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OPTIMIZATION PROBLEMS ON THE SIERPINSKI GASKET

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Abstract. We investigate the existence of an optimal process for such an optimal control problem in which the dynamics is given by the Dirichlet problem driven by weak Laplacian on the Sierpinski gasket. To accomplish this task using a direct variational approach with no global growth conditions on the nonlinear term, we consider the existence of solutions, their uniqueness and their dependence on a functional parameter for mentioned Dirichlet problem. This allows us to prove that the optimal control problem admits at least one solution.

1. Introduction

Let V stand for the Sierpiński gasket, V_0 its intrinsic boundary. Let Δ denote the weak Laplacian on V and let measure μ denote the restriction to V of normalized $\log N/\log 2$-dimensional Hausdorff measure, so that $\mu(V) = 1$. Let M be a compact interval of $\mathbb{R}$ and denote $L_M = \{u \in L^2(V, \mu) : u(y) \in M$ for a.e. $y \in V\}$. The aim of this article is to consider an optimal control problem of minimizing the action functional

$$J_0 = \int_V f_0(y, x(y), u(y))d\mu$$

where the admissible pairs satisfy

$$\begin{gathered}\Delta x(y) + a(y)x(y) = f(y, x(y), u(y)) + g(y) \quad \text{for a.e. } y \in V \setminus V_0, \\ x|_{V_0} = 0,\end{gathered} \tag{1.1}$$

and where we assume, apart from the growth conditions, that $g \in L^1(V, \mu)$, $g \neq 0$ a.e. on V, $f : V \times \mathbb{R} \times M \to \mathbb{R}$ is a Caratheodory function and $a \in L^1(V, \mu)$ and $a \leq 0$ almost everywhere in V. Solutions to (1.1) are understood in the weak sense which we will describe in a more detail later. Define $F : V \times \mathbb{R} \times M \to \mathbb{R}$ by $F(y, \xi, u) = \int_0^\xi f(y, x, u)dx$, for a.e. $y \in V$ and every $u \in M$. Concerning the nonlinear term, we will employ the following conditions

- (H1a) for any fixed $u \in M$ and a.e. $y \in V$ the function $x \to F(y, u, v)$ is convex on $\mathbb{R}$;
- (H1b) there exists constants $0 < A < \frac{1}{2(2N+3)}$, $B \in \mathbb{R}$ such that

$$F(y, x, u) \geq -A|x|^2 + B$$

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for all $x \in \mathbb{R}$, $u \in M$ and a. e. $y \in V$;

(H2) for each $r > 0$ there exist functions f_r, $g_r \in L^1(V,\mu)$ such that for all $(x,u) \in H_0^1(V) \times L_M$ satisfying $\|x\|_{H_0^1(V)} \le r$ and for a.e. $y \in V$ it holds
$$|F(y,x(y),u(y))| \le f_r(y), |f(y,x(y),u(y))| \le g_r(y);$$

(H3) $f_0 : V \times \mathbb{R} \times M \to \mathbb{R}$ is measurable with respect to the first variable and continuous with respect to the two last variables and convex in u. Moreover, for any $d > 0$ there exists a function $\psi_d \in L^1(V,\mu)$ such that $|f_0(y,x,u)| \le \psi_d(y)$ a.e. on V for all $x \in [-d,d]$ and $u \in M$.

In case of weakly convergent sequence of parameters, we would require some structure condition on a nonlinear term, i.e. $f(y,x,u) = f_1(y,x) + f_2(y)u$. We define
$$F_1(y,x) = \int_0^x f_1(y,s)ds \quad \text{for a.e. } y \in V.$$
Now we replace (H2) with the assumption

(H4) $f_1 : V \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, $f_2 \in L^2(V,\mu)$; for each $r > 0$ there exist functions f_r, $g_r \in L^1(V,\mu)$ such that for all $x \in H_0^1(V)$ satisfying $\|x\|_{H_0^1(V)} \le r$ and for a.e. $y \in V$ it holds
$$|F_1(y,x(y))| \le f_r(y), \quad |f_1(y,x(y))| \le g_r(y).$$

The main idea to tackle this optimization problem is first to examine the continuous dependence on a functional parameter of problem (1.1) in case of strongly and weakly convergent sequence of parameters. Having obtained these results we can construct the set on which the optimization problem can be minimized, i.e. the set containing all admissible pairs and then minimize J_0 on this set. In order to get the solution to the optimization problem considered, we require only weak convergence of the sequence of parameters. However, we believe that the continuous dependence on parameters results are of independent interest on fractal domains since these to the best knowledge of the author have not been investigated yet. Similar problems for a system described by second order PDE's of the elliptic type considered on domains in $\mathbb{R}^n$ with Dirichlet boundary data were investigated in [18] and [19] using direct variational methods and for second order ODE in [17]. We base on the approach used in the sources mentioned with necessary modifications due to the setting of Sierpinski gasket. However we modify the ideas from these works by putting emphasis on working mainly with weak solutions and using a kind of a iterative technique. To the authors' knowledge such problem was not studied in the setting of Sierpinski gasket before. Necessary conditions of optimality for second-order systems of ordinary differential equations with Dirichlet boundary conditions were given by Goebel and Raitums in [10] and by Idczak in [14] (see also the references therein).

The Sierpinski gasket has the origin in a paper by Sierpinski [23]. This fractal domain can be described as a subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of 1/4 of the area, removing the corresponding open triangle from each of the three constituent triangles and continuing in this way.

The study of the Laplacian on fractals started in physical sciences in [1] and [21, 22]. The Laplacian on the Sierpiński gasket was first constructed in [16] and[11]. Among the contributions to the theory of nonlinear elliptic equations on fractals we mention [4, 6, 8] and [15], [25]. Concerning some recent results by variational methods and critical point theory pertaining to the existence and the multiplicity of solutions by the recently developed variational tools we must mention the following sources [2], [3], [20], [5].

2. Remarks on the abstract setting

Concerning the Sierpinski gasket we employ the following definition and remarks, these follow remarks collected in [2]. Let $N \geq 2$ be a natural number and let $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, \dots, N\}$, the map $S_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}\, x + \frac{1}{2}\, p_i\,.$$

Let $\mathcal{S} := \{S_1, \dots, S_N\}$ and denote by $G\colon \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$G(A) = \cup_{i=1}^{N} S_i(A).$$

It is known that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the attractor of the family $\mathcal{S}$, such that $G(V) = V$; see, Falconer [7, Theorem 9.1].

The set V is called the *Sierpiński gasket* in $\mathbb{R}^{N-1}$. It can be constructed inductively as follows:

Put $V_0 := \{p_1, \dots, p_N\}$ which is called the *intrinsic boundary* of V and define $V_m := G(V_{m-1})$, for $m \geq 1$, and put $V_* := \cup_{m \geq 0} V_m$. Since $p_i = S_i(p_i)$ for $i \in \{1, \dots, N\}$, we have $V_0 \subseteq V_1$, hence $G(V_*) = V_*$. Taking into account that the maps S_i, $i \in \{1, \dots, N\}$, are homeomorphisms, we conclude that $\overline{V_*}$ is a fixed point of G. On the other hand, denoting by C the convex hull of the set $\{p_1, \dots, p_N\}$, we observe that $S_i(C) \subseteq C$ for $i = 1, \dots, N$. Thus $V_m \subseteq C$ for every $m \in \mathbb{N}$, so $\overline{V_*} \subseteq C$. It follows that $\overline{V_*}$ is non-empty and compact, hence $V = \overline{V_*}$.

The set V is endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$.

Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V) := \{u \in C(V) : du|_{V_0} = 0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $\|\cdot\|_\infty$. The space $L^2(V, \mu)$ equipped with the product

$$\langle v, h \rangle = \int_V v(y) h(y) d\mu$$

is obviously a Hilbert space.

For a function $u : V \to \mathbb{R}$ and for $m \in \mathbb{N}$ let

$$W_m(u) = \big(\frac{N+2}{N}\big)^m \sum_{|x-y|=2^{-m},\, x,y \in V_m} (u(x) - u(y))^2. \tag{2.1}$$

Since $W_m(u) \leq W_{m+1}(u)$ for every natural m, we can put

$$W(u) = \lim_{m \to \infty} W_m(u). \tag{2.2}$$

Define now

$$H_0^1(V) := \{u \in C_0(V) \mid W(u) < \infty\}.$$

$H_0^1(V)$ is a dense linear subset of $L^2(V, \mu)$ equipped with the $\|\cdot\|_2$ norm. We now endow $H_0^1(V)$ with the norm

$$\|u\| = \sqrt{W(u)}.$$

There is an inner product defining this norm: for $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$ let

$$\mathcal{W}_m(u,v) = (\frac{N+2}{N})^m \sum_{|x-y|=2^{-m},\, x,y\in V_m} (u(x)-u(y))(v(x)-v(y)).$$

Put

$$\mathcal{W}(u,v) = \lim_{m\to\infty} \mathcal{W}_m(u,v).$$

Note that $\mathcal{W}(u,v) \in \mathbb{R}$ and the space $H_0^1(V)$, equipped with the inner product $\mathcal{W}$, which induces the norm $\|\cdot\|$, becomes real Hilbert spaces. Moreover,

$$\|u\|_\infty \leq (2N+3)\|u\|, \quad \text{for every } u \in H_0^1(V), \tag{2.3}$$

and the embedding

$$(H_0^1(V), \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_\infty) \tag{2.4}$$

is compact, see [9] for further details.

Note that $(H_0^1(V), \|\cdot\|)$ is a Hilbert space which is dense in $L^2(V,\mu)$, that $\mathcal{W}$ is a Dirichlet form on $L^2(V,\mu)$. Let Z be a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$. Then, in [8] it is defined a linear self-adjoint operator $\Delta\colon Z \to L^2(V,\mu)$, the *(weak) Laplacian* on V, such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v d\mu, \quad \text{for every } (u,v) \in Z \times H_0^1(V).$$

Let $H^{-1}(V)$ be the closure of $L^2(V,\mu)$ with respect to the pre-norm

$$\|u\|_{-1} = \sup_{\|h\|=1,\, h\in H_0^1(V)} |\langle u,h\rangle|,$$

$v \in L^2(V,\mu)$ and $h \in H_0^1(V)$. Then $H^{-1}(V)$ is a Hilbert space. Then the relation

$$-\mathcal{W}(u,v) = \langle \Delta u, v\rangle, \quad \forall v \in H_0^1(V),$$

uniquely defines a function $\Delta u \in H^{-1}(V)$ for every $u \in H_0^1(V)$.

3. Existence and uniqueness

A function $x \in H_0^1(V)$ is called a *weak solution* of (1.1) if

$$\begin{aligned} &\mathcal{W}(x,v) - \int_V a(y)x(y)v(y)d\mu \\ &+ \int_V f(y,x(y),u(y))v(x)d\mu + \int_V g(y)x(y)d\mu = 0, \end{aligned} \tag{3.1}$$

for every $v \in H_0^1(V)$. Further on whenever we write that we obtain a solution to (1.1) we mean the weak one. The functional $J : H_0^1(V) \to \mathbb{R}$ given by

$$J(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}\int_V a(y)x^2(y)d\mu + \int_V F(y,x(y),u(y))d\mu + \int_V g(y)x(y)d\mu, \tag{3.2}$$

for all $x \in H_0^1(V)$, is the Euler action functional attached to problem (1.1). Employing calculations as in the classical case on the domain in $\mathbb{R}^n$ and the reasoning from [8, Proposition 2.19] and further in [2] we obtain the following result.

Lemma 3.1. *Assume either* (H2) *or* (H4)*. Then, the functional* $J\colon H_0^1(V)\to\mathbb{R}$ *defined by relation* (3.2) *is a* $C^1(H_0^1(V),\mathbb{R})$ *functional. Moreover,*

$$J'(x)(w)=\mathcal{W}(u,w)-\int_V a(y)x(y)w(x)d\mu+\int_V f(y,x(y),u(y))d\mu+\int_V g(y)w(y)d\mu,$$

for all $w\in H_0^1(V)$ *for each point* $x\in H_0^1(V)$*. In particular,* $x\in H_0^1(V)$ *is a weak solution of problem* (1.1) *if and only if* x *is a critical point of* J*.*

Lemma 3.2. *Assume that either* (H1a), (H2) *or* (H1b), (H2) *hold. Let* $u\in L^2(V,\mu)$ *be fixed. Then* J *is continuously Gâteaux differentiable, weakly lower semicontinuous and coercive and its critical points correspond to the weak solutions of* (1.1)*.*

