Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 04, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NON-EXISTENCE OF PERIODIC SOLUTIONS TO NON-AUTONOMOUS SECOND-ORDER DIFFERENTIAL EQUATION WITH DISCONTINUOUS NONLINEARITY

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Abstract. We consider a second-order differential equation with discontinuous nonlinearity and sinusoidal external influence, and obtain conditions for the non-existence of periodic solutions.

1. Introduction

Relay control systems have been studied for a long time (see, e.g. [1]–[3]). Nevertheless automatic control systems with relay nonlinearity and external influence are of interest nowadays [4], since there are still open questions. In this article the automatic system is described by a second-order differential equation with time-independent coefficients. There are both a control function and an external influence function in the right-hand side of the equation. We consider the signum function as a relay control model. Since the time of Andronov [5] this model have been used in automatic control systems. The system may be free from periodic modes in case if there is no external influence. Moreover, it is possible that the system does not have periodic modes when relay control is absent. However stabilization of the system may occur under both external influence and control. By stabilization we mean existence of stable oscillations of the certain configuration and given period. If we can not affect on the dynamics of the object and the parameters of the external influence, then at switching the step height c of function $u = \frac{c}{2}\operatorname{sgn} x$ influences significantly on the system dynamics. Taking account the stabilization type above, we set the task for choosing parameter c that depends on the other parameters of the automatic control system.

In recent years the second-order differential equations with discontinuous nonlinearities have been considered in [6]–[10]. Applied problems for such equations are discussed in [11, 12]. Control problems for systems with distributed parameters and discontinuous nonlinearity are studied in [13]. On signification of the research for non-existence of the solutions to the problem with singular potential, see [14]. Classification of discontinuities for real-valued functions is well described in [15]

2010 *Mathematics Subject Classification.* 34A34, 34A36, 34H05, 93C73.
Key words and phrases. Non-autonomous differential equation; discontinuous nonlinearity; periodic solution; non-existence of solutions.

Submitted June 1, 2015. Published January 4, 2016.

giving as an example for the function with discontinuity of the first kind at zero the signum function to be defined as follows:

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

This article continues research stated above. We study the problem when there are no periodic solutions of the automatic control system with a relay and a sine function as external influence. The system dynamics is described by the equation

$$x'' + a_1 x' + a_0 x = \frac{c}{2} \operatorname{sgn} x + \beta \sin(\gamma t), \tag{1.1}$$

where a_1, a_0, β are real constants, parameter $c > 0$, the external influence frequency $\gamma > 0$. Function $\operatorname{sgn} x$ describes the relay control.

We note that equation (1.1) is investigated when $a_1 = a_0 = 0$ and $c < 0$ in [6]. The autonomous equations of the form (1.1) are studied in [7]–[11]. In [12], we consider nonperiodic external influence for the one-dimensional Lavrent'ev model described by the equation $-x'' = \mu \operatorname{sgn} x$, where parameter $\mu > 0$ means a vorticity. Hence this paper develops [6]–[12].

2. Solution of the problem

Let λ_1, λ_2 be the real roots of the equation $\lambda^2 + a_1\lambda + a_0 = 0$ and $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_1 \neq \lambda_2$, i.e. $a_0 \neq 0$ and $a_1^2 > 4a_0$. Also, let $a_1 \neq 0$ and $\lambda_1 < \lambda_2$. Then the general solution of equation (1.1) has the form

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \pm C_0 + Q_1 \cos(\gamma t) + Q_2 \sin(\gamma t), \tag{2.1}$$

where C_1, C_2 are the constants we have to define,

$$C_0 = \frac{c}{2a_0}, \quad Q_1 = -\frac{a_1 \beta \gamma}{(\gamma^2 - a_0)^2 + a_1^2 \gamma^2}, \quad Q_2 = -\frac{\beta(\gamma^2 - a_0)}{(\gamma^2 - a_0)^2 + a_1^2 \gamma^2}.$$

Since $a_1 \neq 0$ and $\gamma > 0$, then $(\gamma^2 - a_0)^2 + a_1^2\gamma^2 \neq 0$ and so that solution (2.1) exists. Thus the following theorem holds.

Theorem 2.1. *Let $a_1 \neq 0$, $a_0 \neq 0$, $a_1^2 > 4a_0$, $c > 0$, $\beta \in \mathbb{R}$, and $\gamma > 0$. Then equation* (1.1) *has a solution of the form* (2.1) *on half-planes $x > 0$ and $x < 0$ from phase plane* (xOx').

We assume that (1.1) has a periodic solution with period T. If we are interested in the periodic solution with a desirable period, we set period T. The closed trajectory relating to a jump on axis Ox_2 at switching of $u = \frac{c}{2} \operatorname{sgn} x$ corresponds to the periodic solution on plane $(x_1 O x_2)$, where $x_1 = x$, $x_2 = x'$. The closed trajectory consists of two phase trajectory pieces by virtue of the different systems

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -a_0 x_1 - a_1 x_2 \pm \frac{c}{2} + \beta \sin(\gamma t). \end{aligned} \tag{2.2}$$

Let $T = t_1 + t_2$, where t_1 corresponds with the trajectory part at $x_1 > 0$, and t_2 corresponds with the trajectory part at $x_1 < 0$. Axis Ox_2 is a straight line on which there is a gluing together of the closed trajectory pieces and the other trajectories by virtue of systems (2.2). Therefore we suppose that the closed trajectory consists of two pieces that are located on half-planes $x_1 > 0$ or $x_1 < 0$.

Let the point with coordinates $x_1 = 0$, $x_2 = x_0'$ belong to the closed trajectory the image point passes for T. We take this point as initial. Then, taking into account that

$$x'(t) = \lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} - \gamma Q_1 \sin(\gamma t) + \gamma Q_2 \cos(\gamma t),$$

we can find C_1 and C_2 corresponding to interval $[0, t_1]$.

On interval $[0, t_1]$, we denote $C_1 = C_1^1$, $C_2 = C_2^1$ and, on interval $[0, t_2]$, we do $C_1 = C_1^2$, $C_2 = C_2^2$ respectively. We obtain

$$C_1^1 = C_2 - C_0 - Q_1, \quad C_2^1 = \frac{x_0' + \lambda_1(C_0 + Q_1) - \gamma Q_2}{\lambda_2 - \lambda_1},$$

where x_0' is unknown. Then we calculate

$$\begin{gathered} x(t_1) = C_1^1 e^{\lambda_1 t_1} + C_2^1 e^{\lambda_2 t_1} + C_0 + Q_1 \cos(\gamma t_1) + Q_2 \sin(\gamma t_1) = 0, \\ x'(t_1) = \lambda_1 C_1^1 e^{\lambda_1 t_1} + \lambda_2 C_2^1 e^{\lambda_2 t_1} - \gamma Q_1 \sin(\gamma t_1) + \gamma Q_2 \cos(\gamma t_1). \end{gathered} \tag{2.3}$$

These values are initial for the solution on interval $[0, t_2]$

$$\begin{gathered} x(0) = 0 = C_1^2 + C_2^2 - C_0 + Q_1, \\ x'(0) = x'(t_1) - c = \lambda_1 C_1^2 + \lambda_2 C_2^2 + \gamma Q_2, \end{gathered}$$

from where it follows that

$$C_1^2 = C_0 - C_2^2 - Q_1, \quad C_2^2 = \frac{x'(t_1) - c - \lambda_1(C_0 - Q_1) - \gamma Q_2}{\lambda_2 - \lambda_1}.$$

Completing the circle along the closed trajectory and remembering the gluing together on axis Ox_2, we have

$$\begin{gathered} x(t_2) = 0 = C_1^2 e^{\lambda_1 t_2} + C_2^2 e^{\lambda_2 t_2} - C_0 + Q_1 \cos(\gamma t_2) + Q_2 \sin(\gamma t_2), \\ x'(t_2) = \lambda_1 C_1^2 e^{\lambda_1 t_2} + \lambda_2 C_2^2 e^{\lambda_2 t_2} - \gamma Q_1 \sin(\gamma t_2) + \gamma Q_2 \cos(\gamma t_2) = x_0' - c. \end{gathered} \tag{2.4}$$

Constants C_1^1, C_2^1 in (2.3) contain x_0' and constants C_1^2, C_2^2 in (2.4) contain $x'(t_1)$. Further, we transform equalities (2.3), (2.4) after excluding x_0' and $x'(t_1)$ from them. Thus $Q_1 \cos(\gamma t_1) + Q_2 \sin(\gamma t_1) = \sin(\gamma t_1 + \delta)$, where $\delta = \arctan(Q_1/Q_2)$, $Q_2 \neq 0$, i.e. $\beta \neq 0$, $\gamma^2 \neq a_0$. After tedious transformations, we get two transcendental equations with respect to t_1 and t_2,

$$\begin{aligned} &\frac{1}{\lambda_2 - \lambda_1}\Big(- \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_1 t_1} + \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_2 t_1}\Big) \Big[(\gamma Q_2 - \lambda_1 Q_1 - \lambda_1 C_0) e^{\lambda_1 t_1} \\ &- (\lambda_2 - \lambda_1)(C_0 + Q_1) e^{\lambda_1 t_1} - (\gamma Q_2 - \lambda_1 Q_1 - \lambda_1 C_0) e^{\lambda_2 t_1} + (\lambda_2 - \lambda_1) C_0 \\ &+ (\lambda_2 - \lambda_1) \sin(\gamma t_1 + \delta) \Big] + \frac{1}{\lambda_2 - \lambda_1} (e^{\lambda_1 t_2} - e^{\lambda_2 t_2}) \\ &\times \Big[\frac{\lambda_1}{\lambda_2 - \lambda_1} (\gamma Q_2 - \lambda_1 Q_1 - \lambda_1 C_0) e^{\lambda_1 t_1} - \lambda_1 (C_0 + Q_1) e^{\lambda_1 t_1} \\ &- \frac{\lambda_2}{\lambda_2 - \lambda_1} (\gamma Q_2 - \lambda_1 Q_1 - \lambda_1 C_0) e^{\lambda_2 t_1} + \gamma \cos(\gamma t_1 + \delta) \Big] \\ &= \frac{1}{\lambda_2 - \lambda_1} (c + \lambda_1 (C_0 - Q_1) + \gamma Q_2) e^{\lambda_1 t_2} + (C_0 - Q_1) e^{\lambda_1 t_2} \\ &\quad - \frac{1}{\lambda_2 - \lambda_1} (c + \lambda_1 (C_0 + Q_1) + \gamma Q_2) e^{\lambda_2 t_2} - C_0 + \sin(\gamma t_2 + \delta), \end{aligned} \tag{2.5}$$

