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ROBIN BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH SUBCRITICAL AND CRITICAL NONLINEARITIES

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Abstract. By using variational methods we study the existence of positive solutions for a class of quasilinear elliptic problems with Robin boundary conditions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. In this article we study the nonlinear Robin problem:

$$\begin{gathered}
-\Delta_p u = \lambda |u|^{p-2}u + a(x)|u|^{q-2}u \quad \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \eta} + b(x)|u|^{p-2}u = \mu\rho(x)|u|^{r-2}u \quad \text{on } \partial\Omega,
\end{gathered}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, denotes the p-Laplace operator, $\frac{\partial u}{\partial \eta}(x)$ denotes the outward unit normal at $x \in \partial\Omega$, λ, μ are parameters, $\mu > 0$, $a : \Omega \to \mathbb{R}$, $b, \rho : \partial\Omega \to \mathbb{R}$ are essentially bounded functions, with $b(x) \geq 0$ and $mx \in \partial\Omega : b(\cdot) > 0\} > 0$. Restrictions on q, r are given in the subsequent sections. With respect to the parameter μ, we notice that its role is crucial in the critical case examined in Section 3.

Quasilinear problems of the form $-\Delta_p u = f(x, u)$ with Dirichlet boundary conditions have received considerable attention; see [2, 8, 16, 20, 23]. This equation with Neumann boundary conditions (i.e. $b(\cdot) \equiv 0$ and $\rho(\cdot) \equiv 0$) and $a(\cdot)$ being a constant has been studied in [4], where existence of solutions has been provided for $\lambda \in (0, \lambda^*)$, for a suitable $\lambda^* > 0$. The same authors in [3] provide positive solutions to the aforementioned problem but with a critical term added to the right hand side of (1). In [5] the existence of solutions is proved for (1)-(1) when λ appears on the boundary condition, $a(\cdot) \equiv 0$, and r can be subcritical, critical or supercritical. Multiplicity of solutions is examined in [18] where the right hand side of (1) is a real Carathéodory function $f(x, u, \lambda)$ defined on $\Omega \times \mathbb{R} \times (0, +\infty)$ and the boundary condition is Neumann. Multiplicity of solutions is also proved in [17] for $\lambda > \lambda_2$,

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for λ_2 being the second eigenvalue of the p-Laplacian operator with Robin boundary conditions, while in [19] existence of positive solutions is shown for $\lambda < \lambda_1$. Existence of solutions depending on the Fučik spectrum of the p-Laplace operator is examined in [24]. When Ω is an exterior domain, existence and nonexistence of solutions is examined in [10]. In case the potential is nonsmooth we refer to [11]. The fibering method, attributed to Pohozaev, is useful when the right hand sides of the equation and the boundary condition are power-like, see [7], [21]. For systems of equations the interested reader may see [6].

Our aim in this work is to provide existence results concerning positive solutions to (1)-(1) when q is either subcritical or critical, r is subcritical and $\lambda \le \lambda_1$, where λ_1 is the first eigenvalue of the associated eigenvalue problem. When the exponents are subcritical, our proofs rely on the fibering method and the mountain pass theorem developed in Ambrosetti-Rabinowitz [1], while in the case of q being critical we use the concentration-compactness principle of Lions [13, 14]. A useful survey of results concerning the mountain pass theorem is provided in [22].

As usual $X := W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{1,p} = \Big(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx\Big)^{1/p}.$$

The action functional $I(\cdot)$ corresponding to problem (1)-(1) is defined on X by

$$I_\lambda(u) = \frac{1}{p}\Big[\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega |u|^p dx + \int_{\partial\Omega} b(x)|u|^p d\sigma\Big] - \frac{1}{q}A(u) - \frac{\mu}{r}P(u),$$

where $P(u) := \int_{\partial\Omega} \rho(x)|u|^r d\sigma$ and $A(u) := \int_\Omega a(x)|u|^q dx$.

