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MULTI-PEAK SOLUTIONS FOR A PLANAR ROBIN NONLINEAR ELLIPTIC PROBLEM WITH LARGE EXPONENT

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Abstract. We consider the elliptic equation $\Delta u+u^p=0$ in a bounded smooth domain Ω in $\mathbb{R}^2$ subject to the Robin boundary condition $\frac{\partial u}{\partial\nu}+\lambda b(x)u=0$. Here ν denotes the unit outward normal vector on $\partial\Omega$, $b(x)$ is a smooth positive function defined on $\partial\Omega$, $0<\lambda<+\infty$, and p is a large exponent. For any fixed λ large we find topological conditions on Ω which ensure the existence of a positive solution with exactly m peaks separated by a uniform positive distance from the boundary and each from other as $p\to+\infty$ and $\lambda\to+\infty$. In particular, for a nonsimply connected domain such solution exists for any $m\geq 1$.

1. Introduction

In this article we consider the boundary-value problem

$$
\begin{gathered}
\Delta u+u^p=0 \quad \text{in } \Omega,\\
u>0 \quad \text{in } \Omega,\\
\frac{\partial u}{\partial \nu}+\lambda b(x)u=0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, ν denotes the unit outward normal vector on $\partial\Omega$, $b(x)$ is a smooth positive function defined on $\partial\Omega$, $0<\lambda<+\infty$, and p is a large exponent.

The boundary condition in problem (1.1) is called Robin boundary condition. Such an Robin boundary condition is particularly interesting in various branches of biological models (see [8, 15]).

When $\lambda=0$, from integration by parts it is trivial to observe that (1.1) has no solution. On the other hand, if $0<\lambda\leq+\infty$, it is easy to prove via standard variational methods that (1.1) always has a least energy solution. Moreover, in the case $\lambda=+\infty$, problem (1.1) is reduced to the problem

$$
\begin{gathered}
\Delta u+u^p=0 \quad \text{in } \Omega,\\
u>0 \quad \text{in } \Omega,\\
u=0 \quad \text{on } \partial\Omega.
\end{gathered}
\tag{1.2}
$$

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Ren and Wei [16, 17] showed that the least energy solution u_p of (1.2) develops one interior peak, namely u_p approaches zero except one interior point where it has an L^∞-norm bounded and bounded away from zero, uniformly in p as $p \to +\infty$. More precisely, the authors prove that, up to a subsequence, both $p|\nabla u_p|^2$ and pu_p^{p+1} behave as a Dirac mass near a critical point of the Robin function $H_\infty(x,x)$, where H_∞ is the regular part of Green's function G_∞ of the Dirichlet Laplacian in Ω, i.e. $H_\infty(x,y) = G_\infty(x,y) + \frac{1}{2\pi}\log|x-y|$. Successively, in [1, 9] the authors give a further description of the asymptotic behavior of u_p, as $p \to +\infty$, by identifying a limit profile problem of Liouville-type $\Delta u + e^u = 0$ in $\mathbb{R}^2$, $\int_{\mathbb{R}^2} e^u < +\infty$, and showing that $\|u_p\|_\infty \to \sqrt{e}$ as $p \to +\infty$. Furthermore, Esposito, Musso and Pistoia [11] prove that if Ω is not simply connected, (1.2) can have many other positive solutions which, as p tends to infinity, concentrate at m different points of Ω, i.e.

$$pu_p^{p+1} \rightharpoonup 8\pi e \sum_{i=1}^m \delta_{\xi_i} \quad \text{weakly in the sense of measure in } \overline{\Omega}, \tag{1.3}$$

where points $\xi = (\xi_1, \dots, \xi_m) \in \Omega^m$ corresponds to a critical point of the function

$$\varphi_m^\infty(\xi_1, \dots, \xi_m) = \sum_{j=1}^m H_\infty(\xi_j, \xi_j) + \sum_{j \neq k} G_\infty(\xi_j, \xi_k). \tag{1.4}$$

In contrast, Grossi and Takahashi [12] prove that when Ω is convex, problem (1.2) has no multi-peak solutions satisfying (1.3). Thus the assumption on the domain in [11] is sharp for the construction of multiple concentrating solutions of (1.2).

The purpose of our research is to give the construction of multi-peak solutions to the so called Robin problem (1.1) with sufficiently large p and λ, and to point out that in general the set of multi-peak solutions of this problem exhibits a richer structure than the problem with Dirichlet boundary condition, which we will finish in this paper and in [18]. In this paper we prove that if Ω is not simple connected, then given any $m \geq 1$, for p and λ large enough problem (1.1) has a positive solution $u_{p,\lambda}$ concentrating at exactly m points that stay uniformly separated from the boundary and from each other as $p \to +\infty$ and $\lambda \to +\infty$. In particular, we recover existence results already known in [11] when $\lambda = +\infty$ and p is large enough.

To state our results, we need to introduce some notation. Let $G_\lambda(x,y)$ be the Green's function satisfying

$$\begin{gathered} -\Delta_x G_\lambda(x,y) = \delta_y(x) \quad x \in \Omega, \\ \frac{\partial G_\lambda}{\partial \nu}(x,y) + \lambda b(x) G_\lambda(x,y) = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.5}$$

then its regular part can be decomposed as

$$H_\lambda(x,y) = G_\lambda(x,y) - \frac{1}{2\pi}\log\frac{1}{|x-y|}. \tag{1.6}$$

Furthermore, let

$$\varphi_m^\lambda(\xi_1, \dots, \xi_m) = \sum_{j=1}^m H_\lambda(\xi_j, \xi_j) + \sum_{j \neq k} G_\lambda(\xi_j, \xi_k). \tag{1.7}$$

Our main result reads as follows.

Theorem 1.1. *Assume that Ω is not simply connected. Then given any $m \geq 1$, there exist $p_m > 0$ and $\lambda_m > 0$ such that for any $p > p_m$ and $\lambda > \lambda_m$, problem*

(1.1) has a solution $u_{p,\lambda}$ with m concentration points $\xi_{1,p,\lambda},\dots,\xi_{m,p,\lambda}$ separated at a uniform positive distance from the boundary and each other as $p\to+\infty$ and $\lambda\to+\infty$. More precisely,

$$u_{p,\lambda}(x)=\sum_{j=1}^{m}\frac{1}{\gamma\mu_j^{2/(p-1)}}\Big[\log\frac{1}{(\delta_j^2+|x-\xi_{j,p,\lambda}|^2)^2}+8\pi H_\lambda(x,\xi_{j,p,\lambda})\Big]+O(\frac{1}{p}),$$

where the parameters γ, δ_j and μ_j satisfy

$$\gamma=p^{\frac{p}{p-1}}\rho^{\frac{2}{p-1}},\quad \delta_j=\mu_j\rho,\quad \rho=e^{-\frac{1}{4}p},\quad \frac{1}{C}<\mu_j<C,$$

for some $C>0$, and $\xi_{p,\lambda}=(\xi_{1,p,\lambda},\dots,\xi_{m,p,\lambda})\in\Omega^m$ satisfies

$$\lim_{p\to+\infty,\ \lambda\to+\infty}\nabla\varphi_m^\lambda\big(\xi_{1,p,\lambda},\dots,\xi_{m,p,\lambda}\big)=0,$$

and

$$\operatorname{dist}(\xi_{j,p,\lambda},\partial\Omega)\ge 2\varepsilon,\quad |\xi_{j,p,\lambda}-\xi_{k,p,\lambda}|\ge 2\varepsilon\quad \forall j,k=1,\dots,m;\ j\neq k,$$

for any $\varepsilon>0$ small. In particular, as $p\to+\infty$ and $\lambda\to+\infty$,

$$pu_{p,\lambda}^{p+1}-8\pi e\sum_{j=1}^{m}\delta_{\xi_{j,p,\lambda}}\rightharpoonup 0\quad \textit{weakly in the sense of measure in } \overline{\Omega},$$
$$u_{p,\lambda}\to 0\quad \textit{uniformly in } \overline{\Omega}\setminus\cup_{j=1}^{m}B_\varepsilon(\xi_{j,p,\lambda}),$$
$$\sup_{B_\varepsilon(\xi_{j,p,\lambda})}u_{p,\lambda}\to\sqrt{e}.$$

The rest of this article is devoted to the proof of Theorem 1.1. Our proof relies on a Lyapunov-Schmidt process as in [7, 10, 11, 14], but we now have to confront some difficulties that are brought by the presence of Robin boundary condition, which can be successfully overcome by making use of some versions of the maximum principle with Robin boundary condition. This is the delicate ingredient during we construct multi-peak solutions of problem (1.1) through performing the finite-dimensional reduction and using the notion of a nontrivial critical level.

This article is organized as follows. In Section 2 we exactly describe the ansatz for the solution of problem (1.1) and estimate the error. Then we rewrite problem (1.1) in terms of a linearized operator for which a solvability theory, subject to suitable orthogonality conditions, is performed through solving a linearized problem in Section 3. In Section 4 we solve an auxiliary nonlinear problem. In Section 5 we reduce (1.1) to a finite system, as we will see in Section 5. In the last section, we use the notion of a nontrivial critical level to give the proof of Theorem 1.1.

