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GROWTH OF SOLUTIONS TO SYSTEMS OF q-DIFFERENCE DIFFERENTIAL EQUATIONS

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Abstract. In this article we study the growth and poles of solutions to systems of complex q-difference differential equations. We give growth estimates for the solutions, and give examples showing the existence of solutions to such systems.

1. Introduction and statement of main results

The purpose of this paper is to study the growth of meromorphic solutions to systems of complex q-difference differential equations. We use the fundamental results and the standard notation of the Nevanlinna value distribution theory for meromorphic functