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PULLBACK ATTRACTORS FOR A CLASS OF SEMILINEAR NONCLASSICAL DIFFUSION EQUATIONS WITH DELAY

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Abstract. In this article, we analyze the existence of solutions for a nonclassical reaction-diffusion equation with critical nonlinearity, a time-dependent force with exponential growth and delayed force term, where the delay term can be entrained by a function under assumptions of measurability. Using a priori estimates we obtain the pullback $\mathcal{D}$-absorbing process and the pullback ω-$\mathcal{D}$-limit compactness that allow us to prove the existence of the pullback $\mathcal{D}$-attractors for the associated process to the problem.

1. Introduction and statement of the problem

The nonclassical diffusion equations occur as models in mechanics, soil mechanics and heat conduction theory (see for example [1, 2, 8, 9, 14]). In recent years, the existence of pullback attractors has been proved for some nonclassical diffusion equations, see for example [16, 19, 21, 22, 23]. Functinal partial differential equations is the subject of intensive studies.

For the functional partial differential equation

$$\begin{gathered}\frac{\partial}{\partial t}u(t,x)-\Delta\frac{\partial}{\partial t}u(t,x)-\Delta u(t,x)=b(t,u(t-\rho(t))(x)+g(t,x)\quad\text{in }(\tau,\infty)\times\Omega\,,\\ u=0\quad\text{on }(\tau,\infty)\times\partial\Omega\,,\\ u(\tau+\theta,x)=\varphi(\theta,x),\quad\tau\in\mathbb{R}\,,\ \theta\in[-r,0],\ x\in\Omega\,,\end{gathered}\tag{1.1}$$

without critical non-linearity, the long-time behavior, and especially the pullback attractors has been studied in [7]. There the author studied the pullback asymptotic behavior of solutions in the phase-spaces $C([-r,0];H_0^1(\Omega))$ and $C([-r,0];H_0^1(\Omega)\cap H^2(\Omega))$.

In [16], the equation without delay

$$\begin{gathered}\frac{\partial}{\partial t}u(t,x)-\varepsilon\Delta\frac{\partial}{\partial t}u(t,x)-\Delta u(t,x)+f(u(t,x))=g(t,x)\quad\text{in }(\tau,\infty)\times\Omega\\ u=0\quad\text{on }(\tau,\infty)\times\partial\Omega\\ u(\tau,x)=u^0(x),\quad\tau\in\mathbb{R},\ x\in\Omega\,,\end{gathered}\tag{1.2}$$

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in the phase-space $H_0^1(\Omega)$ is treated. It is proved that the existence of a pullback attractor where the non-linearity f has a critical exponent in the interval $\big(0, \min\{\frac{N+2}{N-2}, 2+\frac{4}{N}\}\big)$ with $N \geq 3$. On unbounded domain, in [23], the existence of pullback attractor to the solutions in $H^1(\mathbb{R}^N)$ of the following equation without delay

$$
\begin{gathered}
\frac{\partial}{\partial t}u(t,x) - \Delta\frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) + u(t,x) + f(u(t,x)) = g(t,x) \\
\text{in } (\tau,\infty)\times\mathbb{R}^N\,, \\
u(\tau,x) = u_\tau(x), \quad \tau\in\mathbb{R},\ x\in\mathbb{R}^N\,,
\end{gathered} \tag{1.3}
$$

is treated, where the nonlinearity has a critical exponent $p \leq \frac{2}{N-2}$ for $N \geq 3$.

In this article, we consider the functional partial differential equation

$$
\frac{\partial}{\partial t}u(t,x) - \Delta\frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) + f(u(t,x)) = b(t,u_t)(x) + g(t,x) \quad \text{in } (\tau,\infty)\times\Omega\,,
$$

with with the boundary and initial conditions

$$
\begin{gathered}
u = 0 \quad \text{on } (\tau,\infty)\times\partial\Omega\,, \\
u(\tau,x) = u^0(x), \quad \tau\in\mathbb{R},\ x\in\Omega\,, \\
u(\tau+\theta,x) = \varphi(\theta,x),\ \theta\in(-r,0),\ x\in\Omega\,,
\end{gathered} \tag{1.4}
$$

where $\Omega\subset\mathbb{R}^N$ ($N\geq 3$) is a bounded domain with smooth boundary $\partial\Omega$. The equation (1.4) without the term $\Delta\frac{\partial u}{\partial t}$, is a classical equation with delay. Many works have dealed with such equation, see for example [4, 5, 10, 11, 15, 17, 18]. It has been treated in different phase-spaces and the delay term is driven by a function under measurability condition and the nonlinearity is given by different assumptions. For more details on differential equations with delay we refer the reader to [6] and [20].

It is well known that the compact Sobolev embedding can be applied to obtain the existence of pullback $\mathcal{D}$-attractor as well as the higher regularity of the solution of the equation, e.g., although the initial conditions only belong to a weaker topological space, the solution will belong to a stronger topological space with higher regularity. The equation (1.4) contains the term $\Delta\frac{\partial u}{\partial t}$, this involves that the solution has no higher regularity and so the compact Sobolev embedding can not be applied to obtain the existence of a pullback $\mathcal{D}$-attractor. This is similar to the hyperbolic case.

In this article, we prove the existence of a pullback $\mathcal{D}$-attractor. It is well known that for the existence of pullback $\mathcal{D}$-attractors, the key point is to find a bounded family of pullback $\mathcal{D}$-absorbing sets then the pullback w-$\mathcal{D}$-limit compactness for the process corresponding to the solution of our problem. As noticed before, because of the term $\Delta\frac{\partial u}{\partial t}$, the pullback w-$\mathcal{D}$-limit compactness for the process can not be proved by the compact Sobolev embedding. The nonlinearity with critical exponent makes also some barriers. To overcome these difficulties, we apply the decomposition techniques and a method used in [17] to satisfy the pullback w-$\mathcal{D}$-limit compactness of the process with delay. It is based on the concept of the Kuratowski measure of noncompactness of a bounded set as well as some new estimates of the equicontinuity of the solutions.

This article is organized as follows. In section 2 useful results on nonautonomous dynamical systems and pullback $\mathcal{D}$-attractor theory are recalled. In section 3 deals with the main results; we will prove the existence of the solutions using the Faedo-Galerkin approximations; also, the uniqueness and the continuous dependence of the solutions with respect to the initial conditions are proved. Then we prove the existence of the pullback $\mathcal{D}$-attractor.

2. Preliminaries

At first, we give some notation which will be used throughout this paper. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$, the norm and the inner product in $L^2(\Omega)$ are denoted by $\|\cdot\|$ and $\langle \cdot,\cdot\rangle$, respectively, and we denote by $\|\nabla\|$ and $\langle \nabla\cdot, \nabla\cdot\rangle$ the norm and the inner product of $H_0^1(\Omega)$, respectively. The norm in the Banach space Y will be denoted by $\|\cdot\|_Y$. Let c be an arbitrary positive constant, which may be different from line to line and even in the same line.

To study problem (1.4), we need some assumptions: The nonlinear function $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies

$$f(u)u \geq -c_1u^2 - c_2, \tag{2.1}$$

$$f'(u) \geq -c_3,\ f(0) = 0, \tag{2.2}$$

$$|f(u)| \leq k(1 + |u|^\alpha), \tag{2.3}$$

$$\liminf_{|u|\to\infty} \frac{uf(u) - c_4F(u)}{u^2} \geq 0, \tag{2.4}$$

$$\liminf_{|u|\to\infty} \frac{F(u)}{u^2} \geq 0\,, \tag{2.5}$$

where $0 < \alpha < \min\{\frac{N+2}{N-2}, 2+\frac{4}{N}\}$ (is called a critical exponent since the nonlinearity f is not compact in this case i.e. for a bounded subset $B \subset H_0^1(\Omega)$, in general, $f(B)$ is not precompact in $L^q(\Omega)$ where $q = \frac{2N+4}{\alpha N}$), and c_1, c_2, c_3, c_4, k are positive constants, $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in Ω with the homogeneous Dirichlet condition such that $\lambda_1 > \max\{c_1, c_3\}$, and $F(u) = \int_0^u f(s)ds$. We infer from (2.4) and (2.5) that for any $\delta > 0$ there exist positive constants c_δ, c_δ' such that

$$uf(u) - c_4F(u) + \delta u^2 + c_\delta' \geq 0\,, \quad \forall u \in \mathbb{R}\,, \tag{2.6}$$

$$F(u) + \delta u^2 + c_\delta' \geq 0\,, \quad \forall u \in \mathbb{R}\,. \tag{2.7}$$

The operator $b : \mathbb{R} \times L^2((-r, 0); L^2(\Omega)) \to L^2(\Omega)$ is a time-dependent external force with delay; it satisfies:

- (I) For all $\phi \in L^2((-r, 0); L^2(\Omega))$, the function $\mathbb{R} \ni t \mapsto b(t, \phi) \in L^2(\Omega)$ is measurable,
- (II) $b(t, 0) = 0$ for all $t \in \mathbb{R}$;
- (III) there exists $L_b > 0$ such that for all $t \in \mathbb{R}$ and $\phi_1, \phi_2 \in L^2((-r, 0); L^2(\Omega))$,

$$\|b(t, \phi_1) - b(t, \phi_2)\| \leq L_b\|\phi_1 - \phi_2\|_{L^2((-r,0);L^2(\Omega))}\,; \tag{2.8}$$

- (IV) there exists $C_b > 0$ such that for all $t \geq \tau$, and all $u, v \in L^2([\tau-r, t]; L^2(\Omega))$,

$$\int_\tau^t \|b(s, u_s) - b(s, v_s)\|^2 ds \leq C_b \int_{\tau-r}^t \|u(s) - v(s)\|^2 ds\,. \tag{2.9}$$

Remark 2.1. From (I)–(III), for $T > \tau$ the function $\mathbb{R} \ni t \mapsto b(t, \phi) \in L^2(\Omega)$ is measurable and belongs to $L^\infty((\tau, T); L^2(\Omega))$.

The function $g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$ is an another nondelayed time-dependent external force, $u^0 \in H^1_0(\Omega)$ is the initial condition in τ and $\varphi \in L^2((-r,0); L^2(\Omega))$ is also the initial condition in $(\tau - r, \tau)$, $r > 0$ is the length of the delay effect.

In this section, we recall some basic concepts about the pullback attractors and some abstract results about the existence of pullback attractors. Let (Y,d) be a complete metric space. Let us denote $\mathcal{P}(Y)$ the family of all nonempty subsets of Y, and suppose $\mathcal{D}$ is a nonempty class of parameterized sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(Y)$.

Definition 2.2 ([11])**.** A two parameter family of mappings $U(t,\tau) : Y \to Y$, $t \geq \tau$, $\tau \in \mathbb{R}$, is called to be a norm-to-weak continuous process if

(1) $U(\tau,\tau)x = x$ for all $\tau \in \mathbb{R}$, $x \in Y$;
(2) $U(t,s)U(s,\tau)x = U(t,\tau)x$ for all $t \geq s \geq \tau$, $\tau \in \mathbb{R}$, $x \in Y$;
(3) $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$ if $x_n \to x$ in Y.

The following result is useful for satisfying that a process is norm-to-weak continuous.

Proposition 2.3 ([11])**.** *Let Y, Z be two Banach spaces and Y^*, Z^* be their dual spaces. Assume that Y is dense in Z, the injection $i : Y \to Z$ is continuous and its adjoint $i^* : Z^* \to Y^*$ is dense, and $\{U(t,\tau)\}$ is a continuous or weak continuous process on Z. Then $\{U(t,\tau)\}$ is norm-to-weak continuous on Y if and only if for $t \geq \tau$, $\tau \in \mathbb{R}$, $U(t,\tau)$ maps a compact set of Y to a bounded set of Y.*

Definition 2.4 ([12])**.** A family of bounded sets $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is called pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}$ if for any $t \in \mathbb{R}$ and for any $\widehat{D} \in \mathcal{D}$, there exists $\tau_0(t,\widehat{D}) \leq t$ such that

$$U(t,\tau)D(\tau) \subset B(t) \quad \text{for all } \tau \leq \tau_0(t,\widehat{D})\,.$$

Definition 2.5 ([12])**.** A process $\{U(t,\tau)\}$ is called pullback w-$\mathcal{D}$-limit compact if for all $\varepsilon > 0$ and $\widehat{D} \in \mathcal{D}$, there exists $\tau_0(t,\widehat{D}) \leq t$ such that

$$\mathcal{K}\Big(\cup_{\tau \leq \tau_0} U(t,\tau)D(\tau)\Big) \leq \varepsilon\,,$$

where $\mathcal{K}$ is the Kuratowski measure of noncompactness of $B \in \mathcal{P}(Y)$. This measure is defined as

$$\mathcal{K}(B) = \inf\{\delta > 0 : B \text{ has a finite open cover of sets of diameter less than } \delta,$$

and has the following properties.

Lemma 2.6 ([12])**.** *Let B, B_0, B_1 be bounded subsets of Y. Then*

(1) $\mathcal{K}(B) = 0 \Longleftrightarrow \mathcal{K}(N(B,\varepsilon)) \leq 2\epsilon \Longleftrightarrow \overline{B}$ *is compact;*
(2) $\mathcal{K}(B_0 + B_1) \leq \mathcal{K}(B_0) + \mathcal{K}(B_1)$*;*
(3) $\mathcal{K}(B_0) \leq \mathcal{K}(B_1)$ *whenever* $B_0 \subset B_1$*;*
(4) $\mathcal{K}(B_0 \cup B_1) \leq \max\{\mathcal{K}(B_0)\,,\, \mathcal{K}(B_1)\}$*;*
(5) $\mathcal{K}(B) = \mathcal{K}(\overline{B})$*;*
(6) *if B is a ball of radius ε then $\mathcal{K}(B) \leq 2\varepsilon$.*

Definition 2.7 ([12])**.** A family $\widehat{A} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{P}(Y)$ is said to be a pullback $\mathcal{D}$-attractor for $\{U(t,\tau)\}$ if

(1) $A(t)$ is compact for all $t \in \mathbb{R}$;
(2) $\widehat{A}$ is invariant; i.e., $U(t,\tau)A(\tau) = A(t)$, for all $t \geq \tau$;

(3) $\widehat{A}$ is pullback $\mathcal{D}$-attracting ; i.e.,
$$\lim_{\tau\to-\infty} dist(U(t,\tau)D(\tau), A(t)) = 0\,,$$
for all $\widehat{D} \in \mathcal{D}$ and all $t \in \mathbb{R}$;

(4) If $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets then $A(t) \subset C(t)$, for all $t \in \mathbb{R}$.

Theorem 2.8 ([12]). *Let $\{U(t,\tau)\}$ be a norm-to-weak continuous process such that $\{U(t,\tau)\}$ is pullback w-$\mathcal{D}$-limit compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\widehat{B} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ for the process $\{U(t,\tau)\}$, then there exists a pullback $\mathcal{D}$-attractor $\{A(t) : t \in \mathbb{R}\}$ such that*
$$A(t) = w(\widehat{B}, t) = \cap_{s\le t}\overline{\cup_{\tau\le s} U(t,\tau)B(\tau)}\,.$$

3. Existence of pullback D-attractors

3.1. **Existence and uniqueness of weak solutions.** First, we define the concept of weak solution.

Definition 3.1. A function $u \in L^2((\tau - r, T); L^2(\Omega))$ is called a weak solution of (1.4) if for all $T > \tau$ we have
$$u \in C([\tau, T]; H_0^1(\Omega)) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L^2((\tau, T); H_0^1(\Omega))\,,$$
with $u(t) = \varphi(t-\tau)$ for $t \in [\tau - r, \tau]$, and for all test functions $v \in C^1([\tau, T]; H_0^1(\Omega))$ such that $v(T) = 0$, it satisfies
$$\begin{aligned}
&\int_\tau^T -\langle u, v'\rangle + \int_\tau^T\int_\Omega \nabla\frac{\partial u}{\partial t}\nabla v + \int_\tau^T\int_\Omega \nabla u\nabla v + \int_\tau^T\int_\Omega f(u)v \\
&= \int_\tau^T \langle b(t,u_t), v\rangle + \int_\tau^T\int_\Omega gv + \langle u^0, v(\tau)\rangle\,.
\end{aligned} \tag{3.1}$$

Next we have the existence and uniqueness of solutions which are obtained by the usual Faedo-Galerkin approximation and a compactness method.

Theorem 3.2. *For any $\tau \in \mathbb{R}$, $T > \tau$, $u^0 \in H_0^1(\Omega)$, $\varphi \in L^2((-r, 0); L^2(\Omega))$ and if there exist positive constants $\eta, \eta' < 1/2$ such that $\lambda_1 > c_1 + \frac{\eta+\eta'}{2} + \frac{C_b}{2\eta}$, then problem (1.4) has a unique weak solution u on (τ, T).*

Proof. Let $\{e_k\}_{k\ge1}$, be the complete basis of $H_0^1(\Omega)\cap H^2(\Omega)$ given by the orthonormal eigenfunctions of $-\Delta$ in $L^2(\Omega)$. We consider
$$u^m(t) = \sum_{k=1}^m \gamma_{k,m}(t)e_k\,, \quad m \in \mathbb{N}$$
which is the approximate solution of Faedo-Galerkin of order m; that is,
$$\begin{gathered}
\frac{du^m}{dt} - \Delta\frac{\partial u^m}{\partial t} - \Delta u^m + P_m f(u^m) = P_m b(t, u_t^m) + P_m g \\
u^m(\tau) = P_m u^0 = u^0 \quad \text{i.e. } P_m u^m(\tau) \to u^0 \text{ in } H_0^1(\Omega) \\
u^m(\tau+\theta) = P_m\varphi(\theta) = \varphi(\theta)\ \forall\theta \in (-r, 0)
\end{gathered} \tag{3.2}$$
for all $k \in \mathbb{N}$, where $\gamma_{k,m}(t) = \langle u^m(t), e_k\rangle$ denote the Fourier coefficients such that $\gamma_{m,k} \in C^1((\tau, T); \mathbb{R}) \cap L^2((\tau - r, T), \mathbb{R})\,, \gamma'_{k,m}(t)$ is absolutely continuous, and

$P_m u(t) = \sum_{k=1}^{m} \langle u, e_k \rangle e_k$ is the orthogonal projection of $u \in L^2(\Omega)$ or $u \in H_0^1(\Omega)$ in $H_m = span\{e_1, \dots, e_m\}$.

It is well-known that the above finite-dimensional delayed system is well-posed at least locally (see for example [6, Theorem 2.1, p. 14]). Indeed; for fixed m, the system (3.2) defines a linear system of differential equations on $\mathbb{R}^m$. Then we can apply differential equations theory for local existence and uniqueness of solutions to the system (3.2), i.e. for initial conditions $(\varphi, v^m(\tau))$ in $L^2((-r,0);\mathbb{R}^m) \times \mathbb{R}^m$, there exist $t_m > 0$ and a unique solution of (3.2) $v^m(t) = (\gamma_{1,m}(t) \dots \gamma_{m,m}(t))^T$ with $v^m \in L^2((\tau - r, \tau);\mathbb{R}^m)$ such that $v^m|_{[\tau-r,\tau]} = \varphi$ and $v^m(\tau) = a$, and $v^m|_{[\tau,t_m]} \in C^1([\tau, t_m];\mathbb{R}^m)$. Hence, the solution of (3.2) is defined on the interval $[\tau, t_m]$ with $\tau < t_m < T$. The a priori estimates for the Faedo-Galerkin approximate solutions that we obtain will show that $t_m = T$.

Claim 3.3. *$\{u^m\}$ is bounded in $L^\infty((\tau,T);H_0^1(\Omega))$.*

Multiplying (3.2) by u^m and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2) + \|\nabla u^m(t)\|^2 + \int_\Omega f(u^m)u^m = \int_\Omega (b(t, u_t^m)u^m + gu^m)\,.$$

Using (2.1) and the Cauchy inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2) + \|\nabla u^m(t)\|^2 - c_1\|u^m(t)\|^2 - c_2|\Omega| \\ &\le \frac{1}{2\eta}\|b(t, u_t^m)\|^2 + \frac{\eta}{2}\|u^m(t)\|^2 + \frac{1}{2\eta'}\|g(t)\|^2 + \frac{\eta'}{2}\|u^m(t)\|^2\,. \end{aligned}$$

As

$$\lambda_1\|u\|^2 \le \|\nabla u\|^2\,, \tag{3.3}$$

then

$$\begin{aligned} &\frac{d}{dt}(\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2) + (2\lambda_1 - 2c_1 - \eta - \eta')\|u^m(t)\|^2 \\ &\le \frac{1}{\eta}\|b(t, u_t^m)\|^2 + \frac{1}{\eta'}\|g(t)\|^2 + c_2|\Omega|\,. \end{aligned}$$

Integrating this estimate over $[\tau, t]$, $t \le T$, we find that

$$\begin{aligned} &\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2 + (2\lambda_1 - 2c_1 - \eta - \eta')\int_\tau^t \|u^m(s)\|^2 \\ &\le \|u^m(\tau)\|^2 ds + \|\nabla u^m(\tau)\|^2 + \frac{1}{\eta}\int_\tau^t \|b(s, u_s^m)\|^2 ds + \frac{1}{\eta'}\int_\tau^t \|g(s)\|^2 ds \\ &\quad + c_2|\Omega|(t - \tau)\,. \end{aligned}$$

Therefore, by using (2.9), we obtain

$$\begin{aligned} &\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2 + (2\lambda_1 - 2c_1 - \eta - \eta')\int_\tau^t \|u^m(s)\|^2 ds \\ &\le \|u^m(\tau)\|^2 + \|\nabla u^m(\tau)\|^2 + \frac{C_b}{\eta}\int_{\tau-r}^\tau \|u^m(s)\|^2 ds \\ &\quad + \frac{C_b}{\eta}\int_\tau^t \|u^m(s)\|^2 ds + \frac{1}{\eta'}\int_\tau^t \|g(s)\|^2 ds + c_2|\Omega|(t - \tau)\,. \end{aligned}$$

So, one has

$$
\begin{aligned}
&\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2 + \big(2\lambda_1 - 2c_1 - \eta - \eta' - \frac{C_b}{\eta}\big) \int_\tau^t \|u^m(s)\|^2 \\
&\le \|u^m(\tau)\|^2 + \|\nabla u^m(\tau)\|^2 + \frac{C_b}{\eta} \int_{\tau-r}^{\tau} \|u^m(s)\|^2 ds \\
&\quad + \frac{1}{\eta'} \int_\tau^t \|g(s)\|^2 ds + c_2|\Omega|(t-\tau)\,.
\end{aligned} \tag{3.4}
$$

Hence, when $2\lambda_1 - 2c_1 - \eta - \eta' - \frac{C_b}{\eta} > 0$ and $g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$, one gets

$$
\begin{aligned}
\|\nabla u^m(t)\|^2 &\le \|u^m(\tau)\|^2 + \|\nabla u^m(\tau)\|^2 + \frac{C_b}{\eta}\|\varphi\|^2_{L^2((-r,0);L^2(\Omega))} \\
&\quad + \frac{1}{\eta'}\|g\|^2_{L^2([\tau,t];L^2(\Omega))} + c_2|\Omega|(t-\tau)\,.
\end{aligned} \tag{3.5}
$$

By this estimate, for all $T > \tau$, we arrive at

$$
\{u^m\} \text{ is bounded in } L^\infty((\tau,T); H^1_0(\Omega)) \tag{3.6}
$$

Then we deduce that the local solution u^m can be extended to the interval $[\tau, T]$.

Claim 3.4. *$\big\{\frac{\partial}{\partial t}u^m\big\}$ is bounded in $L^2((\tau,T); H^1_0(\Omega))$.*

Multiplying (3.2) by $\frac{\partial u^m}{\partial t}$ and integrating over Ω, we have

$$
\begin{aligned}
&\|\frac{d}{dt}u^m(t)\|^2 + \|\nabla\frac{d}{dt}u^m(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^m(t)\|^2 + \int_\Omega f(u^m)\frac{\partial u^m}{\partial t} \\
&= \int_\Omega b(t,u^m_t)\frac{\partial u^m}{\partial t} + \int_\Omega g\frac{\partial u^m}{\partial t}\,.
\end{aligned} \tag{3.7}
$$

As

$$
\frac{d}{dt}\int_\Omega F(u) = \int_\Omega f(u)\frac{\partial u}{\partial t}\,,
$$

(3.7) becomes

$$
\begin{aligned}
&\|\frac{d}{dt}u^m(t)\|^2 + \|\nabla\frac{d}{dt}u^m(t)\|^2 + \frac{1}{2}\frac{d}{dt}\Big(\|\nabla u^m(t)\|^2 + 2\int_\Omega F(u^m)\Big) \\
&= \int_\Omega b(t,u^m_t)\frac{\partial u^m}{\partial t} + \int_\Omega g\frac{\partial u^m}{\partial t}\,.
\end{aligned}
$$

Using the Young inequality, we find

$$
\|\nabla\frac{d}{dt}u^m(t)\|^2 + \frac{1}{2}\frac{d}{dt}(\|\nabla u^m(t)\|^2 + 2\int_\Omega F(u^m)) \le \frac{1}{2}\|b(t,u^m_t)\|^2 + \frac{1}{2}\|g(t)\|^2\,.
$$

Integrating over $[\tau, t]$, $t \le T$ we obtain

$$
\begin{aligned}
&\int_\tau^t \|\nabla\frac{d}{ds}u^m(s)\|^2 ds + \frac{1}{2}\Big(\|\nabla u^m(t)\|^2 + 2\int_\Omega F(u^m(t,x))\Big) \\
&\le \frac{1}{2}\Big(\|\nabla u^m(\tau)\|^2 + 2\int_\Omega F(u^m(\tau,x))\Big) + \frac{1}{2}\int_\tau^t \|b(s,u^m_s)\|^2 ds \\
&\quad + \frac{1}{2}\int_\tau^t \|g(s)\|^2 ds\,.
\end{aligned}
$$

By (2.9), we have

$$\begin{aligned}
&\int_\tau^t \|\nabla \frac{d}{ds} u^m(s)\|^2 ds + \frac{1}{2}\Big(\|\nabla u^m(t)\|^2 + 2\int_\Omega F(u^m(t,x))\Big) \\
&\le \frac{1}{2}(\|\nabla u^m(\tau)\|^2 + 2\int_\Omega F(u^m(\tau,x))) + \frac{C_b}{2}\int_{\tau-r}^t \|u^m(s)\|^2 ds + \frac{1}{2}\int_\tau^t \|g(s)\|^2 ds \\
&\le \frac{1}{2}\Big(\|\nabla u^m(\tau)\|^2 + 2\int_\Omega F(u^m(\tau,x))\Big) + \frac{C_b}{2}\int_\tau^t \|u^m(s)\|^2 ds \\
&\quad + \frac{C_b}{2}\int_{\tau-r}^\tau \|u^m(s)\|^2 + \frac{1}{2}\|g\|^2_{L^2([\tau,t];L^2(\Omega))}\,.
\end{aligned}$$

From this estimate and (3.4), we deduce that, for all $T > \tau$,

$$\{\frac{\partial}{\partial t} u^m\} \text{ is bounded in } L^2((\tau,T); H_0^1(\Omega))\,. \tag{3.8}$$

Lemma 3.5 ([16, Lemma 3.1])**.** *If $\{u^m\}$ is bounded in $L^\infty((\tau,T), H_0^1(\Omega))$, then*

$$\{f(u^m)\} \text{ is bounded in } L^q((\tau,T); L^q(\Omega))\,, \tag{3.9}$$

where $q = (2N+4)/(\alpha N)$.

By (3.6), (3.8), (3.9), hypothesis (IV) and remark (2.1), we can extract a subsequence (relabeled the same) such that

$$u^m \rightharpoonup u \quad \text{weakly* in } L^\infty((\tau,T); H_0^1(\Omega)), \tag{3.10}$$
$$\Delta u^m \rightharpoonup \Delta u \quad \text{weakly in } L^2((\tau,T); H^{-1}(\Omega)), \tag{3.11}$$
$$\frac{\partial u^m}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly in } L^2((\tau,T); H_0^1(\Omega)), \tag{3.12}$$
$$\Delta(\frac{\partial u^m}{\partial t}) \rightharpoonup \Delta(\frac{\partial u}{\partial t}) \quad \text{weakly in } L^2((\tau,T); H^{-1}(\Omega)), \tag{3.13}$$
$$f(u^m) \rightharpoonup \sigma' \quad \text{weakly in } L^q((\tau,T); L^q(\Omega)), \tag{3.14}$$
$$b(.,u^m_.) \to b(.,u_.) \quad \text{strongly in } L^2((\tau,T); L^2(\Omega))\,. \tag{3.15}$$

By (3.10), we have that $u^m \rightharpoonup u$ weakly in $L^2((\tau,T); L^2(\Omega))$ ($L^\infty((\tau,T); H_0^1(\Omega)) \subset L^2((\tau,T); L^2(\Omega))$), and by (3.12), we have $\frac{\partial u^m}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ weakly in $L^2((\tau,T); L^2(\Omega))$ ($L^2((\tau,T); H_0^1(\Omega)) \subset L^2((\tau,T); L^2(\Omega))$). So, we can extract a subsequence u of u^m that satisfies

$$u^m \to u \quad \text{strongly in } L^2((\tau,T); L^2(\Omega))\,. \quad \text{Thus } u^m \to u \text{ a.e } [\tau,T]\times\Omega\,. \tag{3.16}$$

By (3.16) and the fact that f is continuous, we deduce that $f(u^m) \to f(u)$ a.e $[\tau,T]\times\Omega$, So, from (3.14) and [13, Lemma 1.3, p. 12] we can identify σ' with $f(u)$.

Now, we have to prove that $u(\tau) = u^0$. Recall that (3.1),

$$\begin{aligned}
&\int_\tau^T -\langle u, v'\rangle + \int_\tau^T\int_\Omega \nabla\frac{\partial u}{\partial t}\nabla v + \int_\tau^T\int_\Omega \nabla u\nabla v + \int_\tau^T\int_\Omega f(u)v \\
&= \int_\tau^T \langle b(t,u_t), v\rangle + \int_\tau^T\int_\Omega gv + \langle u(\tau), v(\tau)\rangle\,.
\end{aligned} \tag{3.17}$$

In a similar way, from the Faedo-Galerkin approximations, we have

$$\begin{aligned}
&\int_\tau^T -\langle u^m, v'\rangle + \int_\tau^T \int_\Omega \nabla \frac{\partial u^m}{\partial t}\nabla v + \int_\tau^T \int_\Omega \nabla u^m \nabla v + \int_\tau^T \int_\Omega f(u^m)v \\
&= \int_\tau^T \langle b(t,u_t^m), v\rangle + \int_\tau^T \int_\Omega gv + \langle u^m(\tau), v(\tau)\rangle\,.
\end{aligned} \tag{3.18}$$

Using the fact that $u^m(\tau)\to u^0$ in $H_0^1(\Omega)$ and (3.10)-(3.15) we find that

$$\begin{aligned}
&\int_\tau^T -\langle u, v'\rangle + \int_\tau^T \int_\Omega \nabla \frac{\partial u}{\partial t}\nabla v + \int_\tau^T \int_\Omega \nabla u \nabla v + \int_\tau^T \int_\Omega f(u)v \\
&= \int_\tau^T \langle b(t,u_t), v\rangle + \int_\tau^T \int_\Omega gv + \langle u^0, v(\tau)\rangle\,.
\end{aligned} \tag{3.19}$$

Since $v(\tau)$ is arbitrarily, comparing (3.17) and (3.19) we deduce that $u(\tau)=u^0$.