2. A first approximation of the solution

In this section we provide an ansatz for solutions of problem (1.1). A key ingredient to describe an approximate solution of problem (1.1) is given by the standard bubble:

$$U_{\delta,\xi}(x)=\log\frac{8\delta^2}{(\delta^2+|x-\xi|^2)^2},\quad \delta>0,\quad \xi\in\mathbb{R}^2. \tag{2.1}$$

It is well known (see [4]) that those are all the solutions of the Liouville-type equation

$$\begin{gathered}\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < +\infty. \end{gathered} \tag{2.2}$$

Let us introduce the configuration space in which the concentration points belong to

$$\begin{aligned}\mathcal{O}_\varepsilon := \Big\{ & \xi = (\xi_1, \dots, \xi_m) \in \Omega^m : \operatorname{dist}(\xi_j, \partial\Omega) \geq 2\varepsilon,\ |\xi_j - \xi_k| \geq 2\varepsilon, \\ & j, k = 1, \dots, m;\ j \neq k \Big\}, \end{aligned} \tag{2.3}$$

where $\varepsilon > 0$ is a sufficiently small but fixed number. Furthermore, we set, for each $j = 1, \dots, m$,

$$\gamma = p^{\frac{p}{p-1}} \rho^{\frac{2}{p-1}}, \quad \delta_j = \mu_j \rho, \quad \rho = e^{-\frac{1}{4}p}, \quad \frac{1}{C} < \mu_j < C, \tag{2.4}$$

for some $C > 0$, where the choice of μ_j will be determined later. Define now

$$U_j(x) = \frac{1}{\gamma \mu_j^{2/(p-1)}} \Big[U_{\delta_j, \xi_j}(x) + \frac{1}{p} \omega_1\Big(\frac{x - \xi_j}{\delta_j}\Big) + \frac{1}{p^2} \omega_2\Big(\frac{x - \xi_j}{\delta_j}\Big)\Big]. \tag{2.5}$$

Here, ω_1 and ω_2 are radial solutions of

$$\Delta \omega_i + \frac{8}{(1 + |y|^2)^2} \omega_i = \frac{8}{(1 + |y|^2)^2} f_i(y) \quad \text{in } \mathbb{R}^2, \tag{2.6}$$

for $i = 1, 2$, respectively, with

$$f_1 = \frac{1}{2} U_{1,0}^2, \quad f_2 = \omega_1 U_{1,0} - \frac{1}{3} U_{1,0}^3 - \frac{1}{2} \omega_1^2 - \frac{1}{8} U_{1,0}^4 + \frac{1}{2} \omega_1 U_{1,0}^2, \tag{2.7}$$

having asymptotic properties

$$\begin{gathered} \omega_i(y) = C_i \log|y| + O\Big(\frac{1}{|y|}\Big) \quad \text{as } |y| \to +\infty, \\ \nabla \omega_i(y) = C_i \frac{y}{1 + |y|^2} + O\Big(\frac{1}{1 + |y|^2}\Big) \quad \text{for all } y \in \mathbb{R}^2, \end{gathered} \tag{2.8}$$

for $i = 1, 2$, where

$$C_i = 8 \int_0^\infty t \frac{t^2 - 1}{(t^2 + 1)^3} f_i(t) dt, \tag{2.9}$$

in particular,

$$\begin{aligned} \omega_1(y) = {} & \frac{1}{2} U_{1,0}^2(y) + 6 \log(|y|^2 + 1) + \frac{2 \log 8 - 10}{|y|^2 + 1} + \frac{|y|^2 - 1}{|y|^2 + 1} \\ & \times \Big\{ -\frac{1}{2} \log^2 8 + 2 \log^2(|y|^2 + 1) + 4 \int_{|y|^2}^\infty \frac{ds}{s + 1} \log \frac{s + 1}{s} \\ & - 8 \log|y| \log(|y|^2 + 1) \Big\}, \end{aligned} \tag{2.10}$$

and

$$C_1 = 12 - 4 \log 8 \tag{2.11}$$

(see [3, 11]). Our final ansatz for a solution of (1.1) is

$$U_\xi(x) = \sum_{j=1}^{m} \big[U_j(x) + H_j(x)\big], \tag{2.12}$$

where H_j is a correction term defined as the solution of

$$\begin{gathered} -\Delta H_j = 0 \quad \text{in } \Omega, \\ \frac{\partial H_j}{\partial \nu} + \lambda b(x) H_j = -\frac{\partial U_j}{\partial \nu} - \lambda b(x) U_j \text{on } \partial\Omega. \end{gathered} \tag{2.13}$$

Lemma 2.1. *For any set of points* $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$,

$$\begin{aligned} H_j(x) = \frac{1}{\gamma \mu_j^{2/(p-1)}} \Big[\Big(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2}\Big) 8\pi H_\lambda(x, \xi_j) - \log(8\mu_j^2 \rho^2) \\ + \Big(\frac{C_1}{p} + \frac{C_2}{p^2}\Big) \log(\mu_j \rho) + O\big(\frac{\rho}{p}\big)\Big] \end{aligned} \tag{2.14}$$

in $C(\overline{\Omega})$ *and in* $C^2_{\rm loc}(\Omega)$ *as* p *and* λ *go to* $+\infty$, *where* H_λ *is the regular part of Green's function defined in* (1.6).

Proof. First, on the boundary, by (2.1) and (2.8) we have

$$\begin{aligned} &\frac{\partial H_j}{\partial \nu} + \lambda b(x) H_j \\ &= -\frac{1}{\gamma \mu_j^{2/(p-1)}} \Big\{ \Big(-4 + \frac{C_1}{p} + \frac{C_2}{p^2}\Big) \Big[\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} - \lambda b(x) \log \frac{1}{|x - \xi_j|} \Big] \\ &\quad + \lambda b(x) \Big[\log(8\mu_j^2 \rho^2) - \Big(\frac{C_1}{p} + \frac{C_2}{p^2}\Big) \log(\mu_j \rho) \Big] + O\big(\frac{\lambda \rho}{p}\big) \Big\}. \end{aligned}$$

The regular part of Green's function with Robin boundary condition $H_\lambda(x, \xi_j)$ satisfies

$$\begin{gathered} -\Delta H_\lambda(x, \xi_j) = 0 \quad \text{in } \Omega, \\ \frac{\partial H_\lambda(x, \xi_j)}{\partial \nu} + \lambda b(x) H_\lambda(x, \xi_j) = \frac{1}{2\pi} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} - \frac{1}{2\pi} \lambda b(x) \log \frac{1}{|x - \xi_j|} \quad \text{on } \partial\Omega. \end{gathered} \tag{2.15}$$

So, if we set

$$\begin{aligned} \widetilde{H}_j(x) = \gamma \mu_j^{2/(p-1)} H_j(x) - \Big[\Big(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2}\Big) 8\pi H_\lambda(x, \xi_j) \\ - \log(8\mu_j^2 \rho^2) + \Big(\frac{C_1}{p} + \frac{C_2}{p^2}\Big) \log(\mu_j \rho) \Big], \end{aligned}$$

then $\widetilde{H}_j(x)$ satisfies

$$\begin{gathered} -\Delta \widetilde{H}_j = 0 \quad \text{in } \Omega, \\ \frac{\partial \widetilde{H}_j}{\partial \nu} + \lambda b(x) \widetilde{H}_j = O\Big(\frac{\lambda \rho}{p}\Big) \quad \text{on } \partial\Omega. \end{gathered}$$

From the maximum principle with Robin boundary condition (see [6, Lemma 2.6]), we deduce

$$\max_{\overline{\Omega}} \big|\widetilde{H}_j(x)\big| + \max_{\overline{\Omega}} \big|\operatorname{dist}(x, \partial\Omega) \nabla \widetilde{H}_j(x)\big| \le \frac{C}{\lambda} \Big\| \frac{\partial \widetilde{H}_j}{\partial \nu} + \lambda \widetilde{H}_j \Big\|_{L^\infty(\partial\Omega)} = O\Big(\frac{\rho}{p}\Big).$$

By the interior estimate of derivative of harmonic function, we derive estimate (2.14) in $C(\overline{\Omega})$ and in $C^2_{\rm loc}(\Omega)$. $\square$

From Lemma 2.1, away from the points ξ_j, namely $|x-\xi_j|\ge\varepsilon$ for any $j=1,\dots,m$, one has

$$U_\xi(x)=\sum_{j=1}^m\frac{1}{\gamma\mu_j^{2/(p-1)}}\Big[\Big(1-\frac{C_1}{4p}-\frac{C_2}{4p^2}\Big)8\pi G_\lambda(x,\xi_j)+O\Big(\frac{\rho}{p}\Big)\Big]. \tag{2.16}$$

While for $|x-\xi_j|\le\varepsilon$ with some j, if we write $x=\xi_j+\delta_j y$, then, by (2.4), (2.5), (2.14) and (2.16) we deduce

$$\begin{aligned}
&U_\xi(x)\\
&=\frac{1}{\gamma\mu_j^{2/(p-1)}}\Big\{p+U_{1,0}(y)+\frac{1}{p}\omega_1(y)+\frac{1}{p^2}\omega_2(y)+\Big(1-\frac{C_1}{4p}-\frac{C_2}{4p^2}\Big)8\pi H_\lambda(\xi_j,\xi_j)\\
&\quad-\log(8\mu_j^4)+\Big(\frac{C_1}{p}+\frac{C_2}{p^2}\Big)\log(\mu_j\rho)+O\big(\rho|y|\big)+O\big(\frac{\rho}{p}\big)\Big\}\\
&\quad+\sum_{k\ne j}\frac{1}{\gamma\mu_k^{2/(p-1)}}\Big[\Big(1-\frac{C_1}{4p}-\frac{C_2}{4p^2}\Big)8\pi G_\lambda(\xi_j,\xi_k)+O\big(\rho|y|\big)+O\big(\frac{\rho}{p}\big)\Big].
\end{aligned}$$

We now choose the parameters μ_j: we assume they are defined by the relation

$$\begin{aligned}
\log(8\mu_j^4)&=\Big(1-\frac{C_1}{4p}-\frac{C_2}{4p^2}\Big)\Big[8\pi H_\lambda(\xi_j,\xi_j)+\sum_{k\ne j}\frac{\mu_j^{2/(p-1)}}{\mu_k^{2/(p-1)}}8\pi G_\lambda(\xi_j,\xi_k)\Big]\\
&\quad+\Big(\frac{C_1}{p}+\frac{C_2}{p^2}\Big)\log\big(\mu_j e^{-4p/4}\big).
\end{aligned}$$

