p}\Big(\frac{\|u\|^2}{\mathbb{D}(u)}\Big)^{\frac{2}{2p-2}} \\ &= \Big(\frac{1}{2} - \frac{1}{2p}\Big)\Big(\frac{\|u\|^2}{\mathbb{D}^{1/p}(u)}\Big)^{\frac{p}{p-1}} \\ &= \Big(\frac{1}{2} - \frac{1}{2p}\Big) J^{\frac{p}{p-1}}(u). \end{aligned}$$

(ii) Define $\mathcal{M} = \{u \in H^1(\mathbb{R}^N) : \mathbb{D}(u) > 0\}$. For $u \in \mathcal{M}$, $t_u u \in \mathcal{N}$ and then $I(t_u u) \geq \inf_{u\in\mathcal{N}} I(u)$, which concludes $\inf_{u\in\mathcal{M}} \sup_{t>0} I(tu) \geq \inf_{u\in\mathcal{N}} I(u)$. If $u \in H^1(\mathbb{R}^N)\setminus\mathcal{M}$ and $u \neq 0$, we have $\sup_{t>0} I(tu) = \infty$. In the contrary, if $u \in \mathcal{N}$, then $t_u = 1$ and hence $I(u) \geq \inf_{u\in\mathcal{N}} \sup_{t>0} I(tu)$ which implies that $I(u) \geq \inf_{u\in H^1(\mathbb{R}^N)} \sup_{t>0} I(tu)$. This yields a conclusion.

(iii) Let $\lambda > 0$. For any $u \in \mathcal{N}$, it follows from (i) above that $I(\frac{\lambda}{\|u\|}u) \le I(u)$, and then $I(u) \ge \inf_{v\in\mathcal{S}} I(v)$, where $\mathcal{S} = \{v \in H^1(\mathbb{R}^N) : \|v\| = \lambda\}$. Assume $\lambda = (\frac{p}{2C|q|^2_{L^\infty}})^{\frac{1}{2p-2}}$. According to (2.1) and Sobolev embedding theorem, for any $v \in \mathcal{S}$, we have

$$
\begin{aligned}
I(v) &\ge \frac{1}{2}\|v\|^2 - \frac{C}{2p}|q|^2_{L^\infty}|v|^p_{L^{\frac{2Np}{N+\alpha}}}|v|^p_{L^{\frac{2Np}{N+\alpha}}} \\
&\ge \frac{1}{2}\|v\|^2 - \frac{C}{2p}|q|^2_{L^\infty}\|v\|^{2p} \\
&= \lambda^2(\frac{1}{2} - \frac{\lambda^{2p-2}C|q|^2_{L^\infty}}{2p}) > 0,
\end{aligned}
\tag{3.2}
$$

where $C$ only depends on $p$, $N$ and $\alpha$. This yields $c > 0$.

(iv) Denote $G : H^1(\mathbb{R}^N) \to \mathbb{R}$ by $G(u) = \|u\|^2 - \mathbb{D}(u)$. Applying the same method as Appendix B in [16], we derive $G$ is of class $\mathcal{C}^2$ and its derivative is given by

$$
G'(u)v = 2\langle u, v\rangle - 2p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{q(x)q(y)|u(y)|^p|u(x)|^{p-2}u(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy
$$

for all $u,\ v$ in $H^1(\mathbb{R}^N)$. Since $\mathcal{N} = G^{-1}(0)$ and

$$
G'(u)u = 2\|u\|^2 - 2p\mathbb{D}(u) \ne 0
$$

for all $u \in \mathcal{N}$, then we imply 0 is a regular value of $G$. This, combined with (ii) yields that $\mathcal{N}$ is a submanifold of class $\mathcal{C}^2$ of $H^1(\mathbb{R}^N)$ and $u \notin \ker G'(u)$ for $u \in \mathcal{N}$.