To prove that $u\in C([\tau,T];H_0^1(\Omega))$, we put $w^m=u^m-u$ then we have

$$\frac{\partial}{\partial t}w^m - \Delta\frac{\partial}{\partial t}w^m - \Delta w^m + f(u^m) - f(u) = b(t,u_t^m) - b(t,u_t)\,.$$

Multiplying this equation by w^m and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|w^m(t)\|^2 + \|\nabla w^m(t)\|^2) + 2\|\nabla w^m(t)\|^2 + 2\int_\Omega (f(u^m)-f(u))w^m \\
&= 2\int_\Omega (b(t,u_t^m)-b(t,u_t))(u^m-u)\,.
\end{aligned}$$

By (2.2), (2.8) and Young's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|w^m(t)\|^2 + \|\nabla w^m(t)\|^2) + 2\|\nabla w^m(t)\|^2 \\
&\le (2c_3+L_b)\|w^m(t)\|^2 + L_b\|w_t^m\|^2_{L^2((-r,0);L^2(\Omega))}\,.
\end{aligned}$$

Hence, by (3.3), one gets

$$\begin{aligned}
\frac{d}{dt}(\|w^m(t)\|^2 + \|\nabla w^m(t)\|^2) &\le 2c_3\|w^m(t)\|^2 + L_b\int_{-r}^0 \|w^m(t+\theta)\|^2 d\theta \\
&\le \frac{2c_3+L_b}{\lambda_1}\|\nabla w^m(t)\|^2 + 2L_b\int_{-r}^0 \|w^m(t+\theta)\|^2 d\theta\,.
\end{aligned}$$

Integrating over $[\tau,t]$, we obtain

$$\begin{aligned}
&\|w^m(t)\|^2 + \|\nabla w^m(t)\|^2 - \|w^m(\tau)\|^2 + \|\nabla w^m(\tau)\|^2 \\
&\le \frac{2c_3+L_b}{\lambda_1}\int_\tau^t \|\nabla w^m(s)\|^2 + L_b\int_\tau^t\int_{-r}^0 \|w^m(s+\theta)\|^2 d\theta ds \\
&\le \frac{2c_3+L_b}{\lambda_1}\int_\tau^t \|\nabla w^m(s)\|^2 + L_b\int_{-r}^0\int_{\tau-r}^t \|w^m(s)\|^2 ds d\theta \\
&\le \frac{2c_3+L_b}{\lambda_1}\int_\tau^t \|\nabla w^m(s)\|^2 + L_b r\int_{\tau-r}^\tau \|w^m(s)\|^2 ds + L_b r\int_\tau^t \|w^m(s)\|^2 ds\,.
\end{aligned}$$

Taking $\beta > \max\left\{\frac{2c_3+L_b}{\lambda_1}, L_b r\right\}$, we obtain

$$\|w^m(t)\|^2 + \|\nabla w^m(t)\|^2 \le \|w^m(\tau)\|^2 + \|\nabla w^m(\tau)\|^2$$

$$+ L_b r\int_{\tau-r}^{\tau}\|w^m(s)\|^2ds + \beta\int_{\tau}^{t}(\|\nabla w^m(s)\|^2 + \|w^m(s)\|^2)ds\,.$$

Applying the Gronwall lemma to this estimate, we obtain

$$\begin{aligned}&\|w^m(t)\|^2 + \|\nabla w^m(t)\|^2\\ &\le \Big(\|w^m(\tau)\|^2 + \|\nabla w^m(\tau)\|^2 + L_b r\int_{-r}^{0}\|w^m(\tau+\theta)\|^2d\theta\Big)e^{\beta(t-\tau)}\,.\end{aligned} \tag{3.20}$$

Since $u^m(\tau)\to u^0$ and $u^m(\tau+\theta)\to\varphi(\theta)$, the estimate (3.20) shows that $u^m\to u$ uniformly in $C([\tau,T];H_0^1(\Omega))$.

By concatenation of solutions, it is clear that we obtain at least one global weak solution to (1.4) defined on $(\tau,+\infty)$.

Finally, we prove the uniqueness and continuous dependence of the solution on the data. To do this, we consider u^1,u^2 two solutions of (1.4) with the same initial conditions u^0 and φ. Let $w=u^1-u^2$, and similarly as in the proof of (3.20), we have

$$\begin{aligned}&\|w(t)\|^2 + \|\nabla w(t)\|^2\\ &\le \Big(\|w(\tau)\|^2 + \|\nabla w(\tau)\|^2 + 2L_b r\int_{-r}^{0}\|w(\tau+\theta)\|^2d\theta\Big)e^{\beta(t-\tau)}\,,\end{aligned} \tag{3.21}$$

and this completes the proof of the theorem because $w(\tau)=0$, and $w(\tau+\theta)=0$. ☐

3.2. **Pullback $\boldsymbol{D}$-attractors.** Invoking Theorem 3.2, we will apply the above results in the phase space $X := H_0^1(\Omega)\times L^2((-r,0);L^2(\Omega))$, which is a Hilbert space with the norm

$$\|(u^0,\varphi)\|_X^2 = \|\nabla u^0\|^2 + \int_{-r}^{0}\|\varphi(\theta)\|^2d\theta\,,$$

with a pair (u^0,φ) of X. First, we give the following consequence the theorem of the existence and uniqueness.

Proposition 3.6. *We consider $g\in L^2_{\rm loc}(\mathbb{R};L^2(\Omega))$, $b:\mathbb{R}\times L^2((-r,0);L^2(\Omega))\to L^2(\Omega)$ with the hypotheses (I)–(IV) and $f\in C^1(\mathbb{R};\mathbb{R})$ satisfying* (2.1)–(2.5)*. Then the family of mappings $U(t,\tau):X\to X$,*

$$(u^0,\varphi)\longmapsto U(t,\tau)(u^0,\varphi) = (u(t),u_t)\,, \tag{3.22}$$

with $(t,\tau)\in\mathbb{R}^2$ and u the weak solution to (1.4)*, defines a continuous process.*

Next, we need to consider the Hilbert space

$$X_1 = H_0^1(\Omega)\times L^2((-r,0);H_0^1(\Omega))\,,$$

with the norm

$$\|(u^0,\varphi)\|_{X_1}^2 = \|\nabla u^0\|^2 + \int_{-r}^{0}\|\nabla\varphi(\theta)\|^2d\theta\,.$$

We remark that when $t-\tau\ge r$, $U(t,\tau)$ maps X to X_1. To prove a pullback -absorbing set for the process $U(t,\tau)$, we need the following lemma.

Lemma 3.7. *Assume that f satisfies* (2.3)*,* (2.6) *and* (2.7)*. For all $u,v\in L^2((\tau-r,t);L^2(\Omega))$, b and g satisfy*

$$\int_{\tau}^{t}e^{\sigma s}\|b(s,u_s)-b(s,v_s)\|^2ds \le C_b\int_{\tau-r}^{t}e^{\sigma s}\|u(s)-v(s)\|^2ds\,, \tag{3.23}$$

and

$$\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 ds < \infty\,, \quad \forall t \in \mathbb{R}\,, \tag{3.24}$$

where $0 < \sigma < \delta' < \min\big\{2c_4, \frac{2\lambda_1}{2\lambda_1+1}\big\}$. *Then for all* t *for which* $t \geq \tau + r$ *and all* $(u^0, \varphi) \in X$, *we have the estimates*

$$\|\nabla u(t)\|^2 \leq c\big\{e^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} + 1 + e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 ds\Big\}\,, \tag{3.25}$$

$$\begin{aligned} &\int_{t-r}^{t} \|\nabla u(s)\|^2 ds \\ &\leq c\mu^{-1}\Big\{e^{-\sigma(t-\tau-r)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} + e^{\sigma r}e^{-\sigma(t-r)}\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 ds\Big\}\,, \end{aligned} \tag{3.26}$$

where $\mu := 4(\delta' - \sigma - \frac{C_b}{\lambda_1}) > 0$.

Proof. Multiplying (1.4) by $u + \frac{\partial u}{\partial t}$ and integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + 2\|\nabla u(t)\|^2) + \|\frac{d}{dt}u(t)\|^2 + \|\nabla u(t)\|^2 + \int_\Omega f(u)(u + \frac{\partial u}{\partial t}) \\ &+ \|\nabla \frac{d}{dt}u(t)\|^2 \\ &= \int_\Omega b(t,u_t)(u + \frac{\partial u}{\partial t}) + \int_\Omega g(u + \frac{\partial u}{\partial t})\,. \end{aligned}$$

By (2.6), the Cauchy-Schwarz and Young inequalities, one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + 2\|\nabla u(t)\|^2 + 2\int_\Omega F(u(t))) + \|\nabla u(t)\|^2 + c_4\int_\Omega F(u) \\ &\leq \|b(t,u_t)\|^2 + \|g(t)\|^2 + (1+\delta)\|u(t)\|^2 + c_\delta|\Omega|\,. \end{aligned}$$

By (3.3), we obtain

$$\begin{aligned} &\frac{d}{dt}(\|u(t)\|^2 + 2\|\nabla u(t)\|^2 + 2\int_\Omega F(u(t))) + (2 - \frac{2(1+\delta)}{\lambda_1})\|\nabla u(t)\|^2 + 2c_4\int_\Omega F(u) \\ &\leq 2\|b(t,u_t)\|^2 + 2\|g(t)\|^2 + 2c_\delta|\Omega|\,. \end{aligned}$$

We can choose δ small enough such that

$$\big(2 - \frac{2(1+\delta)}{\lambda_1}\big)\|\nabla u(t)\|^2 \geq \delta'(\|u(t)\|^2 + 2\|\nabla u(t)\|^2)\,, \tag{3.27}$$

where $\delta' < \min\{2c_4, \frac{2\lambda_1}{2\lambda_1+1}\}$. So, we can write

$$\frac{d}{dt}\gamma_1(t) + \delta'\gamma_1(t) \leq 2\|g(t)\|^2 + 2\|b(t,u_t)\|^2 + 2c_\delta|\Omega|\,, \tag{3.28}$$

where

$$\gamma_1(t) = \|u(t)\|^2 + 2\|\nabla u(t)\|^2 + 2\int_\Omega F(u(t))\,. \tag{3.29}$$

Multiplying (3.28) by $e^{\sigma t}$ such that $0 < \sigma < \delta'$, we obtain

$$e^{\sigma t}\frac{d}{dt}\gamma_1(t) + \delta' e^{\sigma t}\gamma_1(t) \leq 2e^{\sigma t}(\|g(t)\|^2 + \|b(t,u_t)\|^2 + c_\delta|\Omega|)\,, \tag{3.30}$$

whereupon

$$\frac{d}{dt}(e^{\sigma t}\gamma_1(t)) \le (\sigma-\delta')e^{\sigma t}\gamma_1(t) + 2e^{\sigma t}(\|g(t)\|^2 + \|b(t,u_t)\|^2 + c_\delta|\Omega|)\,.$$

Integrating this last estimate over $[\tau, t]$, we find that

$$\begin{aligned}
&\gamma_1(t)\\
&\le e^{-\sigma(t-\tau)}\gamma_1(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}\gamma_1(s)ds\\
&\quad + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|b(s,u_s)\|^2 ds + 2e^{-\sigma t}c_\delta|\Omega|\int_\tau^t e^{\sigma s}ds\\
&\le e^{-\sigma(t-\tau)}\gamma_1(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}\gamma_1(s)ds + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds\\
&\quad + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|b(s,u_s)\|^2 ds + 2e^{-\sigma t}c_\delta|\Omega|\sigma^{-1}(1-e^{-\sigma(t-\tau)})\\
&\le e^{-\sigma(t-\tau)}\gamma_1(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}\gamma_1(s)ds\\
&\quad + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|b(s,u_s)\|^2 ds + 2e^{-\sigma t}c_\delta|\Omega|\sigma^{-1}\,.
\end{aligned}$$

Therefore, using (3.23) and (II), one has

$$\begin{aligned}
\gamma_1(t) &\le e^{-\sigma(t-\tau)}\gamma_1(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}\gamma_1(s)ds\\
&\quad + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + 2C_b e^{-\sigma t}\int_{\tau-r}^t e^{\sigma s}\|u(s)\|^2 ds + 2c_\delta|\Omega|\sigma^{-1}\\
&\le e^{-\sigma(t-\tau)}\gamma_1(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}\gamma_1(s)ds\\
&\quad + 2C_b e^{-\sigma(t-\tau)}\int_{\tau-r}^\tau \|u(s)\|^2 ds + 2C_b e^{-\sigma t}\int_\tau^t e^{\sigma s}\|u(s)\|^2 ds\\
&\quad + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + 2c_\delta|\Omega|\sigma^{-1}\,.
\end{aligned} \tag{3.31}$$

We use 2.7 in (3.29) to obtain

$$\begin{aligned}
\gamma_1(t) &\ge \|u(t)\|^2 + 2\|\nabla u(t)\|^2 - 2\delta\|u(t)\|^2 - 2c'_\delta|\Omega|\\
&\ge (1-2\delta)\|u(t)\|^2 + 2\|\nabla u(t)\|^2 - 2c'_\delta|\Omega|\\
&\ge \frac{1}{2}\|\nabla u(t)\|^2\,.
\end{aligned} \tag{3.32}$$

On the other hand,

$$\gamma_1(\tau) = \|u(\tau)\|^2 + 2\|\nabla u(\tau)\|^2 + 2\int_\Omega F(u(\tau))\,. \tag{3.33}$$

By (2.3), we have

$$\int_\Omega F(u) \le k\int_\Omega |u| + \frac{k}{\alpha+1}\int_\Omega |u|^{\alpha+1}\,.$$

Using the Holder inequality and the fact that $\alpha+1 \le \frac{2N}{N-2}$, one has

$$\begin{aligned}\int_\Omega F(u) &\le k\sqrt{|\Omega|}\Big(\int_\Omega |u|^2\Big)^{1/2} + \frac{c}{\alpha+1}\int_\Omega |u|^{\frac{2N}{N-2}}\\ &\le k\sqrt{|\Omega|}\|u\| + \frac{c}{\alpha+1}\|u\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\Omega)}.\end{aligned}$$

By the embedding of $H_0^1(\Omega)$ in $L^{\frac{2N}{N-2}}(\Omega)$, (3.3) and the fact that $1 < \frac{2N}{N-2}$, we have

$$\begin{aligned}\int_\Omega F(u) &\le k\lambda_1^{-1}\sqrt{|\Omega|}\|\nabla u\| + \frac{c}{\alpha+1}\|\nabla u\|^{\frac{2N}{N-2}}\\ &\le c\lambda_1^{-1}\sqrt{|\Omega|}\|\nabla u\|^{\frac{2N}{N-2}} + \frac{c}{\alpha+1}\|\nabla u\|^{\frac{2N}{N-2}}\\ &\le \Big(c\lambda_1^{-1}\sqrt{|\Omega|} + \frac{c}{\alpha+1}\Big)\|\nabla u\|^{\frac{2N}{N-2}}\\ &\le c\|\nabla u\|^{\frac{2N}{N-2}}, \end{aligned}\tag{3.34}$$

where c is a positive constant. Using this inequality in (3.33), we obtain

$$\gamma_1(\tau) \le \|u(\tau)\|^2 + 2\|\nabla u(\tau)\|^2 + c\|\nabla u(\tau)\|^{\frac{2N}{N-2}}.$$

Using (3.3) and $2 < \frac{2N}{N-2}$, one finds that

$$\gamma_1(\tau) \le (\lambda_1^{-1}+2)\|\nabla u(\tau)\|^2 + c\|\nabla u(\tau)\|^{\frac{2N}{N-2}} \le c\|\nabla u(\tau)\|^{\frac{2N}{N-2}}. \tag{3.35}$$

We substitute (3.32) and (3.35) in (3.31), one gets

$$\begin{aligned}\frac{1}{2}\|\nabla u(t)\|^2 &\le ce^{-\sigma(t-\tau)}\|\nabla u(\tau)\|^{\frac{2N}{N-2}} + 2C_b e^{-\sigma(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds\\ &\quad + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}\Big(\|u(s)\|^2 + 2\|\nabla u(s)\|^2 + 2\int_\Omega F(u(s))\Big)ds\\ &\quad + 2e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + 2C_b e^{-\sigma t}\int_\tau^t e^{\sigma s}\|u(s)\|^2 ds + 2c_\delta|\Omega|\sigma^{-1}.\end{aligned}$$

So, by (3.3),

$$\begin{aligned}\|\nabla u(t)\|^2 &\le ce^{-\sigma(t-\tau)}\|\nabla u(\tau)\|^{\frac{2N}{N-2}} + 4C_b e^{-\sigma(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds\\ &\quad + 2(\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}\Big(\|u(s)\|^2 + 2\|\nabla u(s)\|^2 + 2\int_\Omega F(u(s))\Big)ds\\ &\quad + 4e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + \frac{4C_b}{\lambda_1}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|\nabla u(s)\|^2 ds + 4c_\delta|\Omega|\sigma^{-1}.\end{aligned}$$

Then, for $\delta' - \sigma - \frac{C_b}{\lambda_1} > 0$ and the fact that $2 < \frac{2N}{N-2}$, we have

$$\begin{aligned}&\|\nabla u(t)\|^2 + 4\Big(\delta'-\sigma-\frac{C_b}{\lambda_1}\Big)e^{-\sigma t}\int_\tau^t e^{\sigma s}\|\nabla u(s)\|^2 ds\\ &\le ce^{-\sigma(t-\tau)}\|\nabla u(\tau)\|^{\frac{2N}{N-2}} + ce^{-\sigma(t-\tau)}\|\varphi\|^{\frac{2N}{N-2}}_{L^2((-r,0);L^2(\Omega))}\\ &\quad + 4e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + 4c_\delta|\Omega|\sigma^{-1}\\ &\le c\Big\{e^{-\sigma(t-\tau)}\big(\|\nabla u^0\|^{\frac{2N}{N-2}} + \|\varphi\|^{\frac{2N}{N-2}}_{L^2((-r,0);L^2(\Omega))}\big)\end{aligned}$$

$$+ 1 + e^{-\sigma t}\int_{\tau}^{t} e^{\sigma s}\|g(s)\|^2 ds\Big\}\,.$$

By using (3.24), we have

$$\begin{aligned}
&\|\nabla u(t)\|^2 + 4\big(\delta' - \sigma - \frac{C_b}{\lambda_1}\big)e^{-\sigma t}\int_{\tau}^{t} e^{\sigma s}\|\nabla u(s)\|^2 ds \\
&\le c\Big\{e^{-\sigma(t-\tau)}\big(\|\nabla u^0\|^{\frac{2N}{N-2}} + \|\varphi\|^{\frac{2N}{N-2}}_{L^2((-r,0);L^2(\Omega))}\big) \\
&\quad + 1 + e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 ds\Big\}\,.
\end{aligned} \tag{3.36}$$

Whereupon, for all $t \ge \tau$, we obtain (3.25), and

$$\begin{aligned}
&\mu e^{-\sigma t}\int_{\tau}^{t} e^{\sigma s}\|\nabla u(s)\|^2 ds \\
&\le c\Big\{e^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} + 1 + e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 ds\Big\}\,,
\end{aligned} \tag{3.37}$$

where $\mu := 4\big(\delta' - \sigma - \frac{C_b}{\lambda_1}\big) > 0$. Furthermore, for $\tau \le t - r$, we have

$$\int_{\tau}^{t} e^{\sigma s}\|\nabla u(s)\|^2 ds \ge \int_{t-r}^{t} e^{\sigma s}\|\nabla u(s)\|^2 ds \ge e^{\sigma(t-r)}\int_{t-r}^{t}\|\nabla u(s)\|^2 ds\,,$$

as $[t-r,t] \subset [\tau,t]$. Hence, (3.37) becomes

$$\begin{aligned}
&\mu e^{-\sigma r}\int_{t-r}^{t}\|\nabla u(s)\|^2 ds \\
&\le c\Big\{e^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} + 1 + e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 ds\Big\}\,.
\end{aligned}$$

Therefore, for all $t \ge \tau + r$, we obtain (3.26), and this completes the proof. □

Let $\mathcal{R}$ be the set of all functions $\rho : \mathbb{R} \to (0,+\infty)$ such that

$$\lim_{t\to-\infty} e^{\sigma t}\rho^{\frac{2N}{N-2}}(t) = 0\,.$$

By $\mathcal{D}$ we denote the class of all families $\widehat{\mathbf{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ such that $D(t) \subset \overline{\mathbf{B}}_X(0,\rho(t))$, for some $\rho \in \mathcal{R}$, where $\overline{\mathbf{B}}_X(0,\rho(t))$ denotes the closed ball in X centered at 0 with radius $\rho(t)$. Let

$$\rho_1(t) = c\Big(1 + e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 ds\Big)\,,$$

and $R(t) \ge 0$, $R'(t) \ge 0$, where

$$\begin{gathered}
R^2(t) = (1 + \mu^{-1}e^{\sigma r})\rho_1(t)\,, \\
R'^2(t) = c\rho_1^{\frac{N}{N-2}}(t) + c\rho_1(t) + c\|g\|^2_{L^2([t-r,t];L^2(\Omega))}\,.
\end{gathered}$$

Lemma 3.8 (Pullback $\mathcal{D}$-absorbing set)**.** *Under the assumptions of Lemma 3.7, the family $\widehat{\mathbf{B}}$ given by*

$$B(t) = \Big\{(v^0,\phi) \in X_1 : \|(v^0,\phi)\|_{X_1} \le R(t),\ \|\frac{d\phi}{ds}\|_{L^2((-r,0);L^2(\Omega))} \le R'(t)\Big\}\,, \tag{3.38}$$

is pullback $\mathcal{D}$-absorbing for the process $U(.,.)$ defined by (3.22).

Proof. First, we observe that for all $t \in \mathbb{R}$,

$$B(t) \subset \{(v^0, \phi) \in X : \|(v^0, \phi)\|_X \le R(t)\}, \tag{3.39}$$

with

$$\lim_{t\to-\infty} e^{\sigma t} R^{\frac{2N}{N-2}}(t) = 0,$$

and so $\widehat{B} \in \mathcal{D}$.

Now, we prove that $U(t,\tau)D(\tau) \subset B(t)$, for all $\tau \le \tau_0$. To do this, we proceed in two steps.

Step 1. This step concerns the asymptotic estimate using $R(t)$ for $t \in \mathbb{R}$, fixed. It may be proved as follows. By definition, we have

$$\|U(t,\tau)(u^0,\varphi)\|_{X_1}^2 = \|\nabla u(t)\|^2 + \int_{t-r}^t \|\nabla u(s)\|^2 ds. \tag{3.40}$$

From (3.26), for any $t - r \ge \tau$, we have

$$\begin{aligned} \int_{t-r}^t \|\nabla u(s)\|^2 ds &\le c\mu^{-1} e^{-\sigma(t-\tau-r)} \|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} \\ &\quad + \mu^{-1} e^{\sigma r} c(1 + e^{-\sigma t} \int_{-\infty}^t e^{\sigma s} \|g(s)\|^2 ds), \end{aligned} \tag{3.41}$$

for any $(u^0,\varphi) \in X$. We substitute this inequality and (3.25) in (3.40); by the definition of $\rho_1(t)$, we obtain

$$\begin{aligned} \|U(t,\tau)(u^0,\varphi)\|_{X_1}^2 &\le c e^{-\sigma(t-\tau)} \|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} (1 + \mu^{-1} e^{\sigma r}) + (1 + \mu^{-1} e^{\sigma r}) \rho_1(t) \\ &\le c e^{-\sigma(t-\tau)} \|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} (1 + \mu^{-1} e^{\sigma r}) + R^2(t), \end{aligned}$$

for all $t - r \ge \tau$ and all $(u^0,\varphi) \in X$. Hence

$$\|U(t,\tau)(u^0,\varphi)\|_{X_1}^2 \le R^2(t), \tag{3.42}$$

as $e^{\sigma\tau} \to 0$ when $\tau \to -\infty$.

Step 2. This step concerns the asymptotic estimate using $R'(t)$. We assume that $t - 2r \ge \tau$. Multiplying (1.4) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we obtain

$$\|\frac{d}{dt}u(t)\|^2 + \frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|^2 + 2\int_\Omega F(u)) \le \frac{1}{2}\|b(t,u_t)\|^2 + \frac{1}{2}\|g(t)\|^2.$$

Integrating over $[t-r, t]$, we find

$$\begin{aligned} &\int_{t-r}^t \|\frac{d}{ds}u(s)\|^2 ds + \frac{1}{2}\Big(\|\nabla u(t)\|^2 + 2\int_\Omega F(u(t))\Big) \\ &\le \frac{1}{2}(\|\nabla u(t-r)\|^2 + 2\int_\Omega F(u(t-r))) \\ &\quad + \frac{1}{2}\int_{t-r}^t \|g(s)\|^2 ds + \frac{1}{2}\int_{t-r}^t \|b(s,u_s)\|^2 ds. \end{aligned} \tag{3.43}$$

From (II), (IV), and (3.3), one has

$$\int_{t-r}^t \|b(s,u_s)\|^2 ds \le C_b \int_{t-2r}^t \|u(s)\|^2 ds \le C_b \lambda_1^{-1} \int_{t-2r}^t \|\nabla u(s)\|^2 ds.$$

By this estimate and (3.43),

$$\begin{aligned}\int_{t-r}^{t}\|\frac{d}{ds}u(s)\|^2ds &\le \frac{1}{2}(\|\nabla u(t-r)\|^2+2\int_\Omega F(u(t-r)))\\ &\quad+\frac{1}{2}\int_{t-r}^{t}\|g(s)\|^2ds+\frac{C_b\lambda_1^{-1}}{2}\int_{t-2r}^{t}\|\nabla u(s)\|^2ds\,.\end{aligned}$$

By (3.34) and the fact that $2<\frac{2N}{N-2}$, we have

$$\begin{aligned}\int_{t-r}^{t}\|\frac{d}{ds}u(s)\|^2ds &\le \frac{1}{2}(\|\nabla u(t-r)\|^2+2c\|\nabla u(t-r)\|^{\frac{2N}{N-2}})\\ &\quad+\frac{1}{2}\int_{t-r}^{t}\|g(s)\|^2ds+\frac{C_b\lambda_1^{-1}}{2}\int_{t-2r}^{t}\|\nabla u(s)\|^2ds\\ &\le c\|\nabla u(t-r)\|^{\frac{2N}{N-2}}+c\|g\|^2_{L^2([t-r,t];L^2(\Omega))}\\ &\quad+c\int_{t-2r}^{t}\|\nabla u(s)\|^2ds\,. \end{aligned}\tag{3.44}$$

Now, we estimate $\|\nabla u(t-r)\|^2$. Replacing t by $t-r$ in (3.25), we obtain

$$\begin{aligned}\|\nabla u(t-r)\|^2 &\le ce^{-\sigma(t-r-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}\\ &\quad+c\Big(1+e^{-\sigma(t-r)}\int_{-\infty}^{t-r}e^{\sigma(s-r)}\|g(s-r)\|^2ds\Big)\,.\end{aligned}$$

Because $t-r\le t$ and $e^{\sigma r}>1$, we have

$$\begin{aligned}\|\nabla u(t-r)\|^2 &\le ce^{\sigma r}e^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}\\ &\quad+e^{\sigma r}c(1+e^{-\sigma t}\int_{-\infty}^{t}e^{\sigma s}\|g(s)\|^2ds)\\ &\le ce^{\sigma r}e^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}+e^{\sigma r}\rho_1(t)\,.\end{aligned}$$

Hence

$$\|\nabla u(t-r)\|^{\frac{2N}{N-2}}\le\Big(ce^{\sigma r}e^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}+e^{\sigma r}\rho_1(t)\Big)^{\frac{N}{N-2}}\,.$$

Since $\frac{N}{N-2}>1$, using a convexity argument,

$$\|\nabla u(t-r)\|^{\frac{2N}{N-2}}\le ce^{-\frac{N}{N-2}\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N^2}{(N-2)^2}}+c\rho_1^{\frac{N}{N-2}}(t)\,.\tag{3.45}$$

Using (3.26), and taking $2r$ in place of r, we have

$$\int_{t-2r}^{t}\|\nabla u(s)\|^2ds\le ce^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}+c\rho_1(t)\,,\tag{3.46}$$

for any $t-2r\ge\tau$. From (3.44)-(3.46) we conclude that for all $t-2r\ge\tau$ and all $(u^0,\varphi)\in X$,

$$\begin{aligned}\int_{t-r}^{t}\|\frac{d}{ds}u(s)\|^2ds &\le ce^{-\frac{N}{N-2}\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N^2}{(N-2)^2}}+ce^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}\\ &\quad+c\rho_1^{\frac{N}{N-2}}(t)+c\rho_1(t)+c\|g\|^2_{L^2([t-r,t];L^2(\Omega))}\,.\end{aligned}$$

Hence, for all $t-2r \geq \tau$ and for any $(u^0,\varphi) \in X$, we have

$$\begin{aligned}&\int_{t-r}^{t} \|\frac{d}{ds}u(s)\|^2 ds \\ &\leq ce^{-\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} + ce^{-\frac{N}{N-2}\sigma(t-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N^2}{(N-2)^2}} + R'^2(t)\,,\end{aligned}$$

so we obtain

$$\int_{t-r}^{t} \|\frac{d}{ds}u(s)\|^2 ds \leq R'^2(t)\,, \tag{3.47}$$

as $e^{\sigma\tau} \to 0$ when $\tau \to -\infty$. Then, it is clear to see from (3.42), (3.47) and the definition of $\mathcal{D}$ that the family $\widehat{\mathbf{B}}$ given by (3.38) is pullback $\mathcal{D}$-absorbing for the process $U(.,.)$. □

In what follows, we need the following result.

Proposition 3.9. *Let $\{U(t,\tau)\}$ be a process on X, and let $\{B(t) : t \in \mathbb{R}\}$ be a pullback $\mathcal{D}$-absorbing set of $\{U(t,\tau)\}$. Suppose that for each $t \in \mathbb{R}$, any $\widehat{B} \in \mathcal{D}$ and any $\varepsilon > 0$, there exist $\tau_0 = \tau_0(t,\widehat{B},\varepsilon) \leq t$ and $\delta > 0$ such that*

1. *for all $\tau \leq \tau_0$ and $(u(t),u_t) \in U(t,\tau)B(\tau)$, $\|P(u(t),u_t)\|_{X_1}$ is bounded;*
2. *for all $\tau \leq \tau_0$ and $(u(t),u_t) \in U(t,\tau)B(\tau)$, $\|(I-P)(u(t),u_t)\|_{X_1} < \varepsilon$;*
3. *for all $\tau \leq \tau_0$, $u_t \in U(t,\tau)B(\tau)$ and all $l \in \mathbb{R}$ with $|l| < \delta$, we have*
$$\|P(T_l u_t - u_t)\|_{L^2((-r,0);L^2(\Omega))} < \varepsilon\,,$$
where $T_l u_t$ is the translation $(T_l u_t)(\theta) = u(t+\theta+l)$ with $\theta \in (-r,0)$ and P is the canonical projector on the finite dimensional subspace V_n of $H_0^1(\Omega)$ and $L^2(\Omega)$, and I is the identity. Then $\{U(t,\tau)\}$ is pullback ω-$\mathcal{D}$-limit compact in X with respect to each $t \in \mathbb{R}$.