Consider the eigenvalue problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u \quad \text{in } \Omega, \tag{1.1}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \eta} + b(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

It is known that the smallest eigenvalue λ_1 is isolated and positive with corresponding normalized eigenvector $u_1 \in C^1(\Omega)$ (that is, $\|u_1\| = 1$) which is positive in Ω, [12, Lemma 5.3]. Furthermore,

$$\lambda_1 = \inf\Big\{ \frac{\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} b(x)|u|^p d\sigma}{\int_\Omega |u|^p dx} : u \in W^{1,p}(\Omega)\backslash\{0\}\Big\}. \tag{1.3}$$

2. Subcritical exponents

In what follows we assume that $1 < q < p^* := \frac{Np}{N-p}$ and $1 < r < \widehat{p}^* := \frac{p(N-1)}{N-p}$.

2.1. Existence of solutions when $\lambda < \lambda_1$.

Lemma 2.1. *The expression*

$$[u] = \Big[\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega |u|^p dx + \int_{\partial\Omega} b(x)|u|^p d\sigma\Big]^{1/p}$$

is a norm on X and is equivalent to $\|\cdot\|_{1,p}$.

The proof of the above lemma follows from [4, Proposition 2].

Depending on the relative ordering of the exponents p, q, r, we distinguish the following four cases.

Case 1. $p < \min\{q, r\}$. We assume

(H1) $a(\cdot) \geq 0$ and $m\{x \in \Omega : a(\cdot) > 0\} > 0$.
(H2) $\rho(\cdot) \geq 0$ on $\partial\Omega$ and $m\{x \in \partial\Omega : \rho(\cdot) > 0\} > 0$.

Let Y be an Banach space and $\Sigma := \{A \subseteq X \backslash \{0\} : A$ is closed and $A = -A\}$. The genus of a set $A \in \Sigma$ is defined by

$$\gamma(A) := \min\{n \in \mathbb{N} : \exists \varphi \in C(A, \mathbb{R}^n \backslash \{0\}) \text{ with } \varphi(x) = -\varphi(-x)\}.$$

Theorem 2.2. *Suppose that $I : Y \to \mathbb{R}$ is an even $C^1(Y, \mathbb{R})$ function such that:*

(i) *I satisfies the Palais-Smale condition.*
(ii) *$I(u) > 0$ if $0 < \|u\| < r$ and $I(u) \geq c > 0$ if $\|u\| = r$, for some $r > 0$.*
(iii) *There exists a subspace $Y_m \subseteq E$ of dimension m and a compact subset $A_m \subseteq Y_m$ with $I < 0$ on A_m such that 0 lies in a bounded component (in Y_m) of $Y_m \backslash A$.*

Let $\Gamma := \{h \in C(Y, Y) : h(0) = 0, h$ is an odd homeomorhism, $I(h(B_1)) \geq 0\}$, $K_m := \{K \subseteq Y : K$ is compact,$K = -K, \gamma(K \cap h(\partial B_1)) \geq m$ for every $h \in \Gamma\}$, where B_1 denotes the unit ball of Y. Then

$$c_m := \inf_{K \in K_m} \max_{u \in K} I(u)$$

is a critical value of I with $0 < c < c_m \leq c_{m+1} < +\infty$. Furthermore, if $c_m = c_{m+1} = \dots = c_{m+n}$, then $\gamma(K_{c_m}) \geq n + 1$, where $K_{c_m} := \{u \in X : I'(u) = 0, I(u) = c_m\}$.

For the proof of the above Theorem we refer the reader to [1].

Theorem 2.3. *Assume that* (H1) *and* (H2) *hold. Then* (1)-(1) *admits infinitely many solutions.*

Proof. We will show first that I satisfies the Palais-Smale condition. So let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in X such that $|I(u_n)| \leq M$ and $I'(u_n) \to 0$. For $k \in (p, \min\{q, r\})$ we have

$$-M + o_n(1)[u_n] \leq I(u_n) - \frac{1}{k} I'(u_n) u_n \leq M + o_n(1)[u_n],$$

and so

$$\begin{aligned} -M + o_n(1)[u_n] &\leq \Big(\frac{1}{p} - \frac{1}{k}\Big)[u_n]^p + \Big(\frac{1}{k} - \frac{1}{q}\Big) A(u_n) + \mu \Big(\frac{1}{k} - \frac{1}{r}\Big) P(u_n) \\ &\leq M + o_n(1)[u_n], \end{aligned} \tag{2.1}$$

which implies $\{u_n\}_{n \in \mathbb{N}}$ is bounded in X. Without loss of generality we may assume that $u_n \to u$ weakly in X and strongly in $L^p(\Omega)$, $L^q(\Omega)$, $L^p(\partial\Omega)$ and $L^r(\partial\Omega)$. Therefore,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla (u_n - u) dx \to 0, \tag{2.2}$$

$$\int_\Omega a(|u_n|^{q-2} u_n - |u|^{q-2} u)(u_n - u) dx \to 0, \tag{2.3}$$

$$\int_{\partial\Omega} b(|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) d\sigma \to 0, \tag{2.4}$$

$$\int_{\partial\Omega} \rho(|u_n|^{r-2} u_n - |u|^{r-2} u)(u_n - u) d\sigma \to 0 \tag{2.5}$$

as $n \to +\infty$. Since $I'(u_n) \to 0$, (2.3)-(2.5) imply that

$$\langle I'(u_n) - I'(u), u_n - u \rangle \to 0 \quad \text{as } n \to +\infty. \tag{2.6}$$

Thus,

$$\begin{aligned}
&\int_\Omega \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big](\nabla u_n - \nabla u)\,dx \\
&- \lambda \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx \\
&+ \int_{\partial\Omega} b(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)d\sigma \\
&- \int_\Omega a(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n-u)dx \\
&- \mu \int_{\partial\Omega} \rho(|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n-u)d\sigma \to 0 \quad \text{as } n\to+\infty.
\end{aligned}$$

Consequently,

$$\int_\Omega [|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u](\nabla u_n - \nabla u)dx \to 0 \quad \text{as } n\to+\infty.$$

As a consequence of Holder's inequality we have

$$\begin{aligned}
&\int_\Omega [|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u](\nabla u_n - \nabla u)dx \\
&\geq \Big[\Big(\int_\Omega |\nabla u_n|^p dx\Big)^{(p-1)/p} - \Big(\int_\Omega |\nabla u|^p dx\Big)^{(p-1)/p}\Big] \\
&\quad \times \Big[\Big(\int_\Omega |\nabla u_n|^p dx\Big)^{1/p} - \Big(\int_\Omega |\nabla u|^p dx\Big)^{1/p}\Big].
\end{aligned} \tag{2.7}$$

Therefore, $\|u_n\|_{1,p} \to \|u\|_{1,p}$. The uniform convexity of X implies that $u_n \to u$ in X. Note that

$$I(u) = \frac{1}{p}[u]^p - \frac{1}{q}A(u) - \frac{\mu}{r}P(u) \geq \frac{1}{p}[u]^p - c_1[u]^q - c_2[u]^r,$$

by the Sobolev embedding, and so $I(u) > 0$ for $\|u\| = \rho$ and $I(u) \geq c_3 > 0$ for $\|u\| < \rho$, provided ρ is sufficiently small. Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of subspaces of X with dimension $\dim(X_n) = n$ such that $\frac{\partial u}{\partial \eta} \neq 0$ if $u \in X_n \backslash \{0\}$. Then, for $u \in B_1^n := \{v \in X_n : [v] = 1\}$ and ζ sufficiently large

$$I(\zeta u) = \frac{\zeta^p}{p}[u]^p - \frac{\zeta^q}{q}A(u) - \frac{\mu\zeta^r}{r}P(u) < \frac{\zeta^p}{p} - \frac{\zeta^q}{q}\min_{u\in B_1^n} A(u) - \frac{\mu\zeta^r}{r}\min_{u\in B_1^n} P(u) < 0.$$

We can now apply Theorem 2.2 to complete the proof. □

Case 2. $1 < r < q < p$ We assume

(H1') $a(\cdot) \geq 0$ or $a(\cdot) \leq 0$ in Ω and $m\{x \in \Omega : a(\cdot) \neq 0\} > 0$.

Theorem 2.4. *If* $1 < r < q < p$ *and* (H1'), H(2) *hold, then* (1)-(1) *admits a positive solution.*

Proof. Assume first that $a(\cdot) \geq 0$. We consider the open set $Z := \{u \in X : A(u) > 0 \text{ or } P(u) > 0\}$. □

For $u \in Z$, $t \geq 0$, one forms

$$I(tu) = \frac{t^p}{p}H_\lambda(u) - \frac{t^q}{q}A(u) - \frac{\mu t^r}{r}P(u),$$

where $H_\lambda(u) := [u]^p$.

For $t > 0$, let

$$I_t(tu) = t^{p-1}H_\lambda(u) - t^{q-1}A(u) - \mu t^{r-1}P(u).$$

For critical points, we obtain

$$t^pH_\lambda(u) - t^qA(u) - \mu t^rP(u) = 0, \tag{2.8}$$

that has always a unique solution $t = t(u)$. Let $S_\lambda = Z \cap \{u \in X : H_\lambda(u) = 1\}$. We notice that $\{t(u) : u \in S_\lambda\}$ is bounded.

For $u \in Z$, we define $\widehat{I}(u) := I(t(u)u)$. In view of (2.8),

$$\widehat{I}(u) = \big(\frac{1}{p} - \frac{1}{q}\big)t(u)^pH_\lambda(u) + \big(\frac{1}{q} - \frac{1}{r}\big)\mu t(u)^rP(u) < 0. \tag{2.9}$$

Notice that $\widehat{I}(\cdot)$ is bounded below in S_λ. Let $M = \inf_{u\in S_\lambda} \widehat{I}(u)$. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$ be a minimizing sequence for $\widehat{I}/S_\lambda$. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X, we may assume that $u_n \rightharpoonup u$ in X. At the same time, $t(u_n) \to \widehat{t}$ in $\mathbb{R}$. Thus $t(u_n)u_n \rightharpoonup \widehat{t}u$ in X. By weak lower semicontinuity of $I(\cdot)$, we have

$$I(\widehat{t}u) \le \liminf_{n\to+\infty} I(t(u_n)u_n) = M.$$

Thus $\widehat{t}u \neq 0$. Because of the corresponding compact Sobolev embeddings, $A(u_n) \to A(u)$ and $P(u_n) \to P(u)$. Exploiting (2.8) for each n, one has

$$t(u_n)^{p-r} = t(u_n)^{q-r}A(u_n) + \mu P(u_n).$$

Letting $n \to +\infty$, we obtain

$$\widehat{t}^{p-r} = \widehat{t}^{q-r}A(u) + \mu P(u). \tag{2.10}$$

Since $\widehat{t} > 0$, either $A(u) > 0$ or $P(u) > 0$, thus $u \in Z$, and $t(u)$ is well defined. The weak lower semicontinuity of the norm applies to give

$$\widehat{t}^{p-r}H_\lambda(u) \le \widehat{t}^{q-r}A(u) + \mu P(u) \tag{2.11}$$

or

$$H_\lambda(u) \le \widehat{t}^{q-p}A(u) + \widehat{t}^{r-p}\mu P(u).$$

At the same time,

$$H_\lambda(u) = t(u)^{q-p}A(u) + t(u)^{r-p}\mu P(u).$$

Since the map $f(t) = t^{q-p}A(u)+t^{r-p}\mu P(u), t > 0$ is strictly decreasing, the last two relations imply $\widehat{t} \le t(u)$. Let us assume that $\widehat{t} < t(u)$. We set $F(y) := I(yu), y \ge 0$. For $y \in [\widehat{t}, t(u)]$, one has $F'(y) = y^{p-1}H_\lambda(u) - y^{q-1}A(u) - y^{r-1}\mu P(u) = y^{r-1}[y^{p-r}H_\lambda(u)-y^{q-r}A(u)-\mu P(u)]$, which is negative everywhere but at $y = t(u)$, since (2.8) has a unique solution. Thus $F(y)$ is strictly decreasing in $[\widehat{t},\, t(u)]$, so