Thus, by the explicit expression (2.11) of the constant C_1, we observe that for p large, the parameters μ_j satisfies

$$\mu_j=e^{-\frac{3}{4}}e^{2\pi H_\lambda(\xi_j,\xi_j)+2\pi\sum_{k\ne j}G_\lambda(\xi_j,\xi_k)}\big[1+O\big(\frac{1}{p}\big)\big]. \tag{2.17}$$

From this choice of the parameters μ_j, we deduce that for $|x-\xi_j|=\delta_j|y|\le\varepsilon$,

$$U_\xi(x)=\frac{1}{\gamma\mu_j^{2/(p-1)}}\Big[p+U_{1,0}(y)+\frac{1}{p}\omega_1(y)+\frac{1}{p^2}\omega_2(y)+O(\rho|y|)+O\big(\frac{\rho}{p}\big)\Big]. \tag{2.18}$$

Remark 2.2. Let us remark that U_ξ is a positive, uniformly bounded function. Observe that for $|y|\le\varepsilon/\delta_j$,

$$p+U_{1,0}(y)+\frac{1}{p}\omega_1(y)+\frac{1}{p^2}\omega_2(y)\ge 4\log\frac{1}{\varepsilon}+\log(8\mu_j^4)+O\big(\frac{1}{p}\big).$$

Then it is easily checked that choosing $\varepsilon>0$ smaller if necessary, $U_\xi>0$ in $B(\xi_j,\varepsilon)$, and $\sup_{B(\xi_j,\varepsilon)}U_\xi\to\sqrt{e}$ as p and λ go to $+\infty$. Moreover, by the maximum principle, we see that $G_\lambda(x,\xi_j)>0$ in $\overline{\Omega}$ and thus by (2.16), U_ξ is a positive, uniformly bounded function in $\overline{\Omega}$. In conclusion, $0<U_\xi\le 2\sqrt{e}$ in $\overline{\Omega}$.

Let us define

$$S_p(u)=\Delta u+u_+^p,\quad\text{where } u_+=\max\{u,0\}, \tag{2.19}$$

and introduce the functional

$$J_p^\lambda(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\frac{1}{p+1}\int_\Omega u_+^{p+1}+\frac{\lambda}{2}\int_{\partial\Omega}b(x)u^2,\quad u\in H^1(\Omega),\tag{2.20}$$

whose nontrivial critical points are solutions of (1.1). Obviously, by the maximum principle, problem (1.1) is equivalent to

$$S_p(u)=0,\quad u_+\not\equiv 0\quad\text{in }\Omega,\quad \frac{\partial u}{\partial\nu}+\lambda b(x)u=0\quad\text{on }\partial\Omega.\tag{2.21}$$

We will seek solutions of (1.1) in the form $u=U_\xi+\phi$, where ϕ will represent a higher order correction. Observe that

$$S_p(U_\xi+\phi)=L(\phi)+R_\xi+N(\phi)=0,\tag{2.22}$$

where

$$L(\phi)=\Delta\phi+W_\xi\phi\quad\text{with}\quad W_\xi=pU_\xi^{p-1},\tag{2.23}$$

$$R_\xi=\Delta U_\xi+U_\xi^p,\quad N(\phi)=(U_\xi+\phi)_+^p-U_\xi^p-pU_\xi^{p-1}\phi.\tag{2.24}$$

In terms of ϕ, problem (1.1) becomes

$$\begin{gathered}L(\phi)=-\big[R_\xi+N(\phi)\big]\quad\text{in }\Omega,\\ \frac{\partial\phi}{\partial\nu}+\lambda b(x)\phi=0\quad\text{on }\partial\Omega.\end{gathered}\tag{2.25}$$

For any set of point $\xi=(\xi_1,\dots,\xi_m)\in\mathcal{O}_\varepsilon$ and $h\in L^\infty(\Omega)$, define

$$\|h\|_*=\sup_{x\in\Omega}\Big|\Big(\sum_{j=1}^m\frac{\delta_j}{(\delta_j^2+|x-\xi_j|^2)^{3/2}}\Big)^{-1}h(x)\Big|.\tag{2.26}$$

Lemma 2.3. *Let $\varepsilon>0$ be fixed. There exist $C>0$, $D_0>0$, $p_0>0$ and $\lambda_0>0$ such that*

$$\|R_\xi\|_*\le C/p^4,\tag{2.27}$$

$$W_\xi(x)\le D_0\sum_{j=1}^m e^{U_{\delta_j,\xi_j}(x)},\tag{2.28}$$

for any set of point $\xi=(\xi_1,\dots,\xi_m)\in\mathcal{O}_\varepsilon$, any $p\ge p_0$ and $\lambda\ge\lambda_0$. Furthermore,

$$W_\xi(x)=\frac{8}{\delta_j^2(1+|y|^2)^2}\Big[1+\frac{1}{p}\Big(\omega_1-U_{1,0}-\frac{1}{2}U_{1,0}^2\Big)(y)+O\Big(\frac{\log^4(|y|+2)}{p^2}\Big)\Big],\tag{2.29}$$

for any $|x-\xi_j|\le\varepsilon\sqrt{\delta_j}$, where $y=\frac{1}{\delta_j}(x-\xi_j)$.

Since the proof of the above lemma is similar to those of [11, Prop. 2.1 and Lemma 3.1], we omit it.

3. Linear and nonlinear problems

In this section, we shall study first bounded invertibility of the operator L defined in (2.23). Set

$$z_0(y)=\frac{|y|^2-1}{|y|^2+1},\quad z_i(y)=4\frac{y_i}{|y|^2+1},\quad i=1,\,2.\tag{3.1}$$

It is well known [2] that any bounded solution to

$$\Delta\phi + \frac{8}{(1+|y|^2)^2}\phi = 0 \quad \text{in } \mathbb{R}^2 \tag{3.2}$$

is a linear combination of z_i, $i = 0, 1, 2$. Let us consider the linear problem: given $h \in C(\overline{\Omega})$ and the set of points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, we find a function ϕ and scalars c_{ij}, $i = 1, 2$, $j = 1, \dots, m$, such that

$$\begin{gathered} L(\phi) = \Delta\phi + W_\xi\phi = h + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} e^{U_{\delta_j,\xi_j}} Z_{ij} \quad \text{in } \Omega, \\ \frac{\partial\phi}{\partial\nu} + \lambda b(x)\phi = 0 \quad \text{on } \partial\Omega, \\ \int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij}\phi = 0 \quad \text{for } i = 1, 2;\ j = 1, \dots, m. \end{gathered} \tag{3.3}$$

Here, for $i = 0, 1, 2$ and $j = 1, \dots, m$, we denote

$$Z_{ij}(x) := z_i\Big(\frac{x-\xi_j}{\delta_j}\Big) = \begin{cases} \frac{|x-\xi_j|^2-\delta_j^2}{|x-\xi_j|^2+\delta_j^2} & \text{if } i = 0, \\ \frac{4\delta_j(x-\xi_j)_i}{|x-\xi_j|^2+\delta_j^2} & \text{if } i = 1, 2. \end{cases} \tag{3.4}$$

Proposition 3.1. *Let $\varepsilon > 0$ be fixed. There exist $p_0 > 0$, $\lambda_0 > 0$ and $C > 0$ such that for any $h \in C(\overline{\Omega})$, any the set of points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, any $p > p_0$ and $\lambda > \lambda_0$, there is a unique solution ϕ, scalars c_{ij}, $i = 1, 2$, $j = 1, \dots, m$, to problem* (3.3)*, which satisfies*

$$\|\phi\|_\infty \le Cp\|h\|_*. \tag{3.5}$$

Proof. The proof of this result will be divided into six steps.

Step 1: The operator L satisfies the maximum principle in $\widetilde{\Omega} := \Omega \setminus \cup_{j=1}^m B(\xi_j, R\delta_j)$ for R large, independent on p and λ. Specifically, if ψ satisfies

$$\begin{gathered} L(\psi) = \Delta\psi + W_\xi\psi \le 0 \quad \text{in } \widetilde{\Omega}, \\ \psi \ge 0 \quad \text{on } \cup_{j=1}^m \partial B(\xi_j, R\delta_j) \quad \text{and} \quad \frac{\partial\psi}{\partial\nu} + \lambda b(x)\psi \ge 0 \quad \text{on } \partial\Omega, \end{gathered}$$

then $\psi \ge 0$ in $\widetilde{\Omega}$. To prove this, it suffices to construct a positive function Z on $\widetilde{\Omega}$ such that

$$\begin{gathered} L(Z) = \Delta Z + W_\xi Z < 0 \quad \text{in } \widetilde{\Omega}, \\ Z > 0 \quad \text{on } \cup_{j=1}^m \partial B(\xi_j, R\delta_j) \quad \text{and} \quad \frac{\partial Z}{\partial\nu} + \lambda b(x) Z > 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Indeed, let

$$Z(x) = \sum_{j=1}^m z_0\Big(\frac{a(x-\xi_j)}{\delta_j}\Big), \quad a > 0.$$

First, observe that, if $|x - \xi_j| \ge R\delta_j$ for $R > \frac{1}{a}$, then $Z(x) > 0$. On the other hand, since $Z(x) \le m$,

$$W_\xi(x)Z(x) \le D_0 Z(x)\sum_{j=1}^m e^{U_{\delta_j,\xi_j}(x)} \le D_0 Z(x)\sum_{j=1}^m \frac{8\delta_j^2}{|x-\xi_j|^4} \le mD_0\sum_{j=1}^m \frac{8\delta_j^2}{|x-\xi_j|^4},$$

where D_0 is the constant in Lemma 2.3. Further, by the definition of z_0,

$$
\begin{aligned}
-\Delta Z(x) &= \sum_{j=1}^m a^2 \frac{8\delta_j^2(a^2|x-\xi_j|^2-\delta_j^2)}{(a^2|x-\xi_j|^2+\delta_j^2)^3} \\
&\geq \frac{1}{3}\sum_{j=1}^m \frac{8a^2\delta_j^2}{(a^2|x-\xi_j|^2+\delta_j^2)^2} \geq \frac{4}{27}\sum_{j=1}^m \frac{8\delta_j^2}{a^2|x-\xi_j|^4},
\end{aligned}
$$

provided $R>\sqrt{3}/a$. Thus, if a is taken small and fixed, but independent of p and λ, and R is chosen sufficiently large depending on this a, then we have that

$$
L(Z)=\Delta Z+W_\xi Z \leq \Big(-\frac{4}{27}\frac{1}{a^2}+mD_0\Big)\sum_{j=1}^m \frac{8\delta_j^2}{|x-\xi_j|^4}<0.
$$

Moreover,

$$
\begin{gathered}
\big|\frac{\partial}{\partial\nu}Z(x)\big| \leq \sum_{j=1}^m \frac{C\delta_j^2}{a^2|x-\xi_j|^3} = O\Big(\frac{\rho^2}{a^2\varepsilon^3}\Big) \quad \text{on } \partial\Omega, \\
Z(x)\geq \frac{1}{2} \quad \text{on } \partial\Omega\cup\big(\cup_{j=1}^m \partial B(\xi_j,R\delta_j)\big),
\end{gathered}
$$

which, together with (2.4), imply that on $\partial\Omega$,

$$
\frac{\partial Z}{\partial\nu}+\lambda b(x)Z \geq O\Big(\frac{1}{a^2\varepsilon^3}\rho^2\Big)+\frac{1}{2}\lambda b(x) \geq O\big(e^{-p/2}\big)+\frac{1}{2}\lambda\min_{x\in\partial\Omega} b(x)>0 \tag{3.6}
$$

provided that p is chosen sufficiently large. The function $Z(x)$ is what we want.

Step 2: Let R be as before. We define the "inner norm" of ϕ as

$$
\|\phi\|_i = \sup_{x\in\cup_{j=1}^m B(\xi_j,R\delta_j)} |\phi(x)|
$$

and claim that there is a constant $C>0$ such that if $L(\phi)=h$ in Ω, $\frac{\partial\phi}{\partial\nu}+\lambda b(x)\phi=g$ on $\partial\Omega$, then

$$
\|\phi\|_{L^\infty(\Omega)} \leq C\Big(\|\phi\|_i+\|h\|_*+\frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big),
$$

for any $h\in C^{0,\alpha}(\overline{\Omega})$ and $g\in C^{0,\alpha}(\partial\Omega)$. We will establish this estimate with the use of suitable barriers. Let $M=2\operatorname{diam}\Omega$. Consider the problem

$$
\begin{gathered}
-\Delta\psi_j = \frac{2\delta_j}{|x-\xi_j|^3} \quad \text{in } R\delta_j<|x-\xi_j|<M, \\
\psi_j(x)=0 \quad \text{on } |x-\xi_j|=R\delta_j \quad \text{and } |x-\xi_j|=M.
\end{gathered}
$$

Its solution is the positive function

$$
\psi_j = -\frac{2\delta_j}{|x-\xi_j|}+A+B\log|x-\xi_j|,
$$

where

$$
A=\frac{2\delta_j}{M}-B\log M, \quad B=2\Big(\frac{\delta_j}{M}-\frac{1}{R}\Big)\frac{1}{\log\big(\frac{M}{R\delta_j}\big)}<0.
$$

Obviously,

$$
\big|\frac{\partial}{\partial\nu}\psi_j(x)\big| = O\big(\frac{1}{p}\big) \quad \text{on } \partial\Omega. \tag{3.7}
$$

Moreover, for $R\delta_j \le |x-\xi_j| \le M$,

$$\psi_j(x) \le A + B\log(R\delta_j) = \frac{2\delta_j}{M} - B\log\frac{M}{R\delta_j} = \frac{2}{R}. \tag{3.8}$$

Thus, $\psi_j(x)$ is uniformly bounded from above by a constant independent of p and λ. Define now

$$\widetilde{\phi}(x) = C_0\Big(2Z(x) + \sum_{j=1}^m \psi_j(x)\Big)\Big(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big),$$

where Z was defined in the previous step, and $C_0 > 2$ is chosen larger if necessary. First of all, observe that for $x \in \cup_{j=1}^m \partial B(\xi_j, R\delta_j)$, by the definition of Z,

$$\widetilde{\phi}(x) \ge 2C_0\|\phi\|_i Z(x) \ge \|\phi\|_i \ge |\phi(x)|,$$

for $x \in \partial\Omega$, by (3.6), (3.7) and the positivity of $Z(x)$ and $\psi_j(x)$,

$$\begin{aligned}
&\frac{\partial\widetilde{\phi}}{\partial\nu}(x) + \lambda b(x)\widetilde{\phi}(x)\\
&\ge \Big[O\Big(e^{-\frac{1}{2}p} + \frac{1}{p}\Big) + C_0\lambda \min_{x\in\partial\Omega} b(x)\Big]\Big(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big)\\
&\ge \frac{1}{2}C_0\lambda\Big(\min_{x\in\partial\Omega} b(x)\Big)\Big(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big) \ge |g(x)|,
\end{aligned}$$

and for $x \in \Omega\setminus\cup_{j=1}^m B(\xi_j, R\delta_j)$, by (2.28), (3.8) and the definition of $\|\cdot\|_*$ in (2.26),

$$\begin{aligned}
L(\widetilde{\phi}) &\le C_0\Big(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big)\sum_{j=1}^m L(\psi_j)(x)\\
&= C_0\Big(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big)\sum_{j=1}^m \big(-\frac{2\delta_j}{|x-\xi_j|^3} + W_\xi(x)\psi_j(x)\big)\\
&\le C_0\Big(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big)\sum_{j=1}^m \Big(-\frac{2\delta_j}{|x-\xi_j|^3} + \frac{2mD_0}{R}e^{U_{\delta_j,\xi_j}(x)}\Big)\\
&\le -C_0\|h\|_*\sum_{j=1}^m \frac{\delta_j}{(\delta_j^2 + |x-\xi_j|^2)^{3/2}}\\
&\le -|h(x)| \le -|L(\phi)|(x),
\end{aligned}$$

provided $R > 16mD_0$, p and λ large enough. Hence, by the maximum principle in Step 1, we obtain

$$|\phi(x)| \le \widetilde{\phi}(x) \quad \text{for } x \in \widetilde{\Omega},$$

and therefore, since $Z(x) \le m$ and $\psi_j(x) \le \frac{2}{R}$,

$$\|\phi\|_{L^\infty(\Omega)} \le C\Big(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big).$$

Step 3: We prove uniform a priori estimates for solutions ϕ of the problem $L(\phi) = h$ in Ω, $\frac{\partial\phi}{\partial\nu} + \lambda b(x)\phi = g$ on $\partial\Omega$, where $h \in C^{0,\alpha}(\overline{\Omega})$, $g \in C^{0,\alpha}(\partial\Omega)$ and in addition we prove the orthogonality conditions:

$$\int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij}\phi = 0 \quad \text{for } i = 0,1,2;\ j = 1,\dots,m. \tag{3.9}$$

Namely, we prove that there exists $C > 0$ such that for $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, $h \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{0,\alpha}(\partial\Omega)$,

$$\|\phi\|_{L^\infty(\Omega)} \le C\Big(\|h\|_* + \frac{1}{\lambda}\|g\|_{L^\infty(\partial\Omega)}\Big),$$

for p and λ sufficiently large. By contradiction, assume the existence of sequences $p_n \to +\infty$, $\lambda_n \to +\infty$, points $\xi^n = (\xi_1^n, \dots, \xi_m^n) \in \mathcal{O}_\varepsilon$, functions h_n, g_n and associated solutions ϕ_n such that $\|h_n\|_* \to 0$, $\frac{1}{\lambda_n}\|g_n\|_{L^\infty(\partial\Omega)} \to 0$ and $\|\phi_n\|_{L^\infty(\Omega)} = 1$. Since $\|\phi_n\|_{L^\infty(\Omega)} = 1$, Step 2 shows that $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$. Set $\widehat{\phi}_j^n(y) = \phi_n(\delta_j^n y + \xi_j^n)$ for $j = 1, \dots, m$. By (2.29) elliptic estimates imply that $\widehat{\phi}_j^n$ converges uniformly over compact sets to a bounded solution $\widehat{\phi}_j^\infty$ of equation (3.2). Furthermore, $\widehat{\phi}_j^\infty$ is a linear combination of the functions z_i, $i = 0, 1, 2$, defined in (3.1). Since $\|\widehat{\phi}_j^n\|_{L^\infty(\Omega)} \le 1$, by Lebesgue's theorem, the orthogonality conditions on $\widehat{\phi}_j^n$ pass to the limit and give

$$\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_i(y)\widehat{\phi}_j^\infty dy = 0 \quad \text{for } i = 0, 1, 2.$$

Hence, $\widehat{\phi}_j^\infty \equiv 0$ for any $j = 1, \dots, m$ contradicting $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$.

Step 4: We prove that there exists a positive constant $C > 0$ such that any solution ϕ of equation $L(\phi) = h$ in Ω, $\frac{\partial\phi}{\partial\nu} + \lambda b(x)\phi = 0$ on $\partial\Omega$ and in addition the orthogonality conditions:

$$\int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij}\phi = 0 \quad \text{for } i = 1, 2;\ j = 1, \dots, m, \tag{3.10}$$

satisfies

$$\|\phi\|_{L^\infty(\Omega)} \le Cp\|h\|_*,$$

for $h \in C^{0,\alpha}(\overline{\Omega})$. Proceeding by contradiction as in Step 3, we can suppose further that

$$\|\phi_n\|_{L^\infty(\Omega)} = 1, \quad p_n\|h_n\|_* \to 0 \quad \text{as } n \to +\infty. \tag{3.11}$$

but we lose the condition $\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_0(y)\widehat{\phi}_j^\infty = 0$ in the limit. Hence, we have that

$$\widehat{\phi}_j^n \to \widehat{\phi}_j^\infty = C_j \frac{|y|^2 - 1}{|y|^2 + 1} \quad \text{in } C^0_{\rm loc}(\mathbb{R}^2_+), \tag{3.12}$$

for some constants C_j. To reach a contradiction, we have to show that $C_j = 0$ for any $j = 1, \dots, m$. We will obtain it from the stronger condition (3.11) on h_n.