Proof. (i) First, we prove that $\{U(t,\tau)\}$ is pullback ω-$\mathcal{D}$-limit compact in X_1. Note that by (2) in the Lemma 2.6, one has

$$\begin{aligned}\mathcal{K}\Big(\cup_{\tau\leq\tau_0} U(t,\tau)B(\tau)\Big) &\leq \mathcal{K}\Big(P\Big(\cup_{\tau\leq\tau_0} U(t,\tau)B(\tau)\Big)\Big) \\ &\quad + \mathcal{K}\Big((I-P)(\cup_{\tau\leq\tau_0} U(t,\tau)B(\tau))\Big)\,.\end{aligned} \tag{3.48}$$

Assumption (1) gives that $\{P \cup_{\tau\leq\tau_0} U(t,\tau)B(\tau)\}$ is contained in a ball of finite radius. So by (3) in Lemma 2.6, we obtain

$$\mathcal{K}\Big(P\Big(\cup_{\tau\leq\tau_0} U(t,\tau)B(\tau)\Big)\Big) \leq \mathcal{K}(B(0,\varepsilon_0))\,, \tag{3.49}$$

and by (6) in Lemma 2.6, we obtain

$$\mathcal{K}(B(0,\varepsilon_0)) \leq 2\varepsilon_0\,. \tag{3.50}$$

Thus, by (3.49) and (3.50) it follows that

$$\mathcal{K}\Big(P\Big(\cup_{\tau\leq\tau_0} U(t,\tau)B(\tau)\Big)\Big) \leq 2\varepsilon_0\,. \tag{3.51}$$

On the other hand, assumption (2) and property (6) in Lemma 2.6 give

$$\mathcal{K}\Big((I-P)\Big(\cup_{\tau\leq\tau_0} U(t,\tau)B(\tau)\Big)\Big) \leq 2\varepsilon\,. \tag{3.52}$$

Therefore, by (3.48), (3.51) and (3.52) we deduce that

$$\mathcal{K}\Big(\cup_{\tau\leq\tau_0} U(t,\tau)B(\tau)\Big) \leq 2\varepsilon'\,,$$

where $\varepsilon' := \varepsilon_0 + \varepsilon$, and this shows that $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-w-limit compact in X_1, i.e.; for all $\tau \le \tau_0$, any sequences $\tau^{n'} \to -\infty$ and $(u^{0,n'}, \varphi^{n'}) \in B(\tau^{n'})$, the sequence $\{(u^{n'}(t), u_t^{n'})\} = \{U(t,\tau^{n'})(u^{0,n'}, \varphi^{n'})\}$ is relatively compact in X_1.

(ii) Second, we will check the equicontinuity property of u_t in $L^2((-r,0);L^2(\Omega))$. To this end, we need to use the L^p-version of Arzelà-Ascoli theorem (see [3, theorem IV.25, p.72]).

Assumption (2) gives

$$\|(I-P)u_t\|_{L^2((-r,0);H_0^1(\Omega))} < \varepsilon\,. \tag{3.53}$$

Since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ with continuous injection, we have

$$\|(I-P)u_t\|_{L^2((-r,0);L^2(\Omega))} \le c_5 \|(I-P)u_t\|_{L^2((-r,0);H_0^1(\Omega))}\,.$$

So by this estimate and (3.53), one has

$$\|(I-P)u_t\|_{L^2((-r,0);L^2(\Omega))} < \varepsilon'\,, \tag{3.54}$$

where $\varepsilon' := c_5\varepsilon$. From (3.54) and assumption (3), we deduce that for all $\tau \le \tau_0$, $u_t \in U(t,\tau)B(\tau)$ and all $l \in \mathbb{R}^+$ with $l < \delta$, we have

$$\begin{aligned}&\|T_l u_t - u_t\|_{L^2((-r,0);L^2(\Omega))}\\ &\le \|P(T_l u_t - u_t)\|_{L^2((-r,0);L^2(\Omega))} + \|(I-P)(T_l u_t - u_t)\|_{L^2((-r,0);L^2(\Omega))} < \varepsilon''\,,\end{aligned}$$

with $\varepsilon'' := \varepsilon + \varepsilon'$ and this is the desired equicontinuity.

From (i), we deduce that $\{u_t^{n'}\}$ is relatively compact in $L^2((-r,0);H_0^1(\Omega))$, and (ii) gives that $\{u_t^{n'}\}$ is relatively compact in $L^2((-r,0);L^2(\Omega))$. Therefore, we conclude that $\{U(t,\tau)\}$ is pullback ω-$\mathcal{D}$-limit compact in X, which completes the proof. □

Theorem 3.10. *The process $\{U(t,\tau)\}$ corresponding to* (1.4) *has a pullback $\mathcal{D}$-attractor $\widehat{A} = \{A(t) : t \in \mathbb{R}\}$ in X.*

Proof. From Lemma 3.8, $\{U(t,\tau)\}$ has a family of Pullback $\mathcal{D}$-absorbing sets in X. By Theorem 2.8, it remains to show that $\{U(t,\tau)\}$ is Pullback w-$\mathcal{D}$-limit compact. To this end, we need to check conditions (1)-(3) in Proposition 3.9. To this aim, we decompose f as

$$f = f_0 + f_1\,,$$

where $f_0\,, f_1 \in C^1(\mathbb{R},\mathbb{R})$ satisfy

$$f_0(u)u \ge -c_1u^2 - c_2, \tag{3.55}$$

$$f_0'(u) \ge -c_3, \tag{3.56}$$

$$|f_0(u)| \le k(1+|u|^\alpha)\,, \tag{3.57}$$

$$|f_1(u)| \le k(1+|u|^\alpha)\,. \tag{3.58}$$

The delayed forcing term b is decomposed as

$$b = b_0 + b_1\,,$$

where $b_0, b_1 : \mathbb{R} \times L^2((-r,0);L^2(\Omega)) \to L^2(\Omega)$ satisfy

(a) $b_0(t,0) = 0$ for all $t \in \mathbb{R}$;

(b) there exists $C_{b_0} > 0$ such that for all $t \ge \tau$ and all $u, v \in L^2([\tau-r,t];L^2(\Omega))$,

$$\int_\tau^t \|b_0(s,u_s) - b_0(s,v_s)\|^2 ds \le C_{b_0} \int_{\tau-r}^t \|u(s)-v(s)\|^2 ds\,; \tag{3.59}$$

(c) $b_1(t,0)=0$ for all $t\in\mathbb{R}$;
(d) there exists $C_{b_1}>0$ such that for all $t\geq\tau$ and all $u,v\in L^2([\tau-r,t];L^2(\Omega))$,

$$\int_\tau^t e^{\sigma s}\|b_1(s,u_s)-b_1(s,v_s)\|^2ds\leq C_{b_1}\int_{\tau-r}^t e^{\sigma s}\|u(s)-v(s)\|^2ds\,, \tag{3.60}$$

Let $\lambda_1,\lambda_2,\dots$ be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and $w_1,w_2,\dots$ the corresponding eigenfunctions. Then we have $0<\lambda_1\leq\lambda_2\leq\dots\leq\lambda_n\to+\infty$ as $n\to+\infty$. Then $\{w_1,w_2,\dots\}$ form an orthogonal basis in $L^2(\Omega)$ and $H_0^1(\Omega)$. Let $V_n=span\{w_1,w_2,\dots,w_n\}$, P be the canonical projector on V_n and I be the identity. Then we decompose $U(t,\tau)(u^0,\varphi)=(u(t),u_t)$ as

$$u(t)=Pu(t)+(I-P)u(t)=v(t)+w(t),$$

and

$$u_t=Pu_t+(I-P)u_t=v_t+w_t\,.$$

Here v and w solve the following problems:

$$\begin{gathered}\frac{\partial}{\partial t}v-\Delta\frac{\partial}{\partial t}v-\Delta v+f_0(v)=b_0(t,v_t)\\ v=0\quad\text{on }\partial\Omega\\ v(\tau,x)=Pu^0\\ v(\tau+\theta,x)=P\varphi(\theta),\ \theta\in(-r,0)\end{gathered}\tag{3.61}$$

and

$$\begin{gathered}\frac{\partial}{\partial t}w-\Delta\frac{\partial}{\partial t}w-\Delta w+f(u)-f_0(v)=b(t,u_t)-b_0(t,v_t)+g\\ w=0\quad\text{on }\partial\Omega\\ w(\tau,x)=(I-P)u^0\\ w(\tau+\theta,x)=(I-P)\varphi(\theta),\quad\theta\in(-r,0)\end{gathered}\tag{3.62}$$

First, we establish that for all $\tau\leq\tau_0$, $(u(t),u_t)\in U(t,\tau)B(\tau)$ and satisfies $\|P(u(t),u_t)\|_{X_1}<+\infty$. To do this, we multiply (3.61) by v and integrating over Ω; to obtain

$$\frac{d}{dt}(\|v(t)\|^2+\|\nabla v(t)\|^2)+2\|\nabla v(t)\|^2+2\int_\Omega f_0(v)v=2\int_\Omega b_0(t,v_t)v\,.$$

By (3.55) and the Cauchy inequality one obtains

$$\begin{aligned}&\frac{d}{dt}(\|v(t)\|^2+\|\nabla v(t)\|^2)+2\|\nabla v(t)\|^2\\ &\leq 2c_1\|v(t)\|^2+2c_2|\Omega|+\frac{1}{\varepsilon_4}\|b_0(t,v_t)\|^2+\varepsilon_4\|v(t)\|^2\\ &\leq(2c_1+\varepsilon_4)\|v(t)\|^2+2c_2|\Omega|+\frac{1}{\varepsilon_4}\|b_0(t,v_t)\|^2\,.\end{aligned}$$

Integrating from τ to t, for $\tau\leq t\leq T$, we have

$$\begin{aligned}&\|v(t)\|^2+\|\nabla v(t)\|^2+2\int_\tau^t\|\nabla v(s)\|^2ds\\ &\leq\|v(\tau)\|^2+\|\nabla v(\tau)\|^2+(2c_1+\varepsilon_4)\int_\tau^t\|v(s)\|^2ds\end{aligned}$$

$$+\frac{1}{\varepsilon_4}\int_\tau^t \|b_0(s,v_s)\|^2 ds + 2c_2|\Omega|(t-\tau)\,.$$

Using (3.59) and (a), one has

$$\begin{aligned}
&\|v(t)\|^2 + \|\nabla v(t)\|^2 + 2\int_\tau^t \|\nabla v(s)\|^2 ds \\
&\le \|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + (2c_1+\varepsilon_4)\int_\tau^t \|v(s)\|^2 ds \\
&\quad + \frac{C_b}{\varepsilon_4}\int_\tau^t \|v(s)\|^2 ds + 2c_2|\Omega|(t-\tau) \\
&\le \|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + (2c_1+\varepsilon_4)\int_\tau^t \|v(s)\|^2 ds \\
&\quad + \frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^\tau \|v(s)\|^2 ds + \frac{C_{b_0}}{\varepsilon_4}\int_\tau^t \|v(s)\|^2 ds + 2c_2|\Omega|(t-\tau)\,.
\end{aligned}$$

So, one has

$$\begin{aligned}
&\|v(t)\|^2 + \|\nabla v(t)\|^2 + 2\int_\tau^t \|\nabla v(s)\|^2 ds \\
&\le \|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^\tau \|v(s)\|^2 ds \\
&\quad + (2c_1+\varepsilon_4+\frac{C_{b_0}}{\varepsilon_4})\int_\tau^t \|v(s)\|^2 ds + 2c_2|\Omega|(t-\tau)\,.
\end{aligned}$$

By (3.3), one obtains

$$\begin{aligned}
&\|v(t)\|^2 + \|\nabla v(t)\|^2 + 2\int_\tau^t \|\nabla v(s)\|^2 ds \\
&\le \|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^\tau \|v(s)\|^2 ds \\
&\quad + \frac{2c_1+\varepsilon_4+\frac{C_{b_0}}{\varepsilon_4}}{\lambda_1}\int_\tau^t \|\nabla v(s)\|^2 ds + 2c_2|\Omega|(t-\tau)\,.
\end{aligned}$$

Thus, one finds that

$$\begin{aligned}
&\|v(t)\|^2 + \|\nabla v(t)\|^2 + \Big(2-\frac{2c_1+\varepsilon_4+\frac{C_b}{\varepsilon_4}}{\lambda_1}\Big)\int_\tau^t \|\nabla v(s)\|^2 ds \\
&\le \|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^\tau \|v(s)\|^2 ds + 2c_2|\Omega|(t-\tau)\,.
\end{aligned}$$

By the Theorem 3.2, we have $\eta_1 := 2-\frac{2c_1+\varepsilon_4+\frac{C_{b_0}}{\varepsilon_4}}{\lambda_1} > 0$. So the previous estimate gives

$$\|\nabla v(t)\|^2 \le \|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^\tau \|v(s)\|^2 ds + 2c_2|\Omega|(T-\tau)\,, \quad (3.63)$$

and

$$\begin{aligned}&\int_\tau^t \|\nabla v(s)\|^2 ds\\&\le \eta_1^{-1}\Big\{\|v(\tau)\|^2+\|\nabla v(\tau)\|^2+\frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^{\tau}\|v(s)\|^2ds+2c_2|\Omega|(T-\tau)\Big\}\,.\end{aligned}\tag{3.64}$$

Therefore, for $t-r\ge\tau$, with($[t-r,t]\subset[\tau,t]$), we have

$$\begin{aligned}\int_{t-r}^t\|\nabla v(s)\|^2ds &\le \int_\tau^t\|\nabla v(s)\|^2ds\\&\le \eta_1^{-1}(\|v(\tau)\|^2+\|\nabla v(\tau)\|^2+\frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^{\tau}\|v(s)\|^2ds)\\&\quad+2c_2\eta_1^{-1}|\Omega|(T-\tau)\,.\end{aligned}\tag{3.65}$$

We add (3.63) and (3.65) to obtain

$$\begin{aligned}&\|\nabla v(t)\|^2+\int_{t-r}^t\|\nabla v(s)\|^2ds\\&\le(1+\eta_1^{-1})\Big\{\|v(\tau)\|^2+\|\nabla v(\tau)\|^2+\frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^{\tau}\|v(s)\|^2ds+2c_2|\Omega|(T-\tau)\Big\}\\&:=M^2\,.\end{aligned}\tag{3.66}$$

Hence, one obtains

$$\|P(u(t),u_t)\|_{X_1}\le M\,,$$

which means that the condition (1) in Proposition 3.9 holds true.