(v) By Lemma 2.1, we assume $w$ is a ground state solution of (2.2) and $(x_n)_{n\ge1}$ is the unbounded sequence such that $|x_n| \to \infty$, as $n \to \infty$. Then for every $w(\cdot - x_n)$, according to (i) above, there exists $t_n$ such that $t_n w(\cdot - x_n) \in \mathcal{N}$ with

$$
t_n = \Big(\frac{\|w(x-x_n)\|^2}{\mathbb{D}(w(x-x_n))}\Big)^{\frac{1}{2p-2}}.
$$

Since $w \in \mathcal{N}_\infty$, by dominated convergence theorem, we have $t_n \to 1$, and then

$$
\begin{aligned}
c &= \inf_{u\in\mathcal{N}} I(u) \le I(t_n w(x-x_n)) \\
&= \frac{1}{2}t_n^2\|w(x-x_n)\|^2 - \frac{1}{2p}|t_n|^{2p}\mathbb{D}(w(x-x_n)) \\
&\to I_\infty(w) = c_\infty.
\end{aligned}
\tag{3.3}
$$

This completes the proof. □

Next we show that the energy functional satisfies the Mountain-Pass geometry.

**Lemma 3.2.** *Let $N \ge 3$, $\alpha \in (0, N)$ and $2 \le p < \frac{N+\alpha}{N-2}$. Suppose $q$ satisfies* (H1). *Then the functional $I$ satisfies the following conditions.*

(i) *There exists $r > 0$ such that $I(u) \ge \theta > 0$ for all $\|u\| = r$.*
(ii) *There exists $e \in H^1(\mathbb{R}^N)$ such that $e \ge 0$, $\|e\| > r$ and $I(e) < 0$.*

*Proof.* (i)By (2.1), we have

$$
\begin{aligned}
I(u) &\ge \frac{1}{2}\|u\|^2 - \frac{C}{2p}|q|^2_{L^\infty}|u|^p_{L^{\frac{2Np}{N+\alpha}}}|u|^p_{L^{\frac{2Np}{N+\alpha}}} \\
&\ge \frac{1}{2}\|u\|^2 - \frac{C}{2p}|q|^2_{L^\infty}\|u\|^{2p}.
\end{aligned}
\tag{3.4}
$$

where $C$ only depends on $p$, $N$ and $\alpha$. Since $p \geq 2$, we can choose $r, \theta > 0$ such that $I(u) \geq \theta > 0$ for all $\|u\| = r$.

(ii) Note that $I(0) = 0$. In addition, we can find some $u \in H^1(\mathbb{R}^N)$ such that $\mathbb{D}(u) > 0$. Then it follows from $p \geq 2$ that

$$\lim_{t\to+\infty} I(tu) = \lim_{t\to+\infty} \left(\frac{1}{2}t^2\|u\|^2 - \frac{1}{2p}t^{2p}\mathbb{D}(u)\right) = -\infty.$$

Hence, there exists $t_0 > 0$ such that $||t_0 u|| > r$ and $I(t_0 u) < 0$. Take $e = |t_0 u|$. Then the proof is complete. $\square$

Define

$$c_1 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e\}$. In following lemma, we will show the relationship between $c$ and $c_1$ (see [9, Proposition 2,14]).

**Lemma 3.3.** *Let $N \geq 3$, $\alpha \in (0, N)$ and $2 \leq p < \frac{N+\alpha}{N-2}$. Suppose $q$ satisfies* (H1). *Then $c = c_1$.*

*Proof.* According to Lemma 3.2(i), there exists a small ball in $H^1(\mathbb{R}^N)$ containing the origin such that $I(u) \geq 0$ for all $u$ in this component. By Lemma 3.2(ii), we have

$$\langle I'(e), e\rangle = 2I(e) + (\frac{1}{p} - 1)\mathbb{D}(e) < 0.$$

Thus every $\gamma \in \Gamma$ has to cross $\mathcal{N}$ and $c \leq c_1$.

On the other hand, for any $\bar{u} \in \mathcal{N}$, let $l = \{t\bar{u} : t \geq 0\}$ be a half-line and $I(|\bar{u}|) = I(\bar{u}) = \max_{u\in l} I(u)$ due to Lemma 3.1(i). Similarly, we denote $h = \{te : t \geq 0\}$. Let $V^+$ be the set $\{a|\bar{u}| + be : a \geq 0, b \geq 0\}$, let $V$ be the 2-dimensional subsequence of $H^1(\mathbb{R}^N)$ spanned by $|\bar{u}|$ and $e$. Note that $\mathbb{D}(|\bar{u}|) \neq 0$ and $\mathbb{D}(e) \neq 0$. Then for any $v \in V^+\backslash\{0\}$, we have $\mathbb{D}(v) > 0$. Hence there exists a circle $S$ on $V$ with radius $R$ large enough such that $I \leq 0$ on $S \bigcap V^+$. Suppose that $l$ and $h$ intersect $S$ at $v$ and $v_1$, respectively. Thus we can find a path $\bar{\gamma} \in \Gamma$ through $v$ and $v_1$ such that $I(|\bar{u}|) = \max_{u\in\bar{\gamma}} I(u)$. Therefore, $c \geq c_1$. This completes the proof. $\square$

**Proposition 3.4.** *Let $N \geq 3$, $\alpha \in (0, N)$ and $2 \leq p < \frac{N+\alpha}{N-2}$. Suppose $q$ satisfies* (H1)*. If $c < c_\infty$ holds, then $I$ has a critical point $u \in H^1(\mathbb{R}^N)$ such that $I(u) = c$.*

*Proof.* By Lemma 3.2, the energy functional $I$ satisfies the mountain pass geometry. Due to Lemma 3.3 and the mountain pass theorem, there exists a sequence $(u_n)_{n\geq1} \subset H^1(\mathbb{R}^N)$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$. Since $c < c_\infty$, Lemma 2.8 completes the proof. $\square$

Now, we are in a position to prove our main existence result.

*Proof of Theorem 1.1.* First we prove (i). The proof can be split into two cases.

**Case 1.** $q \equiv q_\infty$. By scaling, the conclusion follows from Lemma 2.1.

**Case 2.** $q \not\equiv q_\infty$. For any $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, it is easy to check $I(u) < I_\infty(u)$. Due to Lemma 2.1, we have $c_\infty$ can be attained by a $w \in H^1(\mathbb{R}^N)$. By Lemma 3.1(i), there exists $t_w > 0$ such that $t_w w \in \mathcal{N}$. Hence

$$c \leq I(t_w w) < I_\infty(t_w w) \leq I_\infty(w) = c_\infty.$$

Therefore, Proposition 3.4 yields our conclusion.

Next we prove (ii). By way of contradiction, we assume $c$ can be achieved by $\eta \in H^1(R^N)$. It follows from that Lemma 3.1(i) that there exists $t_\eta > 0$ such that $t_\eta \eta \in \mathcal{N}_\infty$. Therefore,

$$c = I(\eta) \geq I(t_\eta \eta) > I_\infty(t_\eta \eta) \geq c_\infty,$$

a contradiction to Lemma 3.1(v). This completes the proof. □

*Proof of Theorem 1.2.* Set

$$\tilde{c} = \inf\{\|u\|^2 : u \in H^1(\mathbb{R}^N),\ \mathbb{D}(u) = 1\}. \tag{3.5}$$

According to the proof of Lemma 3.1(i), it suffices to prove whether $\tilde{c}$ can be attained by some $u \in H^1(\mathbb{R}^N)$ or not.

First we prove (i). Without loss of generality, there exists a nonnegative minimizing sequence $(u_n)_{n\geq 1}$ such that $\|u_n\|^2 \to \tilde{c}$ and $\mathbb{D}(u_n) = 1$. Then going if necessary to a subsequence, there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. on $\mathbb{R}^N$. It is easy to see $u_0$ is nonnegative, $\|u_0\|^2 \leq \tilde{c}$ and $\mathbb{D}(u_0) \leq 1$.

Since $\lim_{|x|\to\infty} q(x) = 0$, then for any $\varepsilon > 0$, we can find some $R > 0$ such that for any $|x| > R$, we have $|q(x)| < \varepsilon$. By Lemma 2.3, we derive $|u_n|^p \rightharpoonup |u_0|^p$ weakly in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. In addition, $u_n \to u_0$ in $L^{\frac{2Np}{N+\alpha}}_{\rm loc}(\mathbb{R}^N)$. Then by Hardy-Littlewood-Sobolev inequality (2.1), we have

$$\begin{aligned}
&|\mathbb{D}(u_n) - \mathbb{D}(u_0)| \\
&\leq \Big| \int_{\mathbb{R}^N\setminus B_R(0)} \int_{\mathbb{R}^N} \frac{q(x)q(y)|u_n(y)|^p|u_n(x)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N\setminus B_R(0)} \int_{\mathbb{R}^N} \frac{q(x)q(y)|u_0(y)|^p|u_0(x)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\Big| \\
&\quad + \Big| \int_{B_R(0)} \int_{\mathbb{R}^N} \frac{q(x)q(y)|u_n(y)|^p(|u_n(x)|^p - |u_0(x)|^p)}{|x-y|^{N-\alpha}}\,dx\,dy\Big| \\
&\quad + \Big| \int_{B_R(0)} \int_{\mathbb{R}^N} \frac{q(x)q(y)(|u_n(y)|^p - |u_0(y)|^p)|u_0(x)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\Big| \\
&\to 0.
\end{aligned} \tag{3.6}$$

Hence $\mathbb{D}(u_0) = 1$. Let $u_* = \tilde{c}^{\frac{1}{2(p-1)}} u_0$. Then according to the proof of Lemma 3.1(i), we conclude $I(u_*) = c$ and $\langle I'(u_*), u_*\rangle = 0$. Therefore, the Lagrange multiplier rule yields that $u_*$ is a ground state solution of (1.1).

Now we prove (ii). Assume for contradiction, $\tilde{c}$ is attained by some $u \in H^1(\mathbb{R}^N)$ such that $\mathbb{D}(u) = 1$ and $\|u\|^2 = \tilde{c}$. Let $R > 0$ and $u_R(x) = u(x - x_R)$, where

$x_R = (3R, 0, \cdots, 0)$. Clearly, $\|u_R\|^2 = \tilde{c}$, and it is easy to check that

$$\begin{aligned} \mathbb{D}(u_R) &\geq \int_{B_R(x_R)} \int_{B_R(x_R)} \frac{q(x)q(y)|u_R(x)|^p |u_R(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ &\geq \min_{x\in B_R(x_R)} q(x) \min_{y\in B_R(x_R)} q(y) \int_{B_R(0)} \int_{B_R(0)} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ &\geq \min_{2R\leq|x|\leq 4R} q(x) \min_{2R\leq|y|\leq 4R} q(y) \int_{B_R(0)} \int_{B_R(0)} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ &\geq \max_{|x|\leq R} q(x) \max_{|y|\leq R} q(y) \int_{B_R(0)} \int_{B_R(0)} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ &= \int_{B_R(0)} \int_{B_R(0)} \frac{q(x)q(y)|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy + h(R). \end{aligned} \tag{3.7}$$

Here

$$h(R) := \int_{B_R(0)} \int_{B_R(0)} [\max_{|x|\leq R} q(x) \max_{|y|\leq R} q(y) - q(x)q(y)] \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \geq 0.$$

Since $q$ is not constant, we have $h \not\equiv 0$ in $\mathbb{R}^N$. In addition, $h$ is nondecreasing in $R$ and bounded in $\mathbb{R}^N$ due to the fact that $q \in L^\infty(\mathbb{R}^N)$. We assume $h(\infty) = \lim_{R\to\infty} h(R)$. Then there exist $R_1 \geq R_0$ such that $h(R) > \frac{1}{2}h(\infty)$ for all $R \geq R_1$. On the other hand, since $\mathbb{D}(u) = 1$, we can find $R_2 \geq R_0$ such that

$$\int_{B_R(0)} \int_{B_R(0)} \frac{q(x)q(y)|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy > 1 - \frac{1}{2}h(\infty),$$

for all $R \geq R_2$. Take $R = \max\{R_1, R_2\}$. Then $\mathbb{D}(u_R) > 1$. Let $\tilde{u}_R = (\frac{1}{\mathbb{D}(u_R)})^{\frac{1}{2p}} u_R$. Then $\mathbb{D}(\tilde{u}_R) = 1$. But $\|\tilde{u}_R\|^2 < \tilde{c}$, which contradicts the definition of $\tilde{c}$. This completes the proof. $\square$