and

$$
\begin{aligned}
&(e^{\lambda_1 t_1}-e^{\lambda_2 t_2})^{-1}[(\gamma Q_2-\lambda_2 Q_1-\lambda_2 C_0)e^{\lambda_1 t_1}-(\gamma Q_2-\lambda_1 Q_1-\lambda_1 C_0)e^{\lambda_2 t_1}\\
&+(\lambda_2-\lambda_1)C_0+(\lambda_2-\lambda_1)\sin(\gamma t_1+\delta)]-c\\
&=\Big(-\frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_1 t_2}+\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_2 t_2}\Big)\Big\{\Big(-\frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_1 t_1}+\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_2 t_1}\Big)\\
&\quad\times(e^{\lambda_1 t_1}-e^{\lambda_2 t_1})^{-1}\Big[(\gamma Q_2-\lambda_2 Q_1-\lambda_2 C_0)e^{\lambda_1 t_1}\\
&\quad-(\gamma Q_2-\lambda_1 Q_1-\lambda_1 C_0)e^{\lambda_2 t_1}+(\lambda_2-\lambda_1)C_0+(\lambda_2-\lambda_1)\sin(\gamma t_1+\delta)\Big]\\
&\quad+\frac{\lambda_1}{\lambda_2-\lambda_1}(\gamma Q_2-\lambda_1 Q_1-\lambda_1 C_0)e^{\lambda_1 t_1}-\lambda_1(C_0+Q_1)e^{\lambda_1 t_1}\\
&\quad-\frac{\lambda_2}{\lambda_2-\lambda_1}(\gamma Q_2-\lambda_1 Q_1-\lambda_1 C_0)e^{\lambda_2 t_1}+\gamma\cos(\gamma t_1+\delta)\Big\}\\
&\quad+\frac{\lambda_1}{\lambda_2-\lambda_1}(c+\lambda_1(C_0-Q_1)+\gamma Q_2)e^{\lambda_1 t_2}+\lambda_1(C_0-Q_1)e^{\lambda_1 t_2}\\
&\quad-\frac{\lambda_2}{\lambda_2-\lambda_1}(c+\lambda_1(C_0-Q_1)+\gamma Q_2)e^{\lambda_2 t_2}+\gamma\cos(\gamma t_2+\delta).
\end{aligned} \tag{2.6}
$$

If the system (2.5), (2.6) is solvable for t_1, t_2 satisfying (1.1), then there is a periodic solution with period $T=t_1+t_2$. However it is very difficult to solve system (2.5), (2.6). We suppose that the period of the appearing oscillations T is given, i.e. $t_2=T-t_1$. Next we write out equation (2.5) depending only on t_1 and group the expressions at identical multipliers in it. Then we have

$$
\begin{aligned}
&-\frac{\lambda_1}{\lambda_2-\lambda_1}(\gamma Q_2-\lambda_2(C_0+Q_1))e^{2\lambda_1 t_1}+\Big[\frac{\lambda_2}{\lambda_2-\lambda_1}(\gamma Q_2-\lambda_2(C_0+Q_1))\\
&+\frac{\lambda_1}{\lambda_2-\lambda_1}(\gamma Q_2-\lambda_1(C_0+Q_1))\Big]e^{(\lambda_2+\lambda_1)t_1}-\frac{\lambda_2}{\lambda_2-\lambda_1}(\gamma Q_2-\lambda_1(C_0+Q_1))e^{2\lambda_2 t_1}\\
&-\lambda_1 C_0 e^{\lambda_1 t_1}+\lambda_2 C_0 e^{\lambda_2 t_1}-\lambda_1\cos\delta e^{\lambda_1 t_1}\cos(\gamma t_1)-\lambda_1\sin\delta e^{\lambda_1 t_1}\cos(\gamma t_1)\\
&+\lambda_2\cos\delta e^{\lambda_2 t_1}\sin(\gamma t_1)+\lambda_2\sin\delta e^{\lambda_2 t_1}\cos(\gamma t_1)\\
&-\frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_2 T}(\gamma Q_2-\lambda_2(C_0+Q_1))e^{(\lambda_1-\lambda_2)t_1}\\
&-\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1 T}(\gamma Q_2-\lambda_1(C_0+Q_1))e^{(\lambda_2-\lambda_1)t_1}\\
&+e^{\lambda_1 T}\gamma\cos\delta e^{-\lambda_1 t_1}\cos(\gamma t_1)-e^{\lambda_1 T}\gamma\sin\delta e^{-\lambda_1 t_1}\sin(\gamma t_1)\\
&-(c+\lambda_2(C_0-Q_1)+\gamma Q_2)e^{\lambda_1 T}e^{-\lambda_1 t_1}+(c+\lambda_1(C_0-Q_1)+\gamma Q_2)e^{\lambda_2 T}e^{-\lambda_2 t_1}\\
&-\frac{1}{\lambda_2-\lambda_1}(\sin(\gamma T)\cos\delta+\cos(\gamma T)\sin\delta)\cos(\gamma t_1)\\
&-\frac{1}{\lambda_2-\lambda_1}(\sin(\gamma T)\sin\delta-\cos(\gamma T)\cos\delta)\sin(\gamma t_1)\\
&+\Big\{\frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_1 T}[\gamma Q_2-\lambda_2(C_0+Q_1)]+\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_2 T}[\gamma Q_2-\lambda_1(C_0+Q_1)]\\
&+\frac{1}{\lambda_2-\lambda_1}C_0\Big\}=0.
\end{aligned} \tag{2.7}
$$