Now, taking the inner product in $L^2(\Omega)$ of (3.62) with w, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|w(t)\|^2+\|\nabla w(t)\|^2)+2\|\nabla w(t)\|^2+2\int_\Omega(f(u)-f_0(v))w\\&=2\int_\Omega(b(t,u_t)-b_0(t,v_t))w+2\int_\Omega g\,w\,.\end{aligned}\tag{3.67}$$

Since $f_0(v)=f(v)-f_1(v)$ and $b_0(t,v_t)=b(t,v_t)-b_1(t,v_t)$, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|w(t)\|^2+\|\nabla w(t)\|^2)+2\|\nabla w(t)\|^2+2\int_\Omega|f(u)-f(v)||w|+2\int_\Omega|f_1(v)||w|\\&\le2\int_\Omega|b(t,u_t)-b(t,v_t)|\,|w|+2\int_\Omega|b_1(t,v_t)||w|+2\int_\Omega|g|\,|w|\,.\end{aligned}$$

By (2.2), we have

$$\int_\Omega(f(u)-f(v))(u-v)\ge-c_3\|u-v\|^2\,.\tag{3.68}$$

Thus, by (3.68) and the Cauchy inequality, (3.67) leads to

$$\begin{aligned}&\frac{d}{dt}(\|w(t)\|^2+\|\nabla w(t)\|^2)+2\|\nabla w(t)\|^2\\&\le(2c_3+\varepsilon_1+\varepsilon_2+\varepsilon_5)\|w(t)\|^2\\&+2\int_\Omega|f_1(v)|\,|w|+\frac{1}{\varepsilon_1}\|b(t,u_t)-b(t,v_t)\|^2+\frac{1}{\varepsilon_5}\|b_1(t,v_t)\|^2+\frac{1}{\varepsilon_2}\|g(t)\|^2\,.\end{aligned}\tag{3.69}$$

By (3.58) and the Holder inequality,

$$\begin{aligned}\int_\Omega |f_1(v)|\,|w| &\le k\int_\Omega (1+|v|^\alpha)|w| \\ &\le k\Big(\int_\Omega (1+|v|^\alpha)^{\frac{2N}{N+2}}\Big)^{\frac{N+2}{2N}}\Big(\int_\Omega |w|^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{2N}} \\ &\le c\Big(\int_\Omega (1+|v|^{\alpha\frac{2N}{N+2}})\Big)^{\frac{N+2}{2N}}\|w\|_{L^{\frac{2N}{N-2}}(\Omega)}\,.\end{aligned}$$

Since $\alpha<\min\{\frac{N+2}{N-2},2+\frac{4}{N}\}$, one has $\alpha\frac{2N}{N+2}<\frac{2N}{N-2}$ for all $N\ge 3$. Then

$$\begin{aligned}\int_\Omega |f_1(v)|\,|w| &\le c\Big(\int_\Omega (1+c|v|^{\frac{2N}{N-2}})\Big)^{\frac{N+2}{2N}}\|w\|_{L^{\frac{2N}{N-2}}(\Omega)} \\ &\le c\Big(|\Omega|+c\|v\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\Omega)}\Big)^{\frac{N+2}{2N}}\|w\|_{L^{\frac{2N}{N-2}}(\Omega)}\,.\end{aligned}$$

As above, one gets

$$\begin{aligned}\int_\Omega |f_1(v)|\,|w| &\le c(|\Omega|^{\frac{N+2}{2N}}+c\|v\|^{\frac{2N}{N-2}\frac{N+2}{2N}}_{L^{\frac{2N}{N-2}}(\Omega)})\|w\|_{L^{\frac{2N}{N-2}}(\Omega)} \\ &\le c(1+\|v\|^{\frac{N+2}{N-2}}_{L^{\frac{2N}{N-2}}(\Omega)})\|w\|_{L^{\frac{2N}{N-2}}(\Omega)}\,.\end{aligned}$$

By the embedding of $H_0^1(\Omega)$ in $L^{\frac{2N}{N-2}}(\Omega)$, we have

$$\int_\Omega |f_1(v)|\,|w| \le c(1+c\|\nabla v\|^{\frac{N+2}{N-2}})\|\nabla w\| \le c(1+\|\nabla v\|^{\frac{N+2}{N-2}})\|\nabla w\|$$

By (3.66), we obtain

$$\int_\Omega |f_1(v)|\,|w| \le c(1+M^{\frac{N+2}{N-2}})\|\nabla w\| \le c\|\nabla w\|\,,$$

as $\|\nabla v\|^{\frac{N+2}{N-2}}\le M^{\frac{N+2}{N-2}}$, and

$$\int_\Omega |f_1(v)|\,|w| \le \frac{c^2}{2\varepsilon_3}+\frac{\varepsilon_3}{2}\|\nabla w(t)\|^2 \le c+\frac{\varepsilon_3}{2}\|\nabla w(t)\|^2\,,$$

via the Cauchy inequality. By the above estimate and (3.69), one obtains

$$\begin{aligned}&\frac{d}{dt}(\|w(t)\|^2+\|\nabla w(t)\|^2)+(2-\varepsilon_3)\|\nabla w(t)\|^2 \\ &\le (2c_3+\varepsilon_1+\varepsilon_2+\varepsilon_5)\|w(t)\|^2+\frac{1}{\varepsilon_1}\|b(t,u_t)-b(t,v_t)\|^2 \\ &\quad+\frac{1}{\varepsilon_5}\|b_1(t,v_t)\|^2+\frac{1}{\varepsilon_2}\|g(t)\|^2+c\,.\end{aligned}$$

Using (3.3), one has

$$\begin{aligned}&\frac{d}{dt}(\|w(t)\|^2+\|\nabla w(t)\|^2)+(2-\varepsilon_3)\|\nabla w(t)\|^2 \\ &\le \frac{2c_3+\varepsilon_1+\varepsilon_2+\varepsilon_5}{\lambda_1}\|\nabla w(t)\|^2+\frac{1}{\varepsilon_1}\|b(t,u_t)-b(t,v_t)\|^2+\frac{1}{\varepsilon_5}\|b_1(t,v_t)\|^2 \\ &\quad+\frac{1}{\varepsilon_2}\|g(t)\|^2+c\,.\end{aligned}$$

For $\lambda_1 > c_3$, $\varepsilon_3 < 1$, and $\varepsilon_1, \varepsilon_2, \varepsilon_5$ small enough, we have

$$2 - \varepsilon_3 - \frac{2c_3 + \varepsilon_1 + \varepsilon_2 + \varepsilon_5}{\lambda_1} > 0.$$

So one gets

$$\begin{aligned}
&\frac{d}{dt}(\|w(t)\|^2 + \|\nabla w(t)\|^2) + (2 - \varepsilon_3 - \frac{2c_3 + \varepsilon_1 + \varepsilon_2 + \varepsilon_5}{\lambda_1})\|\nabla w(t)\|^2 \\
&\le \frac{1}{\varepsilon_1}\|b(t, u_t) - b(t, v_t)\|^2 + \frac{1}{\varepsilon_5}\|b_1(t, v_t)\|^2 + \frac{1}{\varepsilon_2}\|g(t)\|^2 + c\,.
\end{aligned}$$

Similarly as in (3.27), we can choose the positive constant $\delta' < \min\{2c_4, \frac{2\lambda_1}{2\lambda_1+1}\}$ such that

$$\delta'(\|\nabla w(t)\|^2 + \|w(t)\|^2) \le \Big(2 - \varepsilon_3 - \frac{2c_3 + \varepsilon_1 + \varepsilon_2 + \varepsilon_5}{\lambda_1}\Big)\|\nabla w(t)\|^2\,.$$

In fact,

$$\frac{d}{dt}y(t) + \delta' y(t) \le \frac{1}{\varepsilon_1}\|b(t, u_t) - b(t, v_t)\|^2 + \frac{1}{\varepsilon_5}\|b_1(t, v_t)\|^2 + \frac{1}{\varepsilon_2}\|g(t)\|^2 + c\,,$$

where $y(t) = \|\nabla w(t)\|^2 + \|w(t)\|^2$. Multiplying this last inequality by $e^{\sigma t}$, such that $\sigma < \delta'$, to find that

$$\begin{aligned}
&e^{\sigma t}\frac{d}{dt}y(t) + \delta' e^{\sigma t} y(t) \\
&\le e^{\sigma t}\frac{1}{\varepsilon_1}\|b(t, u_t) - b(t, v_t)\|^2 + e^{\sigma t}\frac{1}{\varepsilon_5}\|b_1(t, v_t)\|^2 + \frac{1}{\varepsilon_2}e^{\sigma t}\|g(t)\|^2 + ce^{\sigma t}\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{d}{dt}(e^{\sigma t}y(t)) &= \sigma e^{\sigma t}y(t) + e^{\sigma t}\frac{d}{dt}y(t) \\
&\le (\sigma - \delta')e^{\sigma t}y(t) + e^{\sigma t}\frac{1}{\varepsilon_1}\|b(t, u_t) - b(t, v_t)\|^2 + e^{\sigma t}\frac{1}{\varepsilon_5}\|b_1(t, v_t)\|^2 \\
&\quad + \frac{1}{\varepsilon_2}e^{\sigma t}\|g(t)\|^2 + ce^{\sigma t}\,.
\end{aligned}$$

Integrating from τ to t, we obtain

$$\begin{aligned}
y(t) &\le e^{-\sigma(t-\tau)}y(\tau) + (\sigma - \delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}y(s)ds \\
&\quad + \frac{1}{\varepsilon_1}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|b(s, u_s) - b(s, v_s)\|^2 ds + \frac{1}{\varepsilon_5}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|b_1(s, v_s)\|^2 ds \\
&\quad + \frac{1}{\varepsilon_2}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + ce^{-\sigma t}\int_\tau^t e^{\sigma s}ds \\
&\le e^{-\sigma(t-\tau)}y(\tau) + (\sigma - \delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}y(s)ds \\
&\quad + \frac{1}{\varepsilon_1}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|b(s, u_s) - b(s, v_s)\|^2 ds + \frac{1}{\varepsilon_5}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|b_1(s, v_s)\|^2 ds \\
&\quad + \frac{1}{\varepsilon_2}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2 ds + c(1 - e^{-\sigma(t-\tau)})\,.
\end{aligned}$$

We use (3.23), (3.60) and (c) to get

$$\begin{aligned}
y(t) &\le e^{-\sigma(t-\tau)}y(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}y(s)ds \\
&\quad + \frac{C_b}{\varepsilon_1}e^{-\sigma t}\int_{\tau-r}^t e^{\sigma s}\|u(s)-v(s)\|^2ds + \frac{C_{b_0}}{\varepsilon_5}e^{-\sigma t}\int_{\tau-r}^t e^{\sigma s}\|v(s)\|^2ds \\
&\quad + \frac{1}{\varepsilon_2}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2ds + c(1-e^{-\sigma(t-\tau)}) \\
&\le e^{-\sigma(t-\tau)}y(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}y(s)ds \\
&\quad + \frac{C_b}{\varepsilon_1}e^{-\sigma t}\int_{\tau-r}^\tau e^{\sigma s}\|w(s)\|^2ds + \frac{C_b}{\varepsilon_1}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|w(s)\|^2ds \\
&\quad + \frac{C_{b_0}}{\varepsilon_5}e^{-\sigma t}\int_{\tau-r}^\tau e^{\sigma s}\|v(s)\|^2ds + \frac{C_{b_0}}{\varepsilon_5}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|v(s)\|^2ds \\
&\quad + \frac{1}{\varepsilon_2}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2ds + c\,.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\tau-r}^t e^{\sigma s}\|w(s)\|^2ds &\le e^{\sigma\tau}\int_{\tau-r}^\tau \|w(s)\|^2ds + \int_\tau^t e^{\sigma s}\|w(s)\|^2ds\,, \\
\int_{\tau-r}^t e^{\sigma s}\|v(s)\|^2ds &\le e^{\sigma\tau}\int_{\tau-r}^\tau \|v(s)\|^2ds + e^{\sigma t}\int_\tau^t \|v(s)\|^2ds\,,
\end{aligned}$$

by (3.3) and (3.64), one obtains

$$\begin{aligned}
y(t) &\le e^{-\sigma(t-\tau)}y(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}y(s)ds \\
&\quad + \frac{C_b}{\varepsilon_1}e^{-\sigma(t-\tau)}\int_{\tau-r}^\tau \|w(s)\|^2ds + \frac{C_b}{\varepsilon_1\lambda_1}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|\nabla w(s)\|^2ds \\
&\quad + \frac{C_{b_0}}{\varepsilon_5}e^{-\sigma(t-\tau)}\int_{\tau-r}^\tau \|v(s)\|^2ds + \frac{C_{b_0}}{\varepsilon_5\lambda_1}\int_\tau^t \|\nabla v(s)\|^2ds \\
&\quad + \frac{1}{\varepsilon_2}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2ds + c \\
&\le e^{-\sigma(t-\tau)}y(\tau) + (\sigma-\delta')e^{-\sigma t}\int_\tau^t e^{\sigma s}y(s)ds \\
&\quad + \frac{C_b}{\varepsilon_1}e^{-\sigma(t-\tau)}\int_{\tau-r}^\tau \|w(s)\|^2ds + \frac{C_b}{\varepsilon_1\lambda_1}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|\nabla w(s)\|^2ds \\
&\quad + \frac{C_{b_0}}{\varepsilon_5}e^{-\sigma(t-\tau)}\int_{\tau-r}^\tau \|v(s)\|^2ds + \frac{1}{\varepsilon_2}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|g(s)\|^2ds + c \\
&\quad + \frac{C_{b_0}}{\varepsilon_5\lambda_1}\eta_1^{-1}\Big\{\|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^\tau \|v(s)\|^2ds + 2c_2|\Omega|(T-\tau)\Big\}\,.
\end{aligned}$$

For $\mu' := \delta' - \sigma - \frac{C_b}{\varepsilon_1\lambda_1} > 0$, one has

$$\|\nabla w(t)\|^2 + \|w(t)\|^2 + \mu' e^{-\sigma t}\int_\tau^t e^{\sigma s}\|\nabla w(s)\|^2ds$$

$$
\begin{aligned}
&\le e^{-\sigma(t-\tau)}(\|\nabla w(\tau)\|^2+\|w(\tau)\|^2+\frac{C_b}{\varepsilon_1}\int_{\tau-r}^{\tau}\|w(s)\|^2ds+\frac{C_{b_0}}{\varepsilon_5}\int_{\tau-r}^{\tau}\|v(s)\|^2ds)\\
&\quad+\frac{C_{b_0}}{\varepsilon_5\lambda_1}\eta_1^{-1}\Big\{\|v(\tau)\|^2+\|\nabla v(\tau)\|^2+\frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^{\tau}\|v(s)\|^2ds+2c_2|\Omega|(T-\tau)\Big\}\\
&\quad+\frac{1}{\varepsilon_2}e^{-\sigma t}\int_{\tau}^{t}e^{\sigma s}\|g(s)\|^2ds+c\,,
\end{aligned}
\tag{3.70}
$$

whereupon

$$
\begin{aligned}
&\|\nabla w(t)\|^2+\|w(t)\|^2\\
&\le e^{-\sigma(t-\tau)}\Big(\|\nabla w(\tau)\|^2+\|w(\tau)\|^2+\frac{C_b}{\varepsilon_1}\int_{\tau-r}^{\tau}\|w(s)\|^2ds+\frac{C_{b_0}}{\varepsilon_5}\int_{\tau-r}^{\tau}\|v(s)\|^2ds\Big)\\
&\quad+\frac{C_{b_0}}{\varepsilon_5\lambda_1}\eta_1^{-1}\Big\{\|v(\tau)\|^2+\|\nabla v(\tau)\|^2+\frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^{\tau}\|v(s)\|^2ds+2c_2|\Omega|(T-\tau)\Big\}\\
&\quad+\frac{1}{\varepsilon_2}e^{-\sigma t}\int_{\tau}^{t}e^{\sigma s}\|g(s)\|^2ds+c\,,
\end{aligned}
\tag{3.71}
$$

and

$$
\begin{aligned}
&\mu' e^{-\sigma t}\int_{\tau}^{t}e^{\sigma s}\|\nabla w(s)\|^2ds\\
&\le e^{-\sigma(t-\tau)}(\|\nabla w(\tau)\|^2+\|w(\tau)\|^2+\frac{C_b}{\varepsilon_1}\int_{\tau-r}^{\tau}\|w(s)\|^2ds+\frac{C_{b_0}}{\varepsilon_5}\int_{\tau-r}^{\tau}\|v(s)\|^2ds)\\
&\quad+\frac{C_{b_0}}{\varepsilon_5\lambda_1}\eta_1^{-1}\Big\{\|v(\tau)\|^2+\|\nabla v(\tau)\|^2+\frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^{\tau}\|v(s)\|^2ds+2c_2|\Omega|(T-\tau)\Big\}\\
&\quad+\frac{1}{\varepsilon_2}e^{-\sigma t}\int_{\tau}^{t}e^{\sigma s}\|g(s)\|^2ds+c\,,
\end{aligned}
\tag{3.72}
$$