$$I(t(u)u) < I(\widehat{t}u) \le M.$$

We take $k > 0$ such that $ku \in S_\lambda$ (actually, combining (2.10) and (2.11) one sees that $k \ge 1$). We have

$$t(ku)^p = t(ku)^qA(ku) + t(ku)^r\mu P(ku)$$

or

$$\big(kt(ku)\big)^pH_\lambda(u) = \big(kt(ku)\big)^qA(u) + \big(kt(ku)\big)^r\mu P(u),$$

thus $kt(ku) = t(u)$. Then

$$I(t(ku)ku) = I(t(u)u) < I(\widehat{t}u) \le M,$$

which is a contradiction. Thus $\widehat{t} = t(u)$, $H_\lambda(u) = 1$, and $t(u)u$ is a nontrivial solution of our problem. Since $|t(u)u|$ will also be a minimizer, by Harnack's inequality we may assume that $t(u)u$ is positive.

If $a(\cdot) \le 0$ in Ω, we define $\hat{Z} := \{u \in X : P(u) > 0\}$. It is clear that (2.8) has a unique positive solution and $M < 0$. Furthermore, since the limit u of a minimizing sequence satisfies (2.10), we have that $P(u) > 0$. Thus $u \in \hat{Z}$ and $|t(u)u|$ is a positive solution of (1),1.

Case 3. $1 < q < r < p$.

Theorem 2.5. *Suppose that* $1 < q < r < p$ *and* (H1), (H2) *hold. Then* (1)-(1) *admits a positive solution.*

Proof. Note that for every $u \in Z$ (2.8) has a unique positive solution $t := t(u)$. Furthermore, the set $\{t(u) : u \in Z\}$ is bounded. Let $\widehat{I}(u) := I(t(u)u)$. Then, in view of (2.8),

$$\begin{aligned} \widehat{I}(u) &= t^p\big(\frac{1}{p} - \frac{1}{q}\big) + \mu t^r\big(\frac{1}{q} - \frac{1}{r}\big)P(u) \\ &\le t^p\big(\frac{1}{p} - \frac{1}{q}\big) + t^p\big(\frac{1}{q} - \frac{1}{r}\big) = t^p\big(\frac{1}{p} - \frac{1}{r}\big) < 0. \end{aligned} \tag{2.12}$$

We can now proceed as in case 2. □

Case 4. $1 < r < p < q$. We assumpe

(H3) $a(\cdot) \le 0$ and $m\{x \in \Omega : a(\cdot) < 0\} > 0$.

Theorem 2.6. *If* $1 < r < p < q$ *and* (H2), (H3) *hold, then* (1)-(1) *admits a positive solution.*

Proof. Once more, (2.8) has a unique positive solution $t := t(u)$ for every $u \in \hat{Z}$. Furthermore, the set $\{t(u) : u \in \hat{Z}\}$ is bounded and $\widehat{I}(u) < 0$ in $\hat{Z}$. We proceed as in case 2. □

2.1.1. *Existence of solutions when* $\lambda = \lambda_1$. In this section we assume that (H2) and (H3) hold.

Case 5. $1 < r < q < p$.