To this end, we perform the following construction. According to [3, 11], there exist radial solutions ω and ζ respectively of equations

$$\Delta\omega + \frac{8}{(1+|y|^2)^2}\omega = \frac{8}{(1+|y|^2)^2} z_0(y), \quad \Delta\zeta + \frac{8}{(1+|y|^2)^2}\zeta = \frac{8}{(1+|y|^2)^2} \quad \text{in } \mathbb{R}^2,$$

such that

$$\omega(y) = \frac{4}{3}\log|y| + O\Big(\frac{1}{|y|}\Big), \quad \zeta(y) = O\big(\frac{1}{|y|}\big) \quad \text{as } |y| \to +\infty,$$

$$\nabla\omega(y) = \frac{4}{3} \cdot \frac{y}{1+|y|^2} + O\big(\frac{1}{1+|y|^2}\big), \quad \nabla\zeta(y) = O\big(\frac{1}{1+|y|^2}\big) \quad \text{for all } y \in \mathbb{R}^2,$$

since $8\int_0^{+\infty} r\frac{(r^2-1)^2}{(r^2+1)^4}dr = \frac{4}{3}$ and $8\int_0^{+\infty} r\frac{r^2-1}{(r^2+1)^3}dr = 0$.

For simplicity in the rest of this article, we omit the dependence on n. For $j=1,\dots,m$, define

$$u_j(x)=\omega\Big(\frac{x-\xi_j}{\delta_j}\Big)+\frac{4}{3}(\log\delta_j)Z_{0j}(x)+\frac{8\pi}{3}H_\lambda(\xi_j,\xi_j)\zeta\Big(\frac{x-\xi_j}{\delta_j}\Big)$$

and denote its projection $Pu_j=u_j+\widetilde{H}_j$, where $\widetilde{H}_j$ is a correction term defined as the solution of

$$\begin{gathered}-\Delta\widetilde{H}_j=0\quad\text{in }\Omega,\\ \frac{\partial\widetilde{H}_j}{\partial\nu}+\lambda b(x)\widetilde{H}_j=-\frac{\partial u_j}{\partial\nu}-\lambda b(x)u_j\quad\text{on }\partial\Omega.\end{gathered}$$

Observe that on $\partial\Omega$,

$$\Big(\frac{\partial}{\partial\nu}+\lambda b(x)\Big)\Big(\widetilde{H}_j+\frac{8\pi}{3}H_\lambda(x,\xi_j)\Big)=O\big(\lambda\rho\big)+(\log\delta_j)O\big(\lambda\rho^2\big)+H_\lambda(\xi_j,\xi_j)O\big(\lambda\rho\big).$$

From the maximum principle with Robin boundary condition we obtain

$$\begin{gathered}Pu_j=u_j-\frac{8\pi}{3}H_\lambda(x,\xi_j)+O\big(\rho\big)\quad\text{in }C(\overline{\Omega}),\\ Pu_j=-\frac{8\pi}{3}G_\lambda(x,\xi_j)+O(\rho)\quad\text{in }C_{\rm loc}(\overline{\Omega}\setminus\{\xi_j\}).\end{gathered}\tag{3.13}$$

The function Pu_j solves

$$\begin{gathered}\Delta Pu_j+W_\xi Pu_j=e^{U_{\delta_j,\xi_j}}Z_{0j}+\big(W_\xi-e^{U_{\delta_j,\xi_j}}\big)Pu_j+R_j\quad\text{in }\Omega,\\ \frac{\partial}{\partial\nu}Pu_j+\lambda b(x)Pu_j=0\quad\text{on }\partial\Omega,\end{gathered}\tag{3.14}$$

where

$$R_j(x)=\Big(Pu_j-u_j+\frac{8\pi}{3}H_\lambda(\xi_j,\xi_j)\Big)e^{U_{\delta_j,\xi_j}}.\tag{3.15}$$

Multiplying (3.14) by ϕ and integrating by parts we obtain

$$\int_\Omega e^{U_{\delta_j,\xi_j}}Z_{0j}\phi+\int_\Omega\big(W_\xi-e^{U_{\delta_j,\xi_j}}\big)Pu_j\phi=\int_\Omega Pu_jh-\int_\Omega R_j\phi.\tag{3.16}$$

We estimate each term of (3.16). First of all, by Lebesgue's theorem and (3.12) we obtain

$$\int_\Omega e^{U_{\delta_j,\xi_j}}Z_{0j}\phi\longrightarrow C_j\int_{\mathbb{R}^2}\frac{8(|y|^2-1)^2}{(|y|^2+1)^4}dy=\frac{8\pi}{3}C_j.\tag{3.17}$$

From (2.15) and the maximum principle with Robin boundary condition, we deduce that $|\nabla H_\lambda(x,\xi_j)|=O(1)$ holds uniformly in Ω. Thus, by (2.28), (2.29) and (3.13), we have

$$\begin{aligned}&\int_\Omega\big(W_\xi-e^{U_{\delta_j,\xi_j}}\big)Pu_j\phi\\ &=\int_{B(\xi_j,\varepsilon\sqrt{\delta_j})}\big(W_\xi-e^{U_{\delta_j,\xi_j}}\big)Pu_j\phi-\frac{8\pi}{3}\sum_{k\neq j}G_\lambda(\xi_k,\xi_j)\int_{B(\xi_k,\varepsilon\sqrt{\delta_k})}W_\xi\phi+O\big(\sqrt{\rho}\big)\\ &=\int_{B(0,\varepsilon/\sqrt{\delta_j})}\frac{8}{(1+|y|^2)^2}\frac{1}{p}\Big(\omega_1-U_{1,0}-\frac{1}{2}U_{1,0}^2\Big)\frac{4}{3}(\log\delta_j)z_0(y)\widehat{\phi}_j\\ &\quad-\frac{8\pi}{3}\sum_{k\neq j}G_\lambda(\xi_k,\xi_j)\int_{B(0,\varepsilon/\sqrt{\delta_k})}\frac{8}{(1+|y|^2)^2}\widehat{\phi}_k+O\big(\frac{1}{p}\big)\end{aligned}$$

$$= -\frac{C_j}{3}\int_{\mathbb{R}^2}\frac{8(|y|^2-1)^2}{(1+|y|^2)^4}\Big(\omega_1 - U_{1,0} - \frac{1}{2}U_{1,0}^2\Big)(y) + o(1)\,.$$

Lebesgue's theorem and (3.12) imply

$$\begin{aligned}&\int_{B(0,\varepsilon/\sqrt{\delta_j})}\frac{8}{(1+|y|^2)^2}\Big(\omega_1 - U_{1,0} - \frac{1}{2}U_{1,0}^2\Big)z_0(y)\widehat{\phi}_j\\ &\to C_j\int_{\mathbb{R}^2}\frac{8(|y|^2-1)^2}{(1+|y|^2)^4}\Big(\omega_1 - U_{1,0} - \frac{1}{2}U_{1,0}^2\Big),\end{aligned}$$

and

$$\int_{B(0,\varepsilon/\sqrt{\delta_k})}\frac{8}{(1+|y|^2)^2}\widehat{\phi}_k \to C_k\int_{\mathbb{R}^2}\frac{8}{(1+|y|^2)^2}\frac{|y|^2-1}{|y|^2+1} = 0.$$

In a straightforward but tedious comptation, by (2.10) we obtain

$$\int_{\mathbb{R}^2}\frac{8(|y|^2-1)^2}{(1+|y|^2)^4}\Big(\omega_1 - U_{1,0} - \frac{1}{2}U_{1,0}^2\Big)(y)dy = -8\pi.$$

Therefore

$$\int_\Omega \big(W_\xi - e^{U_{\delta_j,\xi_j}}\big)Pu_j\phi = \frac{8\pi}{3}C_j + o(1). \tag{3.18}$$

As for the right-hand side of (3.16), that by (2.26) and (3.13), we have

$$\begin{aligned}\Big|\int_\Omega Pu_j h\Big| &\le C\|h\|_*\sum_{k=1}^m\int_{\Omega_{\delta_k}}\frac{1}{(1+|y|^2)^{3/2}}|Pu_j(\delta_k y+\xi_k)|dy\\ &\le C\|h\|_*\int_{\mathbb{R}^2}\frac{\log(|y|+2)}{(1+|y|^2)^{3/2}}dy + Cp\|h\|_*\int_{\mathbb{R}^2}\frac{dy}{(1+|y|^2)^{3/2}}\\ &\le Cp\|h\|_*,\end{aligned} \tag{3.19}$$

where $\Omega_{\delta_k} := \frac{1}{\delta_k}(\Omega - \{\xi_k\})$. Finally, by (3.13) and (3.15) we deduce

$$\int_\Omega R_j\phi = O\Big(\int_\Omega e^{U_{\delta_j,\xi_j}}\big(|x-\xi_j|+\rho\big)dx\Big) = O\big(e^{-\frac{1}{4}p}\big). \tag{3.20}$$

Hence, inserting (3.17)-(3.20) in (3.16) and taking into account (3.11), we conclude that

$$\frac{16\pi}{3}C_j = o(1) \quad \text{for any } j = 1,\dots,m.$$

Necessarily, $C_j = 0$ by contradiction and the claim is proved.