For $t-r\ge\tau$, we have

$$
\int_{\tau}^{t}e^{\sigma s}\|\nabla w(s)\|^2ds\ge\int_{t-r}^{t}e^{\sigma s}\|\nabla w(s)\|^2ds\ge e^{\sigma(t-r)}\int_{t-r}^{t}\|\nabla w(s)\|^2ds\,.
$$

So, by this inequality, (3.72) becomes

$$
\begin{aligned}
&\int_{t-r}^{t}\|\nabla w(s)\|^2ds\\
&\le\mu'^{-1}e^{-\sigma(t-\tau-r)}\Big(\|\nabla w(\tau)\|^2+\|w(\tau)\|^2+\frac{C_b}{\varepsilon_1}\int_{\tau-r}^{\tau}\|w(s)\|^2ds\\
&\quad+\frac{C_{b_0}}{\varepsilon_5}\int_{\tau-r}^{\tau}\|v(s)\|^2ds\Big)+\frac{\mu'^{-1}\eta_1^{-1}C_{b_0}}{\varepsilon_5\lambda_1}e^{\sigma r}\Big\{\|v(\tau)\|^2\\
&\quad+\|\nabla v(\tau)\|^2+\frac{C_{b_0}}{\varepsilon_4}\int_{\tau-r}^{\tau}\|v(s)\|^2ds+2c_2|\Omega|(T-\tau)\Big\}\\
&\quad+\frac{\mu'^{-1}}{\varepsilon_2}e^{-\sigma(t-r)}\int_{\tau}^{t}e^{\sigma s}\|g(s)\|^2ds+ce^{\sigma r}\,.
\end{aligned}
\tag{3.73}
$$

We add (3.71) and (3.73), and we use (3.24) to obtain

$$
\begin{aligned}
&\|\nabla w(t)\|^2 + \int_{t-r}^{t} \|\nabla w(s)\|^2 ds \\
&\le \|\nabla w(t)\|^2 + \|w(t)\|^2 + \int_{t-r}^{t} \|\nabla w(s)\|^2 ds \\
&\le \Big\{ e^{-\sigma(t-\tau)} \Big(\|\nabla w(\tau)\|^2 + \|w(\tau)\|^2 + \frac{C_b}{\varepsilon_1} \int_{\tau-r}^{\tau} \|w(s)\|^2 ds \\
&\quad + \frac{C_{b_0}}{\varepsilon_5} \int_{\tau-r}^{\tau} \|v(s)\|^2 ds \Big) + \frac{\eta_1^{-1} C_{b_0}}{\varepsilon_5 \lambda_1} \Big\{ \|v(\tau)\|^2 \\
&\quad + \|\nabla v(\tau)\|^2 + \frac{C_{b_0}}{\varepsilon_4} \int_{\tau-r}^{\tau} \|v(s)\|^2 ds + 2c_2 |\Omega| (T-\tau) \Big\} \\
&\quad + \big(\frac{1}{\varepsilon_2} e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma s} \|g(s)\|^2 ds + c \big) \Big\} (1 + \mu'^{-1} e^{\sigma r}) .
\end{aligned} \tag{3.74}
$$

Then, (3.74) shows that for all $\varepsilon > 0$, $\tau \le \tau_0$ and all $(u(t), u_t) \in U(t,\tau)B(\tau)$, one has

$$
\|(I-P)(u(t), u_t)\|_{X_1}^2 \le \varepsilon^2 .
$$

Finally, by considering the ordinary functional differential system (3.61), we have

$$
\|\Delta v\|^2 \le \lambda_m \|\nabla v\|^2 \le \lambda_m^2 \|v\|^2 , \tag{3.75}
$$

as

$$
\|\nabla v\|^2 = \langle \Delta v, v \rangle = \langle \sum_{i=1}^{m} \langle v, w_i \rangle \lambda_i w_i, \sum_{j=1}^{m} \langle v, w_j \rangle w_j \rangle = \sum_{i=1}^{m} \lambda_i \langle v, w_i \rangle^2 ,
$$

and

$$
\begin{aligned}
\|\Delta v\|^2 &= \langle \Delta v, \Delta v \rangle \\
&= \langle \sum_{i=1}^{m} \langle v, w_i \rangle \lambda_i w_i, \sum_{j=1}^{m} \langle v, w_j \rangle \lambda_j w_j \rangle \\
&= \sum_{i=1}^{m} \lambda_i^2 \langle v, w_i \rangle^2 \le \lambda_m \sum_{i=1}^{m} \lambda_i \langle v, w_i \rangle^2 .
\end{aligned} \tag{3.76}
$$

Now, we check the equicontinuity property of the solutions $\{v(\cdot)\}$ in the space $L^2([t-r,t]; L^2(\Omega))$. Then, for any $t_1 \in [t-r,t]$, any $l \in \mathbb{R}^+$ with $l < \delta$ and for $[t_1, t_1+l] \subset [t-r,t]$, we have

$$
\begin{aligned}
\|T_l v(t_1) - v(t_1)\| &= \|v(t_1+l) - v(t_1)\| \\
&\le \int_0^1 \|\frac{dv(t_1+s)}{dt_1}\| ds \\
&\le \int_0^1 \|\Delta \frac{d}{dt_1} v(t_1+s)\| ds + \int_0^1 \|\Delta v(t_1+s)\| ds \\
&\quad + \int_0^1 \|f_0(v)\| ds + \int_0^1 \|b_0(t_1+s, v_{t_1+s})\| ds ,
\end{aligned}
$$

and so

$$
\|v(t_1+l) - v(t_1)\|^2 \le \Big(\int_0^1 \|\Delta \frac{d}{dt_1} v(t_1+s)\| ds + \int_0^1 \|\Delta v(t_1+s)\| ds
$$

$$+ \int_0^1 \|f_0(v)\|ds + \int_0^1 \|b_0(t_1+s, v_{t_1+s})\|ds\Big)^2.$$

Consequently,

$$\begin{aligned}
&\|v(t_1+l) - v(t_1)\|^2 \\
&\le 2\Big(\int_0^1 \|\Delta \frac{d}{dt_1} v(t_1+s)\|ds + \int_0^1 \|\Delta v(t_1+s)\|ds\Big)^2 \\
&\quad + 2\Big(\int_0^1 \|f_0(v)\|ds + \int_0^1 \|b_0(t_1+s, v_{t_1+s})\|ds\Big)^2 \\
&\le 4\Big(\int_0^1 \|\Delta \frac{d}{dt_1} v(t_1+s)\|ds\Big)^2 + 4\Big(\int_0^1 \|\Delta v(t_1+s)\|ds\Big)^2 \\
&\quad + 4\Big(\int_0^1 \|f_0(v)\|ds\Big)^2 + 4\Big(\int_0^1 \|b_0(t_1+s, v_{t_1+s})\|ds\Big)^2
\end{aligned}$$

By Holder inequality,

$$\begin{aligned}
&\|v(t_1+l) - v(t_1)\|^2 \\
&\le 4l \int_0^1 \|\Delta \frac{d}{dt_1} v(t_1+s)\|^2 ds + 4l \int_0^1 \|\Delta v(t_1+s)\|^2 ds \\
&\quad + 4l \int_0^1 \|f_0(v)\|^2 ds + 4l \int_0^1 \|b_0(t_1+s, v_{t_1+s})\|^2 ds \\
&\le 4l\Big(\int_0^1 \|\Delta \frac{d}{dt_1} v(t_1+s)\|^2 ds + \int_0^1 \|\Delta v(t_1+s)\|^2 ds + \int_0^1 \|f_0(v)\|^2 ds \\
&\quad + \int_0^1 \|b_0(t_1+s, v_{t_1+s})\|^2 ds\Big),
\end{aligned}$$

which after integration over $[t-r, t]$ leads to

$$\begin{aligned}
&\int_{t-r}^t \|v(t_1+l) - v(t_1)\|^2 dt_1 \\
&\le 4l \int_{t-r}^t \Big(\int_0^1 \|\Delta \frac{d}{dt_1} v(t_1+s)\|^2 ds + \int_0^1 \|\Delta v(t_1+s)\|^2 ds \\
&\quad + \int_0^1 \|f_0(v)\|^2 ds + \int_0^1 \|b_0(t_1+s, v_{t_1+s})\|^2 ds\Big) dt_1 \\
&\le 4l\Big(\int_0^1 \int_{t-r}^t \|\Delta \frac{d}{dt_1} v(t_1+s)\|^2 dt_1 ds + \int_0^1 \int_{t-r}^t \|\Delta v(t_1+s)\|^2 dt_1 ds \\
&\quad + \int_0^1 \int_{t-r}^t \|f_0(v)\|^2 dt_1 ds + \int_0^1 \int_{t-r}^t \|b_0(t_1+s, v_{t_1+s})\|^2 dt_1 ds\Big). \qquad (3.77)
\end{aligned}$$

Next we will estimate the five terms on the right-hand side of the equation.

By (3.75), we have

$$\int_{t-r}^t \|\Delta v(t_1+s)\|^2 dt_1 \le \lambda_m \int_{t-r}^t \|\nabla v(t_1+s)\|^2 dt_1.$$

From (3.25), one has

$$\int_{t-r}^t \|\nabla v(t_1+s)\|^2 dt_1 \le c\|(u^0, \varphi)\|_X^{\frac{2N}{N-2}} \int_{t-r}^t e^{-\sigma(t_1+s-\tau)} dt_1$$

$$
\begin{aligned}
&+ c\int_{t-r}^{t}\Big(1+e^{-\sigma(t_1+s)}\int_{-\infty}^{t_1+s}e^{\sigma s'}\|g(s')\|^2ds'\Big)dt_1\\
&\le \frac{c}{\sigma}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}\Big(e^{-\sigma(t-r+s-\tau)}-e^{-\sigma(t+s-\tau)}\Big)+cr\\
&\quad+\frac{c}{\sigma}\Big(e^{-\sigma(t-r+s)}-e^{-\sigma(t+s)}\Big)\int_{-\infty}^{t}e^{\sigma s'}\|g(s')\|^2ds'\\
&\le \frac{c}{\sigma}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}e^{-\sigma(t-r+s-\tau)}\\
&\quad+\frac{c}{\sigma}e^{-\sigma(t-r+s)}\int_{-\infty}^{t}e^{\sigma s'}\|g(s')\|^2ds'+cr\,.
\end{aligned}\tag{3.78}
$$

Integrating over $[0,l]$, we obtain

$$
\begin{aligned}
&\int_0^1\int_{t-r}^{t}\|\nabla v(t_1+s)\|^2dt_1ds\\
&\le \frac{c}{\sigma^2}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}e^{-\sigma(t-r-\tau)}(1-e^{-\sigma l})\\
&\quad+\frac{c}{\sigma^2}e^{-\sigma(t-r)}(1-e^{-\sigma l})\int_{-\infty}^{t}e^{\sigma s}\|g(s)\|^2ds+crl\,.
\end{aligned}\tag{3.79}
$$

By (3.75) and (3.79), we have

$$
\begin{aligned}
&\int_0^1\int_{t-r}^{t}\|\Delta v(t_1+s)\|^2dt_1ds\le\lambda_m\frac{c}{\sigma^2}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}e^{-\sigma(t-r-\tau)}(1-e^{-\sigma l})\\
&+\lambda_m\frac{c}{\sigma^2}e^{-\sigma(t-r)}(1-e^{-\sigma l})\int_{-\infty}^{t}e^{\sigma s}\|g(s)\|^2ds+\lambda_m crl\to 0 \text{ as } l\to 0\,.
\end{aligned}\tag{3.80}
$$

From (a) and (3.59), we obtain

$$
\int_{t-r}^{t}\|b_0(t_1+s,v_{t_1+s})\|^2dt_1\le C_{b_0}\int_{t-2r}^{t}\|v(t_1+s)\|^2dt_1\,.
$$

Using (3.3), one has

$$
\int_{t-r}^{t}\|b_0(t_1+s,v_{t_1+s})\|^2dt_1\le C_{b_0}\lambda_1^{-1}\int_{t-2r}^{t}\|\nabla v(t_1+s)\|^2dt_1\,.
$$

By (3.25), one gets

$$
\begin{aligned}
&\int_{t-r}^{t}\|b_0(t_1+s,v_{t_1+s})\|^2dt_1\\
&\le C_{b_0}\lambda_1^{-1}c\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}\int_{t-2r}^{t}e^{-\sigma(t_1+s-\tau)}dt_1\\
&\quad+C_{b_0}\lambda_1^{-1}c\int_{t-2r}^{t}dt_1+C_{b_0}\lambda_1^{-1}c\int_{t-2r}^{t}e^{-\sigma(t_1+s)}\int_{-\infty}^{t_1+s}e^{\sigma s'}\|g(s')\|^2ds'dt_1\\
&\le C_{b_0}\lambda_1^{-1}\frac{c}{\sigma}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}(e^{-\sigma(t-2r+s-\tau)}-e^{-\sigma(t+s-\tau)})\\
&\quad+2C_{b_0}\lambda_1^{-1}cr+C_{b_0}\lambda_1^{-1}\frac{c}{\sigma}\Big(e^{-\sigma(t-2r+s)}-e^{-\sigma(t+s)}\Big)\int_{-\infty}^{t}e^{\sigma s'}\|g(s')\|^2ds'\\
&\le C_{b_0}\lambda_1^{-1}\frac{c}{\sigma}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}e^{-\sigma(t-2r+s-\tau)}
\end{aligned}
$$

$$+ 2C_{b_0}\lambda_1^{-1}cr + C_{b_0}\lambda_1^{-1}\frac{c}{\sigma}e^{-\sigma(t-2r+s)}\int_{-\infty}^{t} e^{\sigma s'}\|g(s')\|^2 ds' . \tag{3.81}$$

Integrating over $[0, l]$, we obtain

$$\begin{aligned}
&\int_0^1\int_{t-r}^{t}\|b_0(t_1+s, v_{t_1+s})\|^2 dt_1 ds \\
&\le C_{b_0}\lambda_1^{-1}\frac{c}{\sigma}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}\int_0^1 e^{-\sigma(t-2r+s-\tau)}ds \\
&\quad + C_{b_0}\lambda_1^{-1}\frac{c}{\sigma}\int_0^1 e^{-\sigma(t-2r+s)}\int_{-\infty}^{t} e^{\sigma s'}\|g(s')\|^2 ds' ds + 2C_{b_0}\lambda_1^{-1}cr\int_0^1 ds \\
&\le C_{b_0}\lambda_1^{-1}\frac{c}{\sigma^2}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}e^{-\sigma(t-2r-\tau)}(1-e^{-\sigma l}) \\
&\quad + C_{b_0}\lambda_1^{-1}\frac{c}{\sigma^2}e^{-\sigma(t-2r)}(1-e^{-\sigma l})\int_{-\infty}^{t} e^{\sigma s'}\|g(s')\|^2 ds' + 2C_{b_0}\lambda_1^{-1}crl \\
&\to 0 \quad \text{as } l \to 0 .
\end{aligned} \tag{3.82}$$