## 4. Radial case

It is known to us that if $q$ is radial and bounded in $\mathbb{R}^N$, by standard variational methods and the symmetric criticality principle (see [29, Theorem 1.28]), we can find a nontrivial radial solution to (1.1) in $H^1(\mathbb{R}^N)$. But in this section, we consider the case that $q$ is radial and $q$ may be unbounded in $\mathbb{R}^N$. Applying a similar idea as that of [9], we obtain a nontrivial nonnegative radial solution for (1.1) in $H^1_r(\mathbb{R}^N)$. First, we give the Radial lemma that will play a key role in the proof of Theorem 1.3. Throughout this section, we denote the norm of $H^1_r(\mathbb{R}^N)$ ( or $H^1_{0,r}(B_k)$) by $\|\cdot\|_{H^1_r(\mathbb{R}^N)}$ (or $\|\cdot\|_{H^1_{0,r}(B_k)}$, respectively).

**Lemma 4.1** ([4]). *Let $N \geq 2$. Then for any radial function $u \in H^1(\mathbb{R}^N)$,*

$$|u(r)| \leq C\|u\| r^{\frac{1-N}{2}}, \quad \text{for } r \geq 1,$$

*where $C$ only depends on $N$.*

*Proof of Theorem 1.3.* First, we define

$$M_\infty = \sup_{\|u\|_{H^1_r(\mathbb{R}^N)}=1} \mathbb{D}(u).$$

It follows from Lemma 4.1 and (H4) that $M_\infty < \infty$. Without loss of generality, we assume there exists a nonnegative minimizing sequence $(u_n)_{n\geq 1} \subset H^1_r(\mathbb{R}^N)$ such that $\|u_n\|_{H^1_r(\mathbb{R}^N)}=1$ and $\mathbb{D}(u_n)\to M_\infty$ as $n\to\infty$. Then going if necessary to a subsequence, $u_n\rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N)$ and $u_n\to u_0$ a.e. on $\mathbb{R}^N$. Obviously, $u_0$ is nonnegative, radial and $\|u_0\|_{H^1_r(\mathbb{R}^N)}\le 1$. Applying Lemma 4.1 and (H4) again, we obtain $(qu_n^p)_{n\ge 1}$ is uniformly bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, and then for any $\epsilon>0$, there exists $R>0$ such that $|qu_n^p|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N\setminus B_R(0))}<\epsilon$. Here $R$ is independent of $n$. Since $u_n\to u_0$ strongly in $L^{\frac{2Np}{N+\alpha}}_{\rm loc}(\mathbb{R}^N)$, by (2.1), some standard argument yields that $\mathbb{D}(u_n)\to\mathbb{D}(u_0)=M_\infty$.

Next we show $\|u_0\|_{H^1_r(\mathbb{R}^N)}=1$. If not, we can find $\tilde u_0\in H^1_r(\mathbb{R}^N)$ such that $\|\tilde u_0\|_{H^1_r(\mathbb{R}^N)}=1$ and $\tilde u_0=\lambda u_0$ with $\lambda>1$. This implies $\mathbb{D}(\tilde u_0)>M_\infty$, a contradiction to the definition of $M_\infty$. Let $u_*=(\frac{1}{M_\infty})^{\frac{1}{2p-2}}u_0$. According to Lagrange multiplier rule, we conclude $u_*$ is a nonnegative radial solution of (1.1) in $H^1_r(\mathbb{R}^N)$. This completes the proof. □

Theorem 1.5 can be treated as a by-product of Theorem 1.3. Now we give a simple proof. Let

$$\mathbb{D}_k(u)=\int_{B_k(0)}\int_{B_k(0)}\frac{q(x)q(y)|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy.$$

*Proof of Theorem 1.5.* This proof can be split into two steps.