Step 5: We establish the validity of the a priori estimate

$$\|\phi\|_\infty \le Cp\|h\|_* \tag{3.21}$$

for solutions of problem (3.3) and $h \in C^{0,\alpha}(\overline{\Omega})$. Step 4 gives

$$\|\phi\|_{L^\infty(\Omega)} \le Cp\Big(\|h\|_* + \sum_{i=1}^2\sum_{j=1}^m |c_{ij}|\cdot\|e^{U_{\delta_j,\xi_j}}Z_{ij}\|_*\Big) \le Cp\Big(\|h\|_* + \sum_{i=1}^2\sum_{j=1}^m |c_{ij}|\Big).$$

As before, arguing by contradiction of (3.21), we can proceed as in Step 3 and suppose further that

$$\|\phi_n\|_{L^\infty(\Omega)} = 1, \quad p_n\|h_n\|_* \to 0, \quad p_n\sum_{i=1}^2\sum_{j=1}^m |c_{ij}^n| \ge \delta > 0 \quad \text{as } n\to+\infty. \tag{3.22}$$

We omit the dependence on n. It suffices to estimate the values of the constants c_{ij}. For this aim, we define PZ_{ij} as the projection of Z_{ij} under homogeneous Robin boundary condition, namely

$$\begin{aligned}\Delta PZ_{ij} = \Delta Z_{ij} = -e^{U_{\delta_j,\xi_j}} Z_{ij} \text{in } \Omega,\\ \frac{\partial PZ_{ij}}{\partial \nu} + \lambda b(x) PZ_{ij} = 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{3.23}$$

As in the proof of Lemma 2.1, for $i = 1, 2$ and $j = 1, \dots, m$ we have the expansions:

$$PZ_{ij} = Z_{ij} + 8\pi\delta_j \partial_{(\xi_j)_i} H_\lambda(\cdot, \xi_j) + O\big(\rho^3\big), \quad PZ_{0j} = Z_{0j} - 1 + O\big(\rho^2\big), \tag{3.24}$$

in $C(\overline{\Omega})$ and in $C^2_{\rm loc}(\Omega)$, and

$$PZ_{ij} = 8\pi\delta_j \partial_{(\xi_j)_i} G_\lambda(\cdot, \xi_j) + O(\rho^3), \quad PZ_{0j} = O(\rho^2), \tag{3.25}$$

in $C(\overline{\Omega} \setminus \{\xi_j\})$ and in $C^2_{\rm loc}(\Omega \setminus \{\xi_j\})$. By (3.24), (3.25) and that $|\partial_{(\xi_j)_i} H_\lambda(x, \xi_j)| = O(1)$ uniformly holds in Ω, we can easily deduce the following "orthogonality" relations: for each $i, l = 1, 2$ and $j, k = 1, \dots, m$,

$$\int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij} PZ_{lk} = \Big(64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1 + |y|^2)^4}\Big) \delta_{jk}\delta_{il} + O\big(\rho\big), \tag{3.26}$$

uniformly for any set of points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, where δ_{jk} and δ_{il} denote the Kronecker's symbols.

Multiplying equation (3.3) by PZ_{ij}, $i = 1, 2$, $j = 1, \dots, m$, and integrating by parts we find

$$\sum_{l=1}^{2}\sum_{k=1}^{m} c_{lk} \int_\Omega e^{U_{\delta_k,\xi_k}} Z_{lk} PZ_{ij} + \int_\Omega h PZ_{ij} = \int_\Omega W_\xi \phi PZ_{ij} - \int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij}\phi. \tag{3.27}$$

By (2.28), (2.29) and (3.26), a direct computation shows

$$\begin{aligned}&Dc_{ij} + O\Big(e^{-\frac{p}{2}} \sum_{l,\,k} |c_{lk}| + \|h\|_*\Big)\\ &= \frac{1}{p} \int_{B(0,\varepsilon/\sqrt{\delta_j})} \frac{32 y_i}{(1 + |y|^2)^3} \Big(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2\Big) \widehat{\phi}_j + O\Big(\frac{\|\phi\|_\infty}{p^2}\Big),\end{aligned} \tag{3.28}$$

where $D = 64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}$ and $\widehat{\phi}_j(y) = \phi(\delta_j y + \xi_j)$. Hence, we obtain

$$\sum_{l=1}^{2}\sum_{k=1}^{m} |c_{lk}| = O\Big(\|h\|_* + \frac{1}{p}\|\phi\|_\infty\Big) = o(1). \tag{3.29}$$

As in Step 4, we conclude that for each $j = 1, \dots, m$,

$$\widehat{\phi}_j \to C_j \frac{|y|^2 - 1}{|y|^2 + 1} \quad \text{in } C^0_{\rm loc}(\mathbb{R}^2),$$

with some constant $C_j \in \mathbb{R}$ and thus

$$\begin{aligned}&\int_{B(0,\varepsilon/\sqrt{\delta_j})} \frac{32 y_i}{(1 + |y|^2)^3} \Big(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2\Big) \widehat{\phi}_j\\ &\to C_j \int_{\mathbb{R}^2} \frac{32 y_i (|y|^2 - 1)}{(1 + |y|^2)^4} \Big(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2\Big) = 0.\end{aligned}$$

Therefore,

$$\sum_{l=1}^{2}\sum_{k=1}^{m}|c_{lk}| = o(\frac{1}{p}) + O(\|h\|_*),$$

which is impossible because of (3.22).

Step 6: We prove the solvability of (3.3). To this purpose, we consider the spaces:

$$K_\xi = \big\{ \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}PZ_{ij} : c_{ij} \in \mathbb{R} \text{ for } i=1,2;\ j=1,\dots,m \big\},$$

$$K_\xi^\perp = \big\{ \phi \in L^2(\Omega) : \int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij}\phi = 0 \text{ for } i=1,2;\ j=1,\dots,m \big\}.$$

Define $\Pi_\xi : L^2(\Omega) \to K_\xi$ by

$$\Pi_\xi \phi = \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}PZ_{ij},$$

where the coefficients c_{ij} are uniquely determined (as it follows by (3.26)) by the system

$$\int_\Omega e^{U_{\delta_k,\xi_k}} Z_{lk}\Big(\phi - \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}PZ_{ij}\Big) = 0 \quad \text{for any } l=1,2;\ k=1,\dots,m.$$

Let $\Pi_\xi^\perp = Id - \Pi_\xi : L^2(\Omega) \to K_\xi^\perp$. Moreover, the Hilbert space $K_\xi^\perp \cap H^1(\Omega)$ is endowed with the inner product

$$\langle \phi,\, \psi\rangle_H = \int_\Omega \nabla\phi\nabla\psi + \lambda\int_{\partial\Omega} b(x)\phi\psi.$$

Problem (3.3), expressed in a weak form, is equivalent to find $\phi \in K_\xi^\perp \cap H^1(\Omega)$ such that

$$\langle \phi,\, \psi\rangle_H = \int_\Omega (W_\xi\phi - h)\,\psi \quad \text{for all } \psi \in K_\xi^\perp \cap H^1(\Omega).$$

With the aid of Riesz's representation theorem, this equation gets rewritten in $K_\xi^\perp \cap H^1(\Omega)$ in the operator form $\phi = K(\phi) + \tilde{h}$, where

$$\tilde{h} = -\Pi_\xi^\perp\big[\big(-\Delta\big)|_\Omega + \big(\frac{\partial}{\partial\nu} + \lambda b(x)\big)|_{\partial\Omega}\big]^{-1} h,$$

$$K(\phi) = \Pi_\xi^\perp\big[\big(-\Delta\big)|_\Omega + \big(\frac{\partial}{\partial\nu} + \lambda b(x)\big)|_{\partial\Omega}\big]^{-1} (W_\xi\phi)$$

is a linear compact operator in $K_\xi^\perp \cap H^1(\Omega)$. By the Fredholm's alternative with Robin boundary condition (see [5, 13]), we obtain the unique solvability of this problem for any $\tilde{h} \in K_\xi^\perp$ provided that the homogeneous equation $\phi = K(\phi)$ has only the trivial solution in $K_\xi^\perp \cap H^1(\Omega)$, which in turn follows from the a priori estimate (3.21) in Step 5. Finally, by density we obtain the validity of (3.5) also for $h \in C(\overline{\Omega})$ (not only for $h \in C^{0,\alpha}(\overline{\Omega})$). $\square$

Remark 3.2. Given $h \in C(\overline{\Omega})$, let ϕ be the solution of problem (3.3) given by Proposition 3.1. Multiplying (3.3) against ϕ and integrating by parts, we obtain

$$\|\phi\|_H^2 := \int_\Omega |\nabla\phi|^2 + \lambda\int_{\partial\Omega} b(x)\phi^2 = \int_\Omega W_\xi\phi^2 - \int_\Omega h\phi.$$

By Lemma 2.3, we obtain

$$\|\phi\|_H \le C\left(\|h\|_* + \|\phi\|_\infty\right).$$

Let us solve the nonlinear auxiliary problem: for any set of points $\xi = (\xi_1, \dots, \xi_m)$ in $\mathcal{O}_\varepsilon$, we find a function ϕ, and scalars c_{ij}, $i = 1, 2$, $j = 1, \dots, m$, such that

$$\begin{gathered}
\Delta(U_\xi + \phi) + (U_\xi + \phi)^p = \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} e^{U_{\delta_j,\xi_j}} Z_{ij} \quad \text{in } \Omega, \\
U_\xi + \phi > 0 \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} + \lambda b(x)\phi = 0 \quad \text{on } \partial\Omega, \\
\int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij}\phi = 0 \quad \text{for } i = 1, 2;\ j = 1, \dots, m.
\end{gathered} \tag{3.30}$$

Proposition 3.3. *Let $\varepsilon > 0$ be fixed and small. There exist $C > 0$, $p_0 > 0$ and $\lambda_0 > 0$ such that for any set of points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, any $p > p_0$ and $\lambda > \lambda_0$, problem (3.30) has a unique solution ϕ_ξ, scalars $c_{ij}(\xi)$, $i = 1, 2$, $j = 1, \dots, m$, such that*

$$\|\phi_\xi\|_\infty \le \frac{C}{p^3}, \quad \sum_{i=1}^{2}\sum_{j=1}^{m} |c_{ij}(\xi)| \le \frac{C}{p^4}, \quad \|\phi_\xi\|_H \le \frac{C}{p^3}. \tag{3.31}$$

Furthermore, the map $\xi \to \phi_\xi$ is a C^1-function in $C(\overline{\Omega})$ and $H^1(\Omega)$.