Now, it is clear from (3.57) that

$$\|f_0(v)\|^2 \le \int_\Omega k^2(1+|v|^\alpha)^2 dx .$$

By (3.3) and the fact that $2\alpha < \frac{4N}{N-2}$, one has

$$\begin{aligned}
\|f_0(v)\|^2 &\le \int_\Omega 2k^2(1+|v|^{2\alpha})dx \\
&\le 2k^2|\Omega| + 2k^2\|v(t_1+s)\|^{2\alpha} \\
&\le 2k^2|\Omega| + 2k^2\tilde{\mu}\lambda_1^{-1}\|\nabla v(t_1+s)\|^{\frac{4N}{N-2}} .
\end{aligned} \tag{3.83}$$

By (3.25), we have

$$\begin{aligned}
\|\nabla v(t_1+s)\|^{\frac{4N}{N-2}} \le \Big\{&ce^{-\sigma(t_1+s-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} \\
&+ c(1+e^{-\sigma(t_1+s)}\int_{-\infty}^{t_1+s} e^{\sigma(s')}\|g(s')\|^2 ds')\Big\}^{\frac{2N}{N-2}} .
\end{aligned}$$

Similarly, one obtains

$$\begin{aligned}
&\|\nabla v(t_1+s)\|^{\frac{4N}{N-2}} \\
&\le 2^{\frac{2N}{N-2}-1}\Big(ce^{-\sigma(t_1+s-\tau)}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}}\Big)^{\frac{2N}{N-2}} \\
&\quad + 2^{\frac{2N}{N-2}-1}c^{\frac{2N}{N-2}}\Big(1+e^{-\sigma(t_1+s)}\int_{-\infty}^{t_1+s} e^{\sigma s'}\|g(s')\|^2 ds'\Big)^{\frac{2N}{N-2}} \\
&\le 2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}e^{-\sigma(t_1+s-\tau)\frac{2N}{N-2}}\|(u^0,\varphi)\|_X^{(\frac{2N}{N-2})^2} + 2^{\frac{2N+4}{N-2}}c^{\frac{2N}{N-2}} \\
&\quad + 2^{\frac{2N+4}{N-2}}c^{\frac{2N}{N-2}}e^{-\sigma(t_1+s)\frac{2N}{N-2}}\Big(\int_{-\infty}^{t_1+s} e^{\sigma s'}\|g(s')\|^2 ds'\Big)^{\frac{2N}{N-2}} .
\end{aligned} \tag{3.84}$$

Hence by (3.83) and (3.84), one obtains

$$\int_{t-r}^{t}\|f_0(v)\|^2 dt_1$$

$$
\begin{aligned}
&\le 2k^2|\Omega|\int_{t-r}^{t} 1dt_1 + 2k^2\tilde{\mu}\lambda_1^{-1}\int_{t-r}^{t}\|\nabla v(t_1+s)\|^{\frac{4N}{N-2}}dt_1\\
&\le 2\frac{k^2}{\sigma}\tilde{\mu}\lambda_1^{-1}2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}\big(-e^{-\sigma(t+s-\tau)\frac{2N}{N-2}}+e^{-\sigma(t-r+s-\tau)\frac{2N}{N-2}}\big)\|(u^0,\varphi)\|_X^{(\frac{2N}{N-2})^2}\\
&\quad+2\frac{k^2}{\sigma}\tilde{\mu}\lambda_1^{-1}2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}\big(e^{-\sigma(t-r+s)\frac{2N}{N-2}}-e^{-\sigma(t+s)\frac{2N}{N-2}}\big)\\
&\quad\times\Big(\int_{-\infty}^{t}e^{\sigma s'}\|g(s')\|^2ds'\Big)^{\frac{2N}{N-2}}+2k^2|\Omega|r+2^{\frac{3N+2}{N-2}}c^{\frac{2N}{N-2}}k^2\tilde{\mu}\lambda_1^{-1}r\\
&\le 2\frac{k^2}{\sigma}\tilde{\mu}\lambda_1^{-1}2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}e^{-\sigma(t-r+s-\tau)\frac{2N}{N-2}}\|(u^0,\varphi)\|_X^{(\frac{2N}{N-2})^2}\\
&\quad+2\frac{k^2}{\sigma}\tilde{\mu}\lambda_1^{-1}2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}e^{-\sigma(t-r+s)\frac{2N}{N-2}}\Big(\int_{-\infty}^{t}e^{\sigma s'}\|g(s')\|^2ds'\Big)^{\frac{2N}{N-2}}\\
&\quad+2k^2|\Omega|r+2^{\frac{3N+2}{N-2}}c^{\frac{2N}{N-2}}k^2\tilde{\mu}\lambda_1^{-1}r\,,
\end{aligned}
$$

which integrated from 0 to l gives

$$
\begin{aligned}
&\int_0^1\int_{t-r}^{t}\|f_0(v)\|^2dt_1ds\\
&\le 2\frac{k^2}{\sigma}\tilde{\mu}\lambda_1^{-1}2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}\|(u^0,\varphi)\|_X^{(\frac{2N}{N-2})^2}\int_0^1 e^{-\sigma(t-r+s-\tau)\frac{2N}{N-2}}ds\\
&\quad+2\frac{k^2}{\sigma}\tilde{\mu}\lambda_1^{-1}2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}\Big(\int_{-\infty}^{t}e^{\sigma s'}\|g(s')\|^2ds'\Big)^{\frac{2N}{N-2}}\int_0^1 e^{-\sigma(t-r+s)\frac{2N}{N-2}}ds\\
&\quad+\int_0^1(2k^2|\Omega|r+2^{\frac{3N+2}{N-2}}c^{\frac{2N}{N-2}}k^2\tilde{\mu}\lambda_1^{-1}r)\,.ds
\end{aligned}
$$

Then, we find

$$
\begin{aligned}
&\int_0^1\int_{t-r}^{t}\|f_0(v)\|^2dt_1ds\\
&\le 2\frac{k^2}{\sigma^2}\tilde{\mu}\lambda_1^{-1}2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}\|(u^0,\varphi)\|_X^{(\frac{2N}{N-2})^2}e^{-\sigma(t-r-\tau)\frac{2N}{N-2}}(1-e^{-\sigma l\frac{2N}{N-2}})\\
&\quad+2\frac{k^2}{\sigma^2}\tilde{\mu}\lambda_1^{-1}2^{\frac{N+2}{N-2}}c^{\frac{2N}{N-2}}\Big(\int_{-\infty}^{t}e^{\sigma s'}\|g(s')\|^2ds'\Big)^{\frac{2N}{N-2}}e^{-\sigma(t-r)\frac{2N}{N-2}}(1-e^{-\sigma l\frac{2N}{N-2}})\\
&\quad+2k^2|\Omega|rl+2^{\frac{3N+2}{N-2}}c^{\frac{2N}{N-2}}k^2\tilde{\mu}\lambda_1^{-1}rl\to 0\quad\text{as } l\to 0\,.
\end{aligned}
\tag{3.85}
$$

It remains to estimate $\int_0^1\int_{t-r}^{t}\|\Delta\frac{d}{dt_1}v(t_1+s)\|^2dt_1ds$. To this end, we take the inner product in $L^2(\Omega)$ of (3.61) with $-\Delta\frac{\partial}{\partial t_1}v$, we find

$$
\begin{aligned}
&\|\nabla\frac{d}{dt_1}v(t_1+s)\|^2+\|\Delta\frac{d}{dt_1}v(t_1+s)\|^2+\frac{1}{2}\frac{d}{dt_1}\|\Delta v(t_1+s)\|^2\\
&\le\int_\Omega f_0(v)\Delta\frac{\partial}{\partial t_1}v+\int_\Omega b_0(t_1+s,v_{t_1+s})(-\Delta\frac{\partial}{\partial t_1}v)\,.
\end{aligned}
\tag{3.86}
$$

We have

$$
\int_\Omega f_0(v)\Delta\frac{\partial}{\partial t_1}v=-\int_\Omega f_0'(v)\nabla v\nabla\frac{\partial}{\partial t_1}v\,.
$$

By (3.56), it follows that

$$\int_\Omega f_0(v)\Delta\frac{\partial}{\partial t_1}v \le c_3\int_\Omega |\nabla v|\big|\nabla\frac{\partial}{\partial t_1}v\big| \le c_3\|\nabla v\|\big\|\nabla\frac{\partial}{\partial t_1}v\big\|\,.$$

Using the Young inequality, one has

$$\int_\Omega f_0(v)\Delta\frac{\partial}{\partial t_1}v \le \frac{c_3^2}{2}\|\nabla v(t_1+s)\|^2 + \frac{1}{2}\|\nabla\frac{d}{dt_1}v(t_1+s)\|^2\,. \tag{3.87}$$

By (3.87) and (3.86), one finds

$$\begin{aligned}
&\|\nabla\frac{d}{dt_1}v(t_1+s)\|^2 + \|\Delta\frac{d}{dt_1}v(t_1+s)\|^2 + \frac{1}{2}\frac{d}{dt_1}\|\Delta v(t_1+s)\|^2\\
&\le \frac{c_3^2}{2}\|\nabla v(t_1+s)\|^2 + \frac{1}{2}\|\nabla\frac{d}{dt_1}v(t_1+s)\|^2 + \int_\Omega b_0(t_1+s,v_{t_1+s})(-\Delta\frac{\partial}{\partial t_1}v)\,.
\end{aligned}$$

Which via the Cauchy inequality gives

$$\begin{aligned}
&\|\nabla\frac{d}{dt_1}v(t_1+s)\|^2 + \|\Delta\frac{d}{dt_1}v(t_1+s)\|^2 + \frac{1}{2}\frac{d}{dt_1}\|\Delta v(t_1+s)\|^2\\
&\le \frac{c_3^2}{2}\|\nabla v(t_1+s)\|^2 + \frac{1}{2}\|\nabla\frac{d}{dt_1}v(t_1+s)\|^2 + \frac{1}{2\nu_1}\|b_0(t_1+s,v_{t_1+s})\|^2\\
&\quad + \frac{\nu_1}{2}\|\Delta\frac{d}{dt_1}v(t_1+s)\|^2\,.
\end{aligned}$$

So, one has

$$\begin{aligned}
&\|\Delta\frac{d}{dt_1}v(t_1+s)\|^2\\
&\le \frac{1}{2}\|\nabla\frac{d}{dt_1}v(t_1+s)\|^2 + \|\Delta\frac{d}{dt_1}v(t_1+s)\|^2 + \frac{1}{2}\frac{d}{dt_1}\|\Delta v(t_1+s)\|^2\\
&\le \frac{c_3^2}{2}\|\nabla v(t_1+s)\|^2 + \frac{1}{2\nu_1}\|b_0(t_1+s,v_{t_1+s})\|^2 + \frac{\nu_1}{2}\|\Delta\frac{d}{dt_1}v(t_1+s)\|^2\,.
\end{aligned}$$

Therefore, one gets

$$(1-\frac{\nu_1}{2})\|\Delta\frac{d}{dt_1}v(t_1+s)\|^2 \le \frac{c_3^2}{2}\|\nabla v(t_1+s)\|^2 + \frac{1}{2\nu_1}\|b_0(t_1+s,v_{t_1+s})\|^2\,.$$

For ν_1 small enough, we have $\nu_2 := 1-\frac{\nu_1}{2} > 0$. So we can write

$$\|\Delta\frac{d}{dt_1}v(t_1+s)\|^2 \le \frac{c_3^2}{2\nu_2}\|\nabla v(t_1+s)\|^2 + \frac{1}{2\nu_1\nu_2}\|b_0(t_1+s,v_{t_1+s})\|^2\,.$$

Which integrated over $[t-r,t]$ leads to

$$\begin{aligned}
&\int_{t-r}^t \|\Delta\frac{d}{dt_1}v(t_1+s)\|^2dt_1\\
&\le \frac{c_3^2}{2\nu_2}\int_{t-r}^t \|\nabla v(t_1+s)\|^2dt_1 + \frac{1}{2\nu_1\nu_2}\int_{t-r}^t \|b_0(t_1+s,v_{t_1+s})\|^2dt_1\,;
\end{aligned}$$

integrating the above inequality over $[0,l]$, one obtains

$$\begin{aligned}
&\int_0^1\int_{t-r}^t \|\Delta\frac{d}{dt_1}v(t_1+s)\|^2dt_1ds\\
&\le \frac{c_3^2}{2\nu_2}\int_0^1\int_{t-r}^t \|\nabla v(t_1+s)\|^2dt_1ds + \frac{1}{2\nu_1\nu_2}\int_0^1\int_{t-r}^t \|b_0(t_1+s,v_{t_1+s})\|^2dt_1ds\,.
\end{aligned}$$

By (3.79) and (3.82), it follows that

$$\begin{aligned} &\int_0^1\int_{t-r}^t \|\Delta \frac{d}{dt_1} v(t_1+s)\|^2 dt_1 ds \\ &\le \frac{c_3^2}{2\nu_3}\frac{c}{\sigma^2}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} e^{-\sigma(t-r-\tau)}(1-e^{-\sigma l}) \\ &\quad + \frac{c_3^2}{2\nu_2}\frac{c}{\sigma^2}e^{-\sigma(t-r)}(1-e^{-\sigma l})\int_{-\infty}^t e^{\sigma s}\|g(s)\|^2 ds + \frac{c_3^2}{2\nu_2}crl \\ &\quad + \frac{1}{2\nu_1\nu_2}C_{b_0}\lambda_1^{-1}\frac{c}{\sigma^2}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} e^{-\sigma(t-2r-\tau)}(1-e^{-\sigma l}) \\ &\quad + \frac{1}{2\nu_1\nu_2}C_{b_0}\lambda_1^{-1}\frac{c}{\sigma^2}e^{-\sigma(t-2r)}(1-e^{-\sigma l})\int_{-\infty}^t e^{\sigma s}\|g(s)\|^2 ds + \frac{1}{\nu_1\nu_2}C_{b_0}\lambda_1^{-1}crl\,. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\int_0^1\int_{t-r}^t \|\Delta \frac{d}{dt_1} v(t_1+s)\|^2 dt_1 ds \\ &\le (\frac{c_3^2}{2\nu_2} + \frac{1}{2\nu_1\nu_2}C_{b_0}\lambda_1^{-1}e^{\sigma r})\frac{c}{\sigma^2}\|(u^0,\varphi)\|_X^{\frac{2N}{N-2}} e^{-\sigma(t-r-\tau)}(1-e^{-\sigma l}) \\ &\quad + (\frac{c_3^2}{2\nu_2} + \frac{1}{2\nu_1\nu_2}C_{b_0}\lambda_1^{-1}e^{\sigma r})\frac{c}{\sigma^2}e^{-\sigma(t-r)}(1-e^{-\sigma l})\int_{-\infty}^t e^{\sigma s}\|g(s)\|^2 ds \\ &\quad + \frac{c_3^2}{2\nu_2}crl + \frac{1}{\nu_1\nu_2}C_b\lambda_1^{-1}crl \to 0 \quad \text{as } l\to 0\,. \end{aligned} \tag{3.88}$$

Comprehensively, from (3.77), (3.80), (3.82), (3.85) and (3.88), we have

$$\int_{t-r}^t \|v(t_1+l)-v(t_1)\|^2 dt_1 \to 0 \quad \text{as } l\to 0\,,$$

which implies the needed equicontinuity. This shows that the condition (3) in Proposition 3.9 holds, and thus the process on X is pullback w-$\mathcal{D}$-limit compact. Then from Lemma 3.8 and Theorem 2.8, we conclude the existence of a pullback $\mathcal{D}$-attractor which completes the proof. □

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