Also, we write out equation (2.6) for t_1 and group the terms as above. We obtain

$$
\begin{aligned}
&\Big\{ -c+\gamma Q_2-\lambda_2(C_0+Q_1)-\theta_1(\gamma Q_2-\lambda_2(C_0+Q_1)) \\
&-\frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}e^{\lambda_2 T}(\gamma Q_2-\lambda_1(C_0+Q_1))-\theta_2\frac{\lambda_1}{\lambda_2-\lambda_1}(\gamma Q_2-\lambda_1(C_0+Q_1)) \\
&+\theta_2\lambda_1(C_0+Q_1)-\frac{\lambda_2^2}{(\lambda_2-\lambda_1)^2}e^{\lambda_2 T}(\gamma Q_2-\lambda_1(C_0+Q_1))\Big\}e^{\lambda_1 t_1} \\
&+\Big\{c-\gamma Q_2+\lambda_1(C_0+Q_1)+\theta_1(\gamma Q_2-\lambda_1(C_0+Q_1)) \\
&+\frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}e^{\lambda_1 T}(\gamma Q_2-\lambda_2(C_0+Q_1))+\theta_2\frac{\lambda_2}{\lambda_2-\lambda_1}(\gamma Q_2-\lambda_1(C_0+Q_1)) \\
&+\frac{\lambda_1^2}{\lambda_2-\lambda_1}e^{\lambda_1 T}(C_0+Q_1)-\frac{\lambda_1^2}{(\lambda_2-\lambda_1)^2}e^{\lambda_1 T}(\gamma Q_2-\lambda_1(C_0+Q_1))\Big\}e^{\lambda_2 t_1} \\
&+\big[(\lambda_2-\lambda_1)\cos\delta-\theta_1(\lambda_2-\lambda_1)\cos\delta+\theta_2\gamma\sin\delta\big]\sin(\gamma t_1) \\
&+[(\lambda_2-\lambda_1)\sin\delta-\theta_1(\lambda_2-\lambda_1)\sin\delta-\theta_2\gamma\cos\delta]\cos(\gamma t_1) \\
&+\Big[\frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}e^{\lambda_2 T}(\gamma Q_2-\lambda_2(C_0+Q_1))-\frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}e^{\lambda_2 T}(\gamma Q_2-\lambda_1(C_0+Q_1)) \\
&+\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_2 T}(C_0+Q_1)\Big]e^{(2\lambda_1-\lambda_2)t_1} \\
&+\big[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}C_0e^{\lambda_2 T}+\frac{\lambda_2}{\lambda_2-\lambda_1}(c+\lambda_1(C_0-Q_1)+\gamma Q_2)e^{\lambda_2 T}\big]e^{(\lambda_1-\lambda_2)t_1} \qquad (2.8)\\
&+\Big[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}C_0e^{\lambda_1 T}+\frac{\lambda_1}{\lambda_2-\lambda_1}(c+\lambda_1(C_0-Q_1)+\gamma Q_2)e^{\lambda_1 T} \\
&+\lambda_1(C_0-Q_1)e^{\lambda_1 T}\Big]e^{(\lambda_2-\lambda_1)t_1} \\
&+\big[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\cos\delta e^{\lambda_1 T}+\gamma\sin\delta\frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_1 T}\big]e^{(\lambda_2-\lambda_1)t_1}\sin(\gamma t_1) \\
&+\big[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\sin\delta e^{\lambda_1 T}-\gamma\cos\delta\frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_1 T}\big]e^{(\lambda_2-\lambda_1)t_1}\cos(\gamma t_1) \\
&+\big[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\cos\delta e^{\lambda_2 T}+\gamma\sin\delta\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_2 T}\big]e^{(\lambda_1-\lambda_2)t_1}\sin(\gamma t_1) \\
&+\big[\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\sin\delta e^{\lambda_2 T}-\gamma\cos\delta\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_2 T}\big]e^{(\lambda_1-\lambda_2)t_1}\cos(\gamma t_1) \\
&+\gamma(\cos(\gamma T)\cos\delta-\sin(\gamma T)\sin\delta)e^{\lambda_1 t_1}\cos(\gamma t_1) \\
&-\gamma(\cos(\gamma T)\cos\delta-\sin(\gamma T)\sin\delta)e^{\lambda_2 t_1}\cos(\gamma t_1) \\
&+\gamma(\cos(\gamma T)\sin\delta+\sin(\gamma T)\cos\delta)e^{\lambda_1 t_1}\sin(\gamma t_1) \\
&-\gamma(\cos(\gamma T)\sin\delta+\sin(\gamma T)\cos\delta)e^{\lambda_2 t_1}\sin(\gamma t_1) \\
&+\Big\{(\lambda_2-\lambda_1)C_0-\theta_1(\lambda_2-\lambda_1)C_0-\frac{\lambda_1}{\lambda_2-\lambda_1}[c+\lambda_1(C_0-Q_1)+\gamma Q_2]e^{\lambda_1 T} \\
&-\frac{\lambda_2}{\lambda_2-\lambda_1}[c+\lambda_1(C_0-Q_1)+\gamma Q_2]e^{\lambda_2 T}-\lambda_1(C_0-Q_1)e^{\lambda_1 T}\Big\}=0,
\end{aligned}
$$