Proof. Proposition 3.1 allows us to apply the contraction mapping principle to find a solution for problem (3.30) satisfying (3.31). Since it is a standard procedure, we shall not present the detailed proof, see [11, Lemma 4.1]. □

4. Variational reduction

After problem (3.30) has been solved, we find a solution of (2.25) and hence to the original problem (1.1) if $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$ satisfies

$$c_{ij}(\xi) = 0 \quad \text{for all } i = 1, 2;\ j = 1, \dots, m. \tag{4.1}$$

Equation (1.1) is the Euler-Lagrange equation of the functional $J_p^\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$J_p^\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega u_+^{p+1} + \frac{\lambda}{2}\int_{\partial\Omega} b(x)u^2. \tag{4.2}$$

We introduce the finite-dimensional restriction $F_p^\lambda : \mathcal{O}_\varepsilon \to \mathbb{R}$ given by

$$F_p^\lambda(\xi) = J_p^\lambda(U_\xi + \phi_\xi), \tag{4.3}$$

where ϕ_ξ is the unique solution to problem (3.30) given by Proposition 3.3.

Proposition 4.1. *The function $F_p^\lambda : \mathcal{O}_\varepsilon \to \mathbb{R}$ is of class C^1. Moreover, for all sufficiently large p and λ, if $D_\xi F_p^\lambda(\xi) = 0$, then ξ satisfies (4.1).*

Proof. The function F_p^λ is of class C^1 since $\xi \to \phi_\xi$ is a C^1-map into $H^1(\Omega)$. Then $D_\xi F_p^\lambda(\xi) = 0$ is equivalent to

$$\begin{aligned}
0 &= (DJ_p^\lambda)'(U_\xi + \phi_\xi)(D_\xi U_\xi + D_\xi \phi_\xi) \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}(\xi)\int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij} D_\xi U_\xi + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}(\xi)\int_\Omega D_\xi\big(e^{U_{\delta_j,\xi_j}} Z_{ij}\big)\phi_\xi,
\end{aligned} \tag{4.4}$$

where the seconde equality is due to $\int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij}\phi_\xi = 0$. From the definition of U_ξ in (2.12), we obtain

$$\begin{aligned}\partial_{(\xi_k)_l} U_\xi = \sum_{j=1}^m \frac{1}{\gamma \mu_j^{2/(p-1)}} \Big\{ \partial_{(\xi_k)_l} \Big[U_{\delta_j,\xi_j}(x) + \frac{1}{p}\omega_1\big(\frac{x-\xi_j}{\delta_j}\big) + \frac{1}{p^2}\omega_2\big(\frac{x-\xi_j}{\delta_j}\big) \\ + \gamma \mu_j^{2/(p-1)} H_j(x)\Big] + O(1)\Big\}.\end{aligned}$$

As in the proof of Lemma 2.1, by the maximum principle with Robin boundary condition we can prove that

$$\begin{aligned}\partial_{(\xi_k)_l}\big[\gamma \mu_j^{2/(p-1)} H_j(x)\big] &= \delta_{kj}\Big(1-\frac{C_1}{4p}-\frac{C_2}{4p^2}\Big) 8\pi \partial_{(\xi_k)_l} H_\lambda(x,\xi_j) \\ &\quad - \Big(2-\frac{C_1}{p}-\frac{C_2}{p^2}\Big)\partial_{(\xi_k)_l}\log\mu_j + O\big(\frac{\rho}{p}\big),\end{aligned}$$

where δ_{kj} denote the Kronecker's symbol. Thus, by (2.1), (2.4), (2.8), (3.4) and (3.24) we have that

$$\begin{aligned}\partial_{(\xi_k)_l} U_\xi = \frac{1}{\delta_k \gamma \mu_k^{2/(p-1)}} \Big\{ \Big(1-\frac{C_1}{4p}-\frac{C_2}{4p^2}\Big) P Z_{lk} + O\Big(\rho^3 + \frac{1}{p}\frac{\delta_k^2}{|x-\xi_k|^2+\delta_k^2}\Big)\Big\} \\ + O\big(\frac{1}{\gamma}\big).\end{aligned}$$

On the other hand, it can be shown that $\|D_\xi(e^{U_{\delta_j,\xi_j}} Z_{ij})\|_{L^\infty(\Omega)} = O(1/\delta_j)$ by computing directly. Consequently, (4.4) can be written as, for each $l = 1,2$ and $k = 1,\dots,m$,

$$\begin{aligned}&-\sum_{i,j} \frac{c_{ij}(\xi)\big[1+O(\frac{1}{p})\big]}{\delta_k \gamma \mu_k^{2/(p-1)}} \int_\Omega e^{U_{\delta_j,\xi_j}} Z_{ij} P Z_{lk} \\ &+ \sum_{i,j} |c_{ij}(\xi)| O\Big(\frac{1}{\gamma} + \|\phi_\xi\|_\infty \int_\Omega \big|\partial_{(\xi_k)_l}\big(e^{U_{\delta_j,\xi_j}} Z_{ij}\big)\big|\Big) = 0,\end{aligned}$$

so that, using (3.26) and (3.31),

$$-\frac{64 c_{lk}(\xi)}{\delta_k \gamma \mu_k^{2/(p-1)}} \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4} dy + O\Big(\frac{1}{\delta_k p\gamma} + \frac{1}{\gamma} + \frac{1}{p^3\delta_k}\Big) \sum_{i=1}^2 \sum_{j=1}^m \big|c_{ij}(\xi)\big| = 0,$$

which implies $c_{lk}(\xi) = 0$. □

Proposition 4.2. *Let $\varepsilon > 0$ be fixed. There exist $p_0 > 0$ and $\lambda_0 > 0$ such that for any $p > p_0$ and $\lambda > \lambda_0$,*

$$\begin{aligned}F_p^\lambda(\xi) = \frac{4\pi m p}{\gamma^2} - \frac{32\pi^2}{\gamma^2}\varphi_m^\lambda(\xi_1,\dots,\xi_m) + \frac{4\pi m}{\gamma^2} \\ + \frac{m}{2\gamma^2}\int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2} U_{1,0} - \Delta\omega_1\Big) + o\big(\frac{1}{p^2}\big)\end{aligned} \tag{4.5}$$

C^1-uniformly with respect to $\xi = (\xi_1,\dots,\xi_m) \in \mathcal{O}_\varepsilon$, where $\varphi_m^\lambda(\xi)$ is defined by (1.7).

Proof. According to the proof of [11, Proposition 5.3], it suffices to establish the expansion (4.5) in the C^0-sense. Multiplying the first equation in (3.30) by $U_\xi+\phi_\xi$ and integrating by parts, we obtain

$$\begin{aligned}\int_\Omega (U_\xi+\phi_\xi)^{p+1} &= \int_\Omega |\nabla(U_\xi+\phi_\xi)|^2+\lambda\int_{\partial\Omega} b(x)(U_\xi+\phi_\xi)^2\\ &\quad+\sum_{i=1}^2\sum_{j=1}^m c_{ij}(\xi)\int_\Omega e^{U_{\delta_j,\xi_j}}Z_{ij}U_\xi.\end{aligned}$$

Since U_ξ is a bounded function, by (3.31) we obtain

$$\int_\Omega (U_\xi+\phi_\xi)^{p+1} = \int_\Omega |\nabla(U_\xi+\phi_\xi)|^2+\lambda\int_{\partial\Omega} b(x)(U_\xi+\phi_\xi)^2+O(\frac{1}{p^4})$$

uniformly for any set of points $\xi=(\xi_1,\dots,\xi_m)\in\mathcal{O}_\varepsilon$. Hence, by (2.20)-(4.3) we obtain

$$\begin{aligned}F_p^\lambda(\xi) &= (\frac12-\frac{1}{p+1})\Big[\Big(\int_\Omega|\nabla U_\xi|^2+\lambda\int_{\partial\Omega}b(x)U_\xi^2\Big)\\ &\quad+2\Big(\int_\Omega\nabla U_\xi\nabla\phi_\xi+\lambda\int_{\partial\Omega}b(x)U_\xi\phi_\xi\Big)\\ &\quad+\Big(\int_\Omega|\nabla\phi_\xi|^2+\lambda\int_{\partial\Omega}b(x)\phi_\xi^2\Big)\Big]+O(\frac{1}{p^4}).\end{aligned}\tag{4.6}$$