Theorem 2.7. *Assume that* $1 < r < q < p$ *and* (H2), (H3) *hold. then* (1)-(1) *admits a positive solution.*

Proof. Let $H_\lambda^P(u) := [u]^p - A(u)$ and $S_\lambda^P := \{u \in X : P(u) > 0 \text{ and } H_\lambda^P(u) = 1\}$. If $u \in S_\lambda^P$, then (2.8) has a unique solution $t(u)$ with $\widehat{I}(u) < 0$. Define $M = \inf_{u \in S_\lambda^P} \widehat{I}(u)$ and assume that $u_n \in S_\lambda^P$ is such that $\widehat{I}(u_n) \to M$. We claim that $\|u_n\|_{1,p}$, $n \in \mathbb{N}$, is bounded. Indeed, let us assume that it is not, that is, $\|u_n\|_{1,p} \to +\infty$. Define $z_n := \frac{u_n}{d_n}$, where $d_n = \|u_n\|_{1,p}$. Then

$$d_n^p[z_n]^p - d_n^q A(z_n) = 1.$$

Consequently,

$$[z_n]^p \le \frac{1}{d_n^p} \to 0, \quad 0 \le -A(z_n) \le \frac{1}{d_n^q} \to 0. \tag{2.13}$$

Thus

$$\lambda_1 \int_\Omega |z_n|^p dx \to 1. \tag{2.14}$$

Since $\|z_n\|_{1,p} = 1$, we may assume that $z_n \to z$ weakly in X. Therefore, (2.14) implies that

$$\lambda_1 \int_\Omega |z|^p dx = 1, \tag{2.15}$$

and so $z \neq 0$. By (2.14) and (2.15) we see that $[z] = 0$, that is, z is an eigenvector corresponding to λ_1. On the other hand, since $A(z_n) \to A(z)$, (2.14) yields $A(z) = 0$, contradicting the fact $z > 0$ in Ω. Thus, $\|u_n\|_{1,p}$, $n \in \mathbb{N}$, is indeed bounded. So we may assume that $u_n \to u$ weakly in X. Note that, for an infinite number of $n's$, either $[u_n]^p \geq \frac{1}{2}$, or $-A(u_n) \geq \frac{1}{2}$. In either case, (2.8) implies that $r(u_n)$ is bounded. Since (2.8) implies that $P(u) > 0$, we see that $u \in S_\lambda^P$. We can now proceed as in case 1 to get a solution. □

Case 6. $1 < r < p < q$.

Theorem 2.8. *Assume that* $1 < r < p < q$ *and* (H2), (H3) *hold. Then* (1)-(1) *admits a positive solution.*

Proof. We use the inequality

$$\begin{aligned} \widehat{I}(u) &= t^p\big(\frac{1}{p} - \frac{1}{q}\big) + \mu t^r\big(\frac{1}{q} - \frac{1}{r}\big)P(u) \\ &\leq \mu t^r\big(\frac{1}{p} - \frac{1}{q}\big)P(u) + \mu t^r\big(\frac{1}{q} - \frac{1}{r}\big)P(u) \\ &= \mu t^r\big(\frac{1}{p} - \frac{1}{r}\big)P(u) < 0, \end{aligned}$$

to show that $\inf_{u \in S_\lambda^P} \widehat{I}(u) < 0$ and by following the same steps as in case 5 we obtain a positive solution. □

3. The critical case $q = p^*$

In this section we study the critical problem $q = p^* := \frac{Np}{N-p}$. with $p < r < \frac{p(N-1)}{N-p.}$ and $\lambda < \lambda_1$. The proof follows closely the lines of [9, Theorem 1.4]. Since the embedding $X \hookrightarrow L^{p^*}(\Omega)$ is no longer compact we do not expect that the Palais-Smale condition holds. So we prove a local Palais-Smale condition which is true if $I(\cdot)$ lies below a certain energy value.

In what follows we assume that $a(\cdot) \equiv 1$ and (H2) holds. Consider the problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u + |u|^{q-2}u \quad \text{in } \Omega, \tag{3.1}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \eta} + b(x)|u|^{p-2}u = \mu\rho(x)h(u) \quad \text{on } \partial\Omega. \tag{3.2}$$

Let

$$S = \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^{p^*} dx},$$

be the best Sobolev constant, where $u \in D^{1,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the gradient norm.

Lemma 3.1. *Suppose that* $\{u_n\}_{n\in\mathbb{N}}$ *is a sequence in* X *satisfying the Palais-Smale condition with energy level* $c < \frac{1}{N}S^{\frac{N}{p}}$, *that is*

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

Then $\{u_n\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* X.

Proof. The boundedness of $\{u_n\}_{n\in\mathbb{N}}$ is a consequence of (2.1). Thus, $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, which converges weakly to $u \in X$. By [15, Lemma 3.6] there exists a set of points $\{x_j\}_{j\in J} \subseteq \overline{\Omega}$, J at most countable, and nonnegative numbers μ_j, ν_j satisfying

$$\begin{gathered}|\nabla u_n|^p \to \mu \geq |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j}\,,\\ |u_n|^{p^*} \to \nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}\,,\\ S\nu_j^{\frac{p}{p^*}} \leq \mu_j \quad \text{if } x_j \in \Omega,\\ S\nu_j^{\frac{p}{p^*}} \leq 2^{\frac{p}{N}} \mu_j \text{ if } x_j \in \partial\Omega.\end{gathered}$$

Let $k \in \mathbb{N}$, $\varepsilon > 0$ and take $\varphi \in C^\infty(\Omega)$ such that

$$\varphi \equiv 1 \text{ in } B(x_k, \varepsilon), \quad \varphi \equiv 0 \quad \text{in } X\backslash B(x_k, 2\varepsilon), \quad |\nabla\varphi| \leq \frac{2}{\varepsilon}.$$

Since $I'(u_n)(\varphi u_n) \to 0$ an $n \to +\infty$, we obtain

$$\begin{aligned}&\lim_{n\to+\infty} \Big[\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi u_n dx + \int_\Omega |\nabla u_n|^p \varphi dx\Big]\\ &= \lambda \int_\Omega |u|^p \varphi dx - \int_{\partial\Omega} b(x)|u|^p \varphi d\sigma + \lim_{n\to+\infty} \int_\Omega |u_n|^{p^*} \varphi dx + \mu \int_{\partial\Omega} \rho(x)|u|^r \varphi d\sigma\\ &= \lambda \int_\Omega |u|^p \varphi dx - \int_{\partial\Omega} |u|^p \varphi d\sigma + \int_\Omega \varphi d\nu + \mu \int_{\partial\Omega} \rho(x)|u|^r \varphi d\sigma.\end{aligned}$$

Note that, by the Holder inequality,

$$\begin{aligned}&\lim_{n\to+\infty} \Big| \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi u_n \, dx\Big|\\ &\leq \lim_{n\to+\infty} \Big(\int_\Omega |u_n|^p \varphi dx\Big)^{\frac{p-1}{p}} \lim_{n\to+\infty} \Big(\int_\Omega |\nabla\varphi|^p |u_n|^p dx\Big)^{1/p}\\ &\leq C\Big(\int_{B(x_k,2\varepsilon)\cap\Omega} |\nabla\varphi|^p |u|^p dx\Big)^{1/p}\\ &\leq C\Big(\int_{B(x_k,2\varepsilon)\cap\Omega} |\nabla\varphi|^N dx\Big)^{1/N} \Big(\int_{B(x_k,2\varepsilon)\cap\Omega} |u|^{p^*} dx\Big)^{1/p^*}\\ &\leq C' \int_{B(x_k,2\varepsilon)\cap\Omega} |u|^{p^*} dx \to 0 \quad \text{as } \varepsilon \to 0,\end{aligned}$$

and so

$$\begin{aligned}&\lim_{\varepsilon\to0} \Big[\int_\Omega \varphi d\mu - \lambda \int_\Omega |u|^p \varphi dx + \int_{\partial\Omega} b(x)|u|^p \varphi d\sigma - \int_\Omega \varphi d\nu - \mu \int_{\partial\Omega} \rho(x)|u|^r \varphi d\sigma\Big]\\ &= \mu_k - \nu_k = 0.\end{aligned}$$