where

$$\theta_1 = \frac{\lambda_1^2}{(\lambda_2-\lambda_1)^2}e^{\lambda_1 T} + \frac{\lambda_2^2}{(\lambda_2-\lambda_1)^2}e^{\lambda_2 T}, \quad \theta_2 = -\frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_1 T} - \frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_2 T}.$$

The expressions in braces from (2.7), (2.8), which represent the last terms, are the constants independent of t_1. Obviously, equations (2.7), (2.8) have equal solutions with respect to t_1. Next we find out the conditions under which it is possible. We use the approach based on assumption that the coefficients of equations are equal to each other. We add up the expressions at multipliers $e^{(\lambda_2+\lambda_1)t_1}$, $e^{\lambda_1 t_1}$, $e^{\lambda_2 t_1}$ from (2.7) and set equal to the corresponding multipliers $e^{\lambda_1 t_1}$, $e^{\lambda_2 t_1}$ from (2.8). Then we obtain

$$\begin{aligned}
&-\lambda_1 C_0 + \frac{\lambda_1}{\lambda_2-\lambda_1}(\gamma Q_2 - \lambda_1(C_0+Q_1))e^{\lambda_2 t_1}\\
&= -c + \gamma Q_2 - \lambda_2(C_0+Q_1) - \theta_1(\gamma Q_2 - \lambda_2(C_0+Q_1))\\
&- \frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}e^{\lambda_2 T}(\gamma Q_2 - \lambda_1(C_0+Q_1))\\
&- \theta_2\frac{\lambda_1}{\lambda_2-\lambda_1}(\gamma Q_2 - \lambda_1(C_0+Q_1)) + \theta_2\lambda_1(C_0+Q_1)\\
&- \frac{\lambda_2^2}{(\lambda_2-\lambda_1)^2}e^{\lambda_2 T}(\gamma Q_2 - \lambda_1(C_0+Q_1)),
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\lambda_2 C_0 + \frac{\lambda_2}{\lambda_2-\lambda_1}(\gamma Q_2 - \lambda_2(C_0+Q_1))e^{\lambda_1 t_1}\\
&= c - \gamma Q_2 + \lambda_1(C_0+Q_1) + \theta_1(\gamma Q_2 - \lambda_1(C_0+Q_1))\\
&\quad + \frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}e^{\lambda_1 T}(\gamma Q_2 - \lambda_2(C_0+Q_1))\\
&\quad + \theta_2\frac{\lambda_2}{\lambda_2-\lambda_1}(\gamma Q_2 - \lambda_1(C_0+Q_1)) + \frac{\lambda_1^2}{\lambda_2-\lambda_1}e^{\lambda_1 T}(C_0+Q_1)\\
&\quad - \frac{\lambda_1^2}{(\lambda_2-\lambda_1)^2}e^{\lambda_1 T}(\gamma Q_2 - \lambda_1(C_0+Q_1)),
\end{aligned} \tag{2.10}$$