We expand the term $\int_\Omega|\nabla U_\xi|^2+\lambda\int_{\partial\Omega}U_\xi^2$: in view of (2.18) we deduce

$$\begin{aligned}&\int_\Omega|\nabla U_\xi|^2+\lambda\int_{\partial\Omega}b(x)U_\xi^2=\int_\Omega(-\Delta U_\xi)U_\xi\\ &=\sum_{j=1}^m\frac{1}{\gamma\mu_j^{2/(p-1)}}\int_{B(\xi_j,\varepsilon)}\Big[e^{U_{\delta_j,\xi_j}}-\frac{1}{p\delta_j^2}\Delta\omega_1\big(\frac{x-\xi_j}{\delta_j}\big)-\frac{1}{p^2\delta_j^2}\Delta\omega_2\big(\frac{x-\xi_j}{\delta_j}\big)\Big]U_\xi\\ &\quad+O(\rho^2)\\ &=\sum_{j=1}^m\frac{1}{\gamma^2\mu_j^{4/(p-1)}}\int_{B(0,\varepsilon/\delta_j)}\Big[\frac{8}{(1+|y|^2)^2}-\frac1p\Delta\omega_1(y)-\frac{1}{p^2}\Delta\omega_2(y)\Big]\\ &\quad\times\Big[p+U_{1,0}(y)+\frac1p\omega_1(y)+\frac{1}{p^2}\omega_2(y)+O(\rho|y|)+O\big(\frac{\rho}{p}\big)\Big]dy+O\big(\rho^2\big)\\ &=\sum_{j=1}^m\frac{1}{\gamma^2\mu_j^{4/(p-1)}}\Big[8\pi p+\int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2}U_{1,0}-\Delta\omega_1\Big)+O\big(\frac1p\big)\Big]\\ &=\frac{8\pi mp}{\gamma^2}-\frac{32\pi}{\gamma^2}\sum_{j=1}^m\log\mu_j+\frac{m}{\gamma^2}\int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2}U_{1,0}-\Delta\omega_1\Big)+O\big(\frac{1}{p^3}\big)\end{aligned}$$

since $\mu_j^{-\frac{4}{p-1}}=1-\frac4p\log\mu_j+O(\frac{1}{p^2})$. Recalling the expansion (2.17) of μ_j, then we obtain

$$\begin{aligned}\int_\Omega|\nabla U_\xi|^2+\lambda\int_{\partial\Omega}b(x)U_\xi^2 &= \frac{8\pi mp}{\gamma^2}-\frac{64\pi^2}{\gamma^2}\varphi_m^\lambda(\xi)+\frac{24\pi m}{\gamma^2}\\ &\quad+\frac{m}{\gamma^2}\int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2}U_{1,0}-\Delta\omega_1\Big)+O\big(\frac{1}{p^3}\big).\end{aligned}\tag{4.7}$$

On the other hand, by (3.31), we have

$$\begin{aligned} &2\Big(\int_\Omega \nabla U_\xi \nabla \phi_\xi + \lambda \int_{\partial\Omega} b(x) U_\xi \phi_\xi\Big) + \Big(\int_\Omega |\nabla \phi_\xi|^2 + \lambda \int_{\partial\Omega} b(x)\phi_\xi^2\Big) \\ &= O\big(\frac{1}{p^{7/2}}\big). \end{aligned} \tag{4.8}$$

Consequently, inserting (4.7)-(4.8) in (4.6), we obtain (4.5). □

5. Proof of Theorem 1.1

Definition 5.1. Let $\mathcal{D}$ be an open set compactly contained in Ω^m with smooth boundary. We recall that φ_m^∞ links in $\mathcal{D}$ at critical level $\mathcal{C}$ relative to B and B_0 if B and B_0 are closed subsets of $\overline{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: let us set Γ to be the class of all maps $\Phi \in C(B, \mathcal{D})$ with the property that there exists a function $\Psi \in C([0,1] \times B, \mathcal{D})$ such that

$$\Psi(0,\cdot) = Id_B, \quad \Psi(1,\cdot) = \Phi \quad \Psi(t,\cdot)|_{B_0} = Id_{B_0} \quad \text{for all } t \in [0,1].$$

We assume

$$\sup_{y\in B_0} \varphi_m^\infty(y) < \mathcal{C} \equiv \inf_{\Phi\in\Gamma} \sup_{y\in B} \varphi_m^\infty(\Phi(y)), \tag{5.1}$$

and for all $y \in \partial\mathcal{D}$ such that $\varphi_m^\infty(y) = \mathcal{C}$, there exists a vector τ_y tangent to $\partial\mathcal{D}$ at y such that

$$\nabla \varphi_m^\infty(y) \cdot \tau_y \neq 0. \tag{5.2}$$

Under these conditions a critical point $\bar{y} \in \mathcal{D}$ of φ_m^∞ with $\varphi_m^\infty(\bar{y}) = \mathcal{C}$ exists, as a standard deformation argument involving the negative gradient flow of φ_m^∞ shows. It is easy to check that the above conditions hold if

$$\inf_{x\in\mathcal{D}} \varphi_m^\infty(y) < \inf_{x\in\partial\mathcal{D}} \varphi_m^\infty(x), \quad \text{or} \quad \sup_{x\in\mathcal{D}} \varphi_m^\infty(x) > \sup_{x\in\partial\mathcal{D}} \varphi_m^\infty(x),$$

namely the case of (possibly degenerate) local minimum or maximum points of φ_m^∞. We call $\mathcal{C}$ a nontrivial critical level of φ_m^∞ in $\mathcal{D}$.

Proof of Theorem 1.1. Since Ω is not simply connected, from the proof of [7, Theorem 1] it follows that given any $m \geq 1$, φ_m^∞ has a nontrivial critical level $\mathcal{C}$ in some open set $\mathcal{D}$, compactly contained in Ω^m. Let us consider the set $\mathcal{D}$, the associated critical value $\mathcal{C}$ and $\xi \in \mathcal{D}$. According to Proposition 4.1, the function $u_{p,\lambda} = U_\xi + \phi_\xi$ where U_ξ is defined in (2.12) and ϕ_ξ is the unique solution to problem (3.30) given by Proposition 3.3, is a solution to problem (1.1) if we adjust ξ so that it is a critical point of $F_p^\lambda(\xi)$ defined by (4.3). This is equivalent to finding a critical point of

$$\widetilde{F}_p^\lambda(\xi) = -\frac{\gamma^2}{32\pi^2}\Big[F_p^\lambda(\xi) - \frac{4\pi mp}{\gamma^2} - \frac{4\pi m}{\gamma^2} - \frac{m}{2\gamma^2}\int_{\mathbb{R}^2}\Big(\frac{8}{(1+|y|^2)^2}U_{1,0} - \Delta\omega_1\Big)\Big].$$

On the other hand, from Proposition 4.2, for $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{D} \cap \mathcal{O}_\varepsilon$, we have

$$\widetilde{F}_p^\lambda(\xi) = \varphi_m^\lambda(\xi) + o(1)\Theta_{p,\lambda}(\xi), \tag{5.3}$$

where $\Theta_{p,\lambda}$ and $\nabla_\xi \Theta_{p,\lambda}$ are uniformly bounded in the considered region as p and λ go to $+\infty$.

We claim that

$$\varphi_m^\lambda(\xi) = \varphi_m^\infty(\xi) + O\big(1/\lambda\big) \quad \text{uniformly in } C^1(\mathcal{O}_\varepsilon) \text{ as } \lambda \to +\infty. \tag{5.4}$$

From the definitions of φ_m^λ and φ_m^∞ it suffices to establish that for any set of points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$,

$$H_\lambda(x, \xi_j) = H_\infty(x, \xi_j) + O\big(1/\lambda\big)$$

in $C(\overline{\Omega})$ and in $C^1_{\rm loc}(\Omega)$ as $\lambda \to +\infty$. Indeed, if we set $h(x) = H_\lambda(x, \xi_j) - H_\infty(x, \xi_j)$, then by (2.15),

$$\begin{gathered} -\Delta h = 0 \quad \text{in } \Omega, \\ \frac{\partial h}{\partial \nu} + \lambda b(x) h = -\frac{\partial G_\infty}{\partial \nu}(x, \xi_j) \quad \text{on } \partial\Omega. \end{gathered}$$

Furthermore, by the maximum principle with Robin boundary condition and the definition of $\mathcal{O}_\varepsilon$ in (2.3), we deduce

$$\max_{\overline{\Omega}} \big|h(x)\big| + \max_{\overline{\Omega}} \big|\operatorname{dist}(x, \partial\Omega) \nabla h(x)\big| \le \frac{C}{\lambda} \Big\| \frac{\partial G_\infty}{\partial \nu}(x, \xi_j) \Big\|_{L^\infty(\partial\Omega)} = O\big(1/\lambda\big).$$

According to Definition 5.1, we have that if $M > \mathcal{C}$, then assumptions (5.1), (5.2) still hold for the function $\min\{M, \varphi_m^\infty(\xi)\}$ as well as for $\min\{M, \varphi_m^\infty(\xi) + o(1)\Theta_{p,\lambda}(\xi) + O(1/\lambda)\}$. By (5.3)-(5.4) it follows that the function $\min\{M, \widetilde{F}_p^\lambda(\xi)\}$ satisfies for all p and λ large assumptions (5.1), (5.2) in $\mathcal{D}$ and therefore has a critical value $\mathcal{C}_{p,\lambda} < M$ which is close to $\mathcal{C}$ in this region. If $\xi_{p,\lambda} \in \mathcal{D}$ is a critical point at this level for $\widetilde{F}_p^\lambda(\xi)$, then since

$$\widetilde{F}_p^\lambda(\xi_{p,\lambda}) = \varphi_m^\infty(\xi_{p,\lambda}) + o(1)\Theta_{p,\lambda}(\xi_{p,\lambda}) + O(1/\lambda) \le \mathcal{C}_{p,\lambda} < M, \tag{5.5}$$

we have that there exists $\varepsilon > 0$ such that $|\xi_{i,p,\lambda} - \xi_{j,p,\lambda}| > 2\varepsilon$, $\operatorname{dist}(\xi_{i,p,\lambda}, \partial\Omega) > 2\varepsilon$. This implies C^1-closeness of $\widetilde{F}_p^\lambda(\xi)$ and φ_m^∞ at this level, hence $\nabla\varphi_m^\infty(\xi_{p,\lambda}) \to 0$ and thus $\nabla\varphi_m^\lambda(\xi_{p,\lambda}) \to 0$ as $p \to +\infty$ and $\lambda \to +\infty$. The function $u_{p,\lambda}(x) = U_{\xi_{p,\lambda}}(x) + \phi_{\xi_{p,\lambda}}(x)$ is therefore a solution with the qualitative properties as predicted in Theorem 1.1. □

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