**Step 1.** Define

$$M_{k,r}=\sup_{\|u\|_{H^1_{0,r}(B_k)}=1}\mathbb{D}_k(u),\quad M_k=\sup_{\|u\|_{H^1_0(B_k)}=1}\mathbb{D}_k(u)$$

Fix $k$. Without loss of generality, we assume there exists an nonnegative minimizing sequence $(v_n^k)_{n\ge1}\subset H^1_{0,r}(B_k)$ such that $\|v_n^k\|_{H^1_{0,r}(B_k)=1}$ and $\mathbb{D}_k(v_n^k)\to M_{k,r}$ as $n\to\infty$. Then going if necessary to a subsequence, $v_n^k\rightharpoonup u_k$ weakly in $H^1_{0,r}(B_k)$ and $v_n^k\to u_k$ a.e. on $\mathbb{R}^N$ as $n\to\infty$. Obviously, $u_k$ is nonnegative, radial and $\|u_k\|_{H^1_{0,r}(B_k)}\le 1$. Since $v_n^k\to u_k$ strongly in $L^{\frac{2Np}{N+\alpha}}_{\rm loc}(\mathbb{R}^N)$, by (2.1), some standard arguments can imply that $\mathbb{D}_k(v_n^k)\to\mathbb{D}_k(u_k)$ as $n\to\infty$. Hence $\mathbb{D}_k(u_k)=M_{k,r}$. Similar to the proof of Theorem 1.3, we have $\|u_k\|_{H^1_{0,r}(B_k)}=1$. Therefore, $M_{k,r}$ is attained by $u_k\in H^1_{0,r}(B_k)$. Let $w_k=\Big(\frac{1}{M_{k,r}}\Big)^{\frac{1}{2p-2}}u_k$. It follows from Lagrange multiplier rule and symmetric criticality principle (see [29, Theorem 1.28]) that $w_k$ is a nontrivial nonnegative radial solution of the equation

$$\begin{gathered}-\Delta u+u=q(x)(\int_{B_k(0)}\frac{q(y)u^p(y)}{|x-y|^{N-\alpha}}dy)u^{p-1}\quad\text{in } B_k(0),\\ u\ge 0\quad\text{in } B_k(0),\\ u=0\quad\text{on }\partial B_k.\end{gathered}\tag{4.1}$$

Similarly, $M_k$ is also attained by $u_k^*\in H^1_0(B_k)$ and $w_k^*=(\frac{1}{M_k})^{\frac{1}{2p-2}}u_k^*$.

**Step 2.** Since $M_{k,r}$ is increasing with $k$, according to Theorem 1.3,

$$\Big(\frac{1}{M_\infty}\Big)^{\frac{1}{2p-2}}\le\|w_k\|_{H^1_r(\mathbb{R}^N)}=\Big(\frac{1}{M_{k,r}}\Big)^{\frac{1}{2p-2}}\le\Big(\frac{1}{M_{1,r}}\Big)^{\frac{1}{2p-2}},$$

that is to say, $(w_k)_{k\ge1}$ is uniformly bounded in $H^1_r(\mathbb{R}^N)$.

On the other hand, we choose $u_0 \in H^1(\mathbb{R}^N)$ such that $u_0$ has compact support in $B_1(0)$ and $\|u_0\| = 1$. Then for large $k$, there exists $x_k \in \mathbb{R}^N$ such that $B_1(x_k) \subset B_k(0)$. Without loss of generality, we assume $|x_k| \to \infty$ as $k \to \infty$. Hence

$$\begin{aligned} M_k &\geq \int_{B_k(0)} \int_{B_k(0)} \frac{q(x)q(y)|u_0(y-x_k)|^p |u_0(x-x_k)|^p}{|x-y|^{N-\alpha}} \, dx \, dy \\ &\geq \int_{B_1(x_k)} \int_{B_1(x_k)} \frac{q(x)q(y)|u_0(y-x_k)|^p |u_0(x-x_k)|^p}{|x-y|^{N-\alpha}} \, dx \, dy \\ &\geq \min_{x \in B_1(x_k)} q(x) \min_{y \in B_1(x_k)} q(y) \int_{B_1(0)} \int_{B_1(0)} \frac{|u_0(y)|^p |u_0(x)|^p}{|x-y|^{N-\alpha}} \, dx \, dy. \end{aligned} \tag{4.2}$$

This implies that $M_k \to \infty$ as $k \to \infty$. Then for $k$ large enough, (4.1) has two different weak solutions $w_k$ and $w_k^*$. One is radial and the other is not. This completes the proof. $\square$

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Tao Wang
School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China
*E-mail address*: wt_61003@163.com