where $e^{\lambda_2 t_1}$ is given by (2.9). From (2.10), we obtain $e^{\lambda_1 t_1}$. Then we equate constants in (2.7), (2.8). We have the first condition on the parameters of the system

$$\begin{aligned}
&(2\gamma Q_2 + c)(\lambda_1 e^{\lambda_1 T} + \lambda_2 e^{\lambda_2 T}) - 2\lambda_1\lambda_2 Q_1(e^{\lambda_1 T} + e^{\lambda_2 T})\\
&+ C_0(\lambda_1^2 e^{\lambda_1 T} + \lambda_2^2 e^{\lambda_2 T}) - C_0((\lambda_2-\lambda_1)^2 - 1) = 0.
\end{aligned} \tag{2.11}$$

Further, we consider the terms in (2.7), (2.8) without the sine and cosine functions. We set equal the coefficients at multiplier $e^{(\lambda_1-\lambda_2)t_1}$. Then

$$2\lambda_1\lambda_2 Q_1 - \gamma Q_2(\lambda_1+\lambda_2) = \lambda_2(c + \lambda_1 C_0). \tag{2.12}$$

Next we equate the coefficients at multiplier $e^{(\lambda_2-\lambda_1)t_1}$. Then

$$\lambda_1\lambda_2 Q_1 - \gamma Q_2(\lambda_1+\lambda_2) = \lambda_1 c + \lambda_1(\lambda_1+1)(C_0 - Q_1). \tag{2.13}$$

It should add the condition following from equalities of the sum of the rest terms at sine and cosine multipliers from (2.7), (2.8) to conditions (2.11)–(2.13). These conditions are redundant. First, we have proceeded from the necessary conditions for the existence of the periodic solution to the initial system with period T we do given. Secondly, parameters a_1, a_0 (i.e. λ_1, λ_2), and β, γ are the parameters influencing on the choice of c. All these parameters are in conditions (2.12), (2.13), since $Q_1 = Q_1(a_1, a_0, \beta, \gamma)$, $Q_2 = Q_2(a_1, a_0, \beta, \gamma)$. In particular, from (2.12) and (2.13), we have

$$c = \gamma(\lambda_1 + \lambda_2)Q_2\Big(\frac{1}{2} + \frac{1 + \lambda_2 - \lambda_1}{1 + \lambda_1 - \lambda_2}\Big) \tag{2.14}$$

provided that $\lambda_2 - \lambda_1 \neq 1$ or $a_1^2 - 4a_0 \neq 1$. From (2.11), we obtain

$$\begin{aligned} &2\gamma Q_2(\lambda_1 e^{\lambda_1 T} + \lambda_2 e^{\lambda_2 T}) - 2\lambda_1\lambda_2 Q_1(e^{\lambda_1 T} + e^{\lambda_2 T}) \\ &= c\Big\{\frac{1}{2\lambda_1\lambda_2}[(\lambda_2 - \lambda_1)^2 - 1] - \frac{1}{2\lambda_1\lambda_2}(\lambda_1^2 e^{\lambda_1 T} + \lambda_2^2 e^{\lambda_2 T}) - \lambda_1 e^{\lambda_1 T} \\ &\quad - \lambda_2 e^{\lambda_2 T}\Big\}. \end{aligned} \tag{2.15}$$

From (2.15), it is easy to express c with respect to T.

Now we return to (2.9), (2.10). From (2.9), we get

$$\begin{aligned} e^{\lambda_2 t_1} &= \Big[\frac{\lambda_1}{\lambda_2 - \lambda_1}(\gamma Q_2 - \lambda_1(C_0 + Q_1))\Big]^{-1} \\ &\quad \times \Big\{\lambda_1 C_0 - c + \gamma Q_2 - \lambda_2(C_0 + Q_1) \\ &\quad - e^{\lambda_2 T}\frac{\lambda_2}{(\lambda_2 - \lambda_1)^2}[2\lambda_2\gamma Q_2 - (\lambda_2^2 + \lambda_1^2)(C_0 + Q_1)]\Big\}. \end{aligned} \tag{2.16}$$

From (2.10), we have

$$\begin{aligned} e^{\lambda_1 t_1} &= \Big[\frac{\lambda_2}{\lambda_2 - \lambda_1}(\gamma Q_2 - \lambda_2(C_0 + Q_1))\Big]^{-1} \\ &\quad \times \Big\{ - \lambda_2 C_0 + c - \gamma Q_2 + \lambda_1(C_0 + Q_1) \\ &\quad + e^{\lambda_1 T}\frac{\lambda_1^2}{\lambda_2 - \lambda_1}(C_0 + Q_1)\}. \end{aligned} \tag{2.17}$$

Thus the following two conditions must be satisfied

$$\gamma Q_2 - \lambda_1(C_0 + Q_1) \neq 0, \quad \gamma Q_2 - \lambda_2(C_0 + Q_1) \neq 0. \tag{2.18}$$

Since $\lambda_1 \neq \lambda_2$, conditions (2.18) hold and do not turn into equalities if

$$c \neq \frac{2\gamma\beta\lambda_1\lambda_2(2\lambda_2 + \lambda_1) - 2\lambda_2\gamma^3\beta}{\gamma^4 + \gamma^2(\lambda_1^2 + \lambda_2^2) + \lambda_1^2\lambda_2^2}, \quad c \neq \frac{2\gamma\beta\lambda_1\lambda_2(2\lambda_1 + \lambda_2) - 2\lambda_1\gamma^3\beta}{\gamma^4 + \gamma^2(\lambda_1^2 + \lambda_2^2) + \lambda_1^2\lambda_2^2} \tag{2.19}$$

for the first and second conditions of (2.18) respectively. In (2.19), the condition

$$\gamma^4 + \gamma^2(\lambda_1^2 + \lambda_2^2) + \lambda_1^2\lambda_2^2 \neq 0$$

is carried out automatically owing to the conditions imposed on λ_1, λ_2, i.e. for all $\gamma > 0$, equality to zero does not take place. Besides, the right-hand sides of (2.16), (2.17) have to be more than unit if λ_1, λ_2 are positive, and to belong to interval $(0, 1)$ if λ_1, λ_2 are negative.

So, we formulate the conditions guaranteeing non-existence of the periodic solutions to system (1.1) with given period T.

Theorem 2.2. *Let $a_1 \neq 0$, $a_0 \neq 0$, $0 < a_1^2 - 4a_0 \neq 1$, $c > 0$, $\beta \in \mathbb{R}\backslash\{0\}$, $\gamma > 0$, and $\gamma^2 \neq a_0$. Then equation* (1.1) *has no periodic solutions with given period T, where $T = t_1 + t_2$, t_1 is defined by* (2.16), (2.17) *under* (2.19) *if* (2.14), (2.15) *are not fair.*

If the other coefficients of (2.7), (2.8) are equated, we obtain the other equalities for t_1, t_2 and therefore the other ratios linking c with parameters a_1, a_0, β, γ. In the space of parameters c, a_1, a_0, β, γ the set of such ratios allows us to allocate domains for the possible existence of the periodic solutions with given period T and domains for which such solutions do not exist.

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