Consequently, $S\nu_k^{\frac{p}{p^*}} \leq \nu_k$ if $x_k \in \Omega$ or $2^{-\frac{p}{N}} S\nu_k^{\frac{p}{p^*}} \leq \nu_k$ if $x_k \in \partial\Omega$, implying that $S^{\frac{N}{p}} \leq \nu_k$ if $x_k \in \Omega$ or $\frac{1}{2} S^{\frac{N}{p}} \leq \nu_k$ if $x_k \in \partial\Omega$. On the other hand,

$$c = \lim_{n\to+\infty} I(u_n) = \lim_{n\to+\infty} I(u_n) - \lim_{n\to+\infty} \frac{1}{p} I'(u_n)(u_n)$$

$$
\begin{aligned}
&= \big(\frac{1}{p}-\frac{1}{p^*}\big)\int_\Omega |u|^{p^*} + \big(\frac{1}{p}-\frac{1}{p^*}\big)\int_\Omega \sum_{j\in J}\nu_j\delta_{x_j} + \mu\big(\frac{1}{p}-\frac{1}{r}\big)\int_{\partial\Omega}\rho(x)|u|^r d\sigma \\
&\ge \big(\frac{1}{p}-\frac{1}{p^*}\big)\nu_k = \frac{1}{N}S^{N/p}.
\end{aligned}
$$

Thus $\nu_k = 0$ for every $k \in J$, implying that $\int_\Omega |u_n|^{p^*}dx \to \int_\Omega |u|^{p^*}dx$. The result follows by exploiting the continuity of the inverse p-Laplace operator. □

Theorem 3.2. *There exists $\mu_0 > 0$ such that for $\mu \ge \mu_0$ problem* (1)-(1) *admits a solution.*

Proof. We will first verify the requirements for the mountain pass theorem. By the Sobolev embedding and trace theorems we see that

$$
\begin{aligned}
I(u) &= \frac{1}{p}[u]^p - \frac{1}{p^*}A(u) - \frac{\mu}{r}P(u) \\
&\ge \frac{1}{p}[u]^p - C_1[u]^{p^*} - C_2[u]^r,
\end{aligned}
$$

for some $C_1, C_2 > 0$,and so for a sufficiently small positive number β there exists $a > 0$ such that $I(u) > a > 0$ for $[u] = \beta$. We now take $v \in X\backslash\{0\}$. It is easy to see that $\lim_{s\to+\infty} I(sv) = -\infty$. Thus, $I(s_0v) < 0$ for sufficiently large s_0.

Let $c := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t))$, where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = s_0v\}$. We will show that $c < \frac{1}{N}S^{\frac{N}{p}}$ for large enough μ. Take $z \in X$ such that $\|z\|_{p^*} = 1$. The maximum value of $\eta \to I(\eta z)$, $\eta > 0$, is assumed at the point η_μ satisfying $\frac{d}{d\eta}I(\eta_\mu z) = 0$, that is

$$
\eta_\mu^p[z]^p = \eta_\mu^{p^*}\|z\|_{p^*}^{p^*} + \mu\eta_\mu^r P(z) = \eta_\mu^{p^*} + \mu\eta_\mu^r P(z). \tag{3.3}
$$

Therefore,

$$
\eta_\mu \le [z]^{\frac{p}{p^*-p}},
$$

which, in view of (3.3), yields $\lim_{\mu\to+\infty}\eta_\mu = 0$. On the other hand,

$$
I(\eta_\mu z) = \eta_\mu^p\big(\frac{1}{p}-\frac{1}{p^*}\big)[z]^p + \mu\eta_\mu^r\big(\frac{1}{p^*}-\frac{1}{r}\big)P(z) \le \eta_\mu^p\big(\frac{1}{p}-\frac{1}{p^*}\big)[z]^p,
$$

implying that $\lim_{\mu\to+\infty} I(\eta_\mu z) = 0$. Thus, for large enough μ, say $\mu \ge \mu_0$, $I(\eta_\mu z) < \frac{1}{N}S^{\frac{N}{p}}$. By Lemma 3.1, $I(\cdot)$ satisfies the Palais-Smale condition and the mountain pass theorem provides a solution. □

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