Proof. Let us take any weakly convergent sequence $\{x_k\}_{k=1}^\infty\subset H_0^1(V)$. Then by (2.4) a sequence $\{x_k\}_{k=1}^\infty$ has a subsequence $\{x_{k_n}\}_{n=1}^\infty$ which is strongly convergent in $L^2(V,\mu)$ and also convergent in $C(V)$. Denote by $\overline{x}\in H_0^1(V)$ the weak limit of $\{x_{k_n}\}_{n=1}^\infty$. Since $\{x_{k_n}\}_{n=1}^\infty$ is bounded there exist a constant $r>0$ such that $\|v_{k_n}\|_{H_0^1(V)}\le r$ for all $n\in N$. Thus from (H3) there exists a function $g_r\in L^1(V,\mu)$ such that $|F(y,x_{k_n}(y),u(y))|\le g_r(y)$ for a.e. $y\in V$. Than by the Lebesgue Dominated Convergence Theorem we obtain

$$\int_V F(y,x_{k_n}(y),u(y))d\mu\to\int_V F(y,\overline{x}(y),u(y))d\mu.$$

Therefore, J is weakly l.s.c. on $H_0^1(V)$ since all other terms of J are weakly l.s.c. on $H_0^1(V)$

Consider first case (H1a). Since f is nondecreasing, it follows that F is convex in the second variable. We see that for all $v\in\mathbb{R}$, $u\in M$ and a.e. $y\in V$ it follows

$$F(y,v,u)\ge f(y,0,u)y+F(y,0,u) \tag{3.3}$$

By (H2) there exist functions f_0, $g_0\in L^1(V,\mu)$ such that

$$|F(y,0,u(y))|\le f_0(y),\quad |f(y,0,u(y))|\le g_0(y)$$

We see by (2.3) that for every $y\in V$,

$$|x(y)|\le\|x\|_\infty\le(2N+3)\|x\|_{H_0^1(V)}. \tag{3.4}$$

Then we see that

$$\begin{aligned}\int_V|f(y,0,u(y))||x(y)|d\mu&\le\|x\|_\infty\int_V|f(y,0,u(y))|d\mu\\&\le((2N+3)\int_V|g_0(y)|d\mu)\|x\|_{H_0^1(V)}\end{aligned}$$

for any $x\in H_0^1(V)$. Thus by (3.3),

$$\int_V F(y,x(y),u(y))d\mu\ge-((2N+3)\int_V|g_0(y)|d\mu)\|x\|_{H_0^1(V)}-\int_V g_0(y)d\mu \tag{3.5}$$

for any $x\in H_0^1(V)$. Next we see that

$$\int_V g(y)x(y)d\mu\le\|x\|_\infty\int_V|g(y)|d\mu\le((2N+3)\int_V|g(y)|d\mu)\|x\|_{H_0^1(V)}$$

Since a is non-positive, it now follows that J is coercive in x for any fixed u. Indeed

$$J(x)\ge\frac12\|x\|^2-\Big((2N+3)\int_V|g_0(y)|d\mu+\int_V|g(y)|d\mu\Big)\|x\|_{H_0^1(V)}-\int_V g_0(y)d\mu$$

so $J(x)\to\infty$ as $\|x\|\to\infty$.

To prove coercivity in case (H1b) we see using the first inequality (3.4) and the fact that $\mu(V)=1$

$$\|x\|_{L^2(V,\mu)} \le \|x\|_\infty \le (2N+3)\|x\|_{H_0^1(V)}$$

for any $x \in H_0^1(V)$. Thus

$$\int_V F(y,x(y),u(y))d\mu \ge -A(2N+3)\|x\|^2_{H_0^1(V)} - B$$

Now by the assumptions on A, we see that J is coercive. □

Replacing (H2) with (H4) we obtain the following result.

Corollary 3.3. *Assume that either* (H1a), (H4) *or* (H1b), (H4) *hold. Let $u \in L^2(V,\mu)$ be fixed. Then J is continuously Gâteaux differentiable, weakly l.s.c and coercive and its critical points correspond to the weak solutions of* (1.1).

The above assertion follows from Lemma 3.1. we obtain the last assertion.

Theorem 3.4. *Let $u \in L^2(V,\mu)$ be fixed. Then Problem* (1.1) *has exactly one solution $\overline{x}_u \in H_0^1(V)$ in case* (H1a), (H2) *and at least one solution in case* (H1b), (H2). *Note that all solutions are non-trivial.*

Proof. By Lemma 3.2 J is Gâteaux differentiable, weakly l.s.c. and coercive on $H_0^1(V)$. Therefore there exists $\overline{x}_u \in H_0^1(V)$ such that $J(\overline{x}_u) = \inf_{v \in H_0^1(V)} J(v)$ and thus $\overline{x}_u$ satisfies (1.1). Since in case (H1a), (H2), (H3) functional J is strictly convex, the critical point is unique. Assuming that 0 is a weak solution, we arrive at contradiction since *then we obtain* $g(y)=0$ for a.e. $y \in V$, which contradicts the assumption on g. □

Corollary 3.5. *Let $u \in L^2(V,\mu)$ be fixed. Problem* (1.1) *has exactly one solution $\overline{x}_u \in H_0^1(V)$ in case* (H1a), (H4) *and at least one solution in case* (H1b), (H4). *Note that all solutions are non-trivial.*

4. Continuous dependence on parameters

Having shown the existence and in one case the uniqueness of a solution, we investigate the dependence on a sequence of parameters. These results will be indispensable in proving the existence of solutions to the optimal control problem. We note that it is not necessary to use the uniqueness of solutions in demonstrating the continuous dependence on parameters. Therefore we would not distinguish between the unique and the non-unique case as far as the methods in the proof are used. We mainly use the iterative technique together with the definition of the weak solution together with coercivity which appears to be uniform in u.

4.1. Case of strongly convergent sequence of parameters.

Theorem 4.1. *Assume that either* (H1a), (H2) *or* (H1b), (H2) *hold. Assume that $\{u_n\}_{n=1}^\infty$ satisfies that $u_k \to u_0 \in L^2(V,\mu)$. Then, for any sequence $\{x_k\}_{k=1}^\infty$ of solutions to* (1.1) *corresponding to u_k, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset H_0^1(V)$ and an element $x_0 \in H_0^1(V)$ such that $x_{n_k} \to x_0$ (strongly) in $H_0^1(V)$ and that x_0 is a weak solution to* (1.1) *corresponding to u_0.*

Proof. We define a sequence $\{x_n\}_{n=1}^{\infty}$, where x_n is a solution to (1.1) with $u = u_n$. Thus the following holds

$$\begin{gathered} -\Delta x_n(y) + a(y)x_n(y) = f(y, x_n(y), u_n(y)) \quad \text{for a.e. } y \in V \setminus V_0, \\ x_n|_{V_0} = 0. \end{gathered} \tag{4.1}$$

We shall investigate the convergence of $\{x_n\}_{n=1}^{\infty}$. By definition of J and by (3.5) there exists a constant $r > 0$ such that $\|x_n\|_{H_0^1(V)} \le r$ for $n \in N$. Indeed, each x_n is the argument of a minimum to J, so we see that

$$\begin{aligned} 0 = J(0) &\ge \frac{1}{2}\|x_n\|^2 - \frac{1}{2}\int_V a(y)x_n^2(y)d\mu - \int_V F(y, x_n(y), u_n(y))d\mu \\ &\quad - \int_V g(y)x_n(y)d\mu \\ &\ge \frac{1}{2}\|x_n\|^2 - \Big((2N+3)\int_V |g_0(y)|d\mu + \int_V |g(y)|d\mu\Big)\|x_n\|_{H_0^1(V)} \\ &\quad - \int_V f_0(y)d\mu. \end{aligned}$$

Hence $\{x_n\}_{n=1}^{\infty}$ is weakly convergent in $H_0^1(V)$ to some x_0, possibly up to a subsequence which we assume to be chosen. We shall observe that x_0 is a solution to (1.1) corresponding to u_0. Observe that by by (2.4) $\{x_n\}_{n=1}^{\infty}$ is also convergent in $C(V)$ and therefore in $L^2(V, \mu)$. Note that since $\{x_n\}_{n=1}^{\infty}$ is bounded by some r say, we obtain by (2.3) for any $v \in H_0^1(V)$ with $\|v\| \le r$

$$|f(y, x_m(y), u_m(y))v(y)|d\mu \le |f(y, x_m(y), u_m(y))|(2N+3)r \le r(2N+3)g_r(y)$$

Since $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ are convergent in $L^2(V, \mu)$ it follows by the Lebesgue Dominated Theorem that

$$\int_V f(y, x_m(y), u_m(y))v(y)d\mu \to \int_V f(y, x_0(y), u_0(y))v(y)d\mu \tag{4.2}$$

We see by the definition of the weak solution that

$$\mathcal{W}(x_m, v) - \int_V a(y)x_m(y)v(y)d\mu + \int_V f(y, x_m(y), u_m(y))v(y)d\mu = 0.$$

Since $\mathcal{W}(x_m, v) \to \mathcal{W}(x_0, v)$ and $\int_V a(y)x_m(y)v(y)d\mu \to \int_V a(y)x_0(y)v(y)d\mu$ as $m \to \infty$, we see by (4.2)

$$\mathcal{W}(x_0, v) - \int_V a(y)x_0(y)v(y)d\mu + \int_V f(y, x_0(y), u_0(y))v(y)d\mu = 0$$

for any $v \in H_0^1(V)$, so x_0 is a weak solution to (1.1) corresponding to u_0.

Now we further examine the convergence of $\{x_n\}_{n=1}^{\infty}$. Namely, we shall show that it is in fact strong. Since each x_n for $n \in \mathbb{N}$ is a critical point we see that for any $m \ge k$ we have

$$0 = \langle J'(x_k), x_k\rangle - \langle J'(x_m), x_m\rangle. \tag{4.3}$$

Writing (4.3) explicitly we obtain

$$\begin{aligned} 0 &= \mathcal{W}(x_k, x_k) - \mathcal{W}(x_m, x_m) \\ &\quad - \int_V a(y)x_k^2(y)d\mu + \int_V f(y, x_k(y), u_k(y))x_k(x)d\mu \\ &\quad + \int_V a(y)x_m^2(y)d\mu - \int_V f(y, x_m(y), u_m(y))x_m(x)d\mu \end{aligned}$$

As already mentioned since $\{x_n\}_{n=1}^{\infty}$ is weakly convergent in $H_0^1(V)$ by (2.4) it is strongly convergent in $L^2(V,\mu)$. Thus for some fixed $\varepsilon > 0$ there exists N_ε^1 that for all $m \geq k \geq N_\varepsilon^1$

$$-\frac{\varepsilon}{2} < \int_V a(y)x_m^2(y)d\mu - \int_V a(y)x_k^2(y)d\mu < \frac{\varepsilon}{2}.$$

By (4.2) for all $m \geq k \geq N_\varepsilon^2$, where N_ε^2 is some number

$$-\frac{\varepsilon}{2} < \int_V f(y,x_m(y),u_m(y))x_m(y)d\mu - \int_V f(y,x_k(y),u_k(y))x_k(y)d\mu < \frac{\varepsilon}{2}$$

Therefore for all $m \geq k \geq N_\varepsilon := \max\{N_\varepsilon^1, N_\varepsilon^2\}$

$$-\frac{\varepsilon}{2} < \mathcal{W}(x_k,x_k) - \mathcal{W}(x_m,x_m) < \frac{\varepsilon}{2}$$

This means that $\{\mathcal{W}(x_n,x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence, and since $H_0^1(V)$ is complete we see that

$$\mathcal{W}(x_n,x_n) \to \mathcal{W}(x_0,x_0) \quad \text{as } n \to \infty.$$

Since also $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to x_0, it converges strongly by the properties of the scalar product. □

4.2. **Case of weakly convergent sequence of parameters.** In Theorem 4.1 the convergence of a sequence of parameters was a strong one. We are now interested in the case when this convergence is weak.

Theorem 4.2. *Assume that either* (H1a), (H4) *or* (H1b), (H4) *hold. Let* $\{u_k\}_{k=1}^{\infty}$ *satisfy that* $u_k \rightharpoonup u_0$ *(weakly)* $L^2(V,\mu)$*. Then, for any sequence* $\{x_k\}_{k=1}^{\infty}$ *of solutions to* (1.1) *corresponding to* u_k*, there exists a subsequence* $\{x_{k_n}\}_{n=1}^{\infty} \subset H_0^1(V)$ *and an element* $x_0 \in H_0^1(V)$ *such that* $x_{k_n} \rightharpoonup x_0$ *(weakly) in* $H_0^1(V)$ *and that* x_0 *is a classical solution to* (1.1) *corresponding to* u_0*.*

Proof. Following the proof of Theorem 4.1 we obtain the weak convergence of a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ of solutions corresponding to a subsequence of parameters. The only change is that now we apply the Lebesgue Dominated Theorem to function f_1 and we observe that for any $v \in H_0^1(V)$,

$$\int_V f_2(y)u_{n_k}(y)v(y)d\mu \to \int_V f_2(y)\overline{u}(y)v(y)d\mu$$

by the weak convergence of $\{u_k\}_{k=1}^{\infty}$. Then we obtain that x_0 is a weak solution corresponding to u_0. □

5. Solvability of the optimal control problem

We construct a set $A \subset H_0^1(V) \times L^2(V,\mu)$ consisting of pairs (x_u,u) chosen as follows: we fix a function $u \in L_M$ and next we take x_u as all solutions to (1.1) corresponding to u with assumptions (H1a), (H4), (H3) or (H1b), (H4), (H3) We recall that to some u there may exist more than one solution whether we employ convexity or not.

Remark 5.1. Since the functions from L_M are pointwisely equibounded we obtain $\lim_{k\to\infty} u_k = \overline{u}$ weakly in $L^2(V,\mu)$, up to a subsequence by (3.4), for any sequence $\{u_k\}_{k=1}^{\infty} \subset L_M$. Moreover, any sequence $\{x_k\}_{k=1}^{\infty}$ of solutions to (1.1) corresponding to such $\{u_k\}_{k=1}^{\infty}$ is necessarily bounded in $H_0^1(V)$ as follows from the proof of Theorem 4.2. Thus by relation (2.3) there exists a $d > 0$ such that $x_k(y) \in [-d,d]$

for all $k = 1, 2, \dots$ and for all $y \in V$. Note that the last relation holds for all $y \in V$ since x_k is continuous for all $k = 1, 2, \dots$.

Theorem 5.2. *Assume that either* (H1a), (H4), (H3) *or* (H1b), (H4), (H3) *hold. There exists a pair* $(\overline{x}, \overline{u}) \in A$ *such that* $J(\overline{x}, \overline{u}) = \inf_{(x_u,u)\in A} J(x_u, u)$.

Proof. From Remark 5.1 it follows that any sequence in A is bounded. Any bounded sequence in $H_0^1(V)$ has a uniformly convergent subsequence and by convexity of f_0 with respect to u we see that J_0 is weakly l.s.c. on $H_0^2(V) \times L^2(V,\mu)$. Assumption (H3) and Remark 5.1 provide that the functional J_0 is bounded from below on A. Thus we may choose a minimizing sequence $\{x_u^k, u^k\}_{k=1}^\infty$ for a functional J such that $\{u^k\}_{k=1}^\infty$ is weakly convergent in $L^2(V,\mu)$ to a certain $\overline{u} \in L_M$. Theorem 4.2 asserts that $\{x_u^k\}_{k=1}^\infty$ converges, possibly up to a subsequence, strongly in $C(V)$, weakly in $H_0^1(V)$ to a certain $\overline{x}$ solving (1.1) for $\overline{u}$ in the weak sense. Thus

$$J_0(\overline{x}, \overline{u}) = \lim \inf_{k\to\infty} J(x_u^k, u^k) \geq J(\overline{x}, \overline{u}) \geq \inf_{(x,u)\in A} J(x, u).$$

Therefore $(\overline{x}, \overline{u})$ solves our optimization problem. $\square$

6. Examples

We conclude the paper with some examples on nonlinear terms satisfying our assumptions related to both the continuous dependence on parameters results and the optimization problem.

Example 6.1. Let $g \in L^1(V,\mu)$, $g \neq 0$ a.e. on V, $h \in L^2(V,\mu)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function. Consider

$$\begin{gathered} \Delta x(y) + a(y)x(y) + f(x(y)) + h(y)u(y) = g(y), \\ x|_{V_0} = 0. \end{gathered} \tag{6.1}$$

Then we see that problem (6.1) satisfies the assumptions of Theorem 4.2 in the convex case.

Example 6.2. Let $g \in L^1(V,\mu)$, $g \neq 0$ a.e. on V, $h \in L^2(V,\mu)$. Let m be an odd number arbitrarily fixed. Consider

$$\begin{gathered} -\Delta x(y) + a(y)x(y) + x^m(y)e^{-u^2(y)} + h(y)u(y) = g(y), \\ x|_{V_0} = 0. \end{gathered} \tag{6.2}$$

Again we see that (6.2) satisfies the assumptions of Theorem 4.1 again in the convex case.

We conclude with an example of the integrand f_0.

Example 6.3. Let $f_0(y, x, u) = h(y)e^{-x^2(y)}g(u)$, where $h \in L^1(V,\mu)$ is positive a.e. on V and $g : \mathbb{R} \to \mathbb{R}$ be a convex continuous function. Function ψ_d reads $\psi_d(y) = h(y)e^{-d^2} \max_{u\in M} g(u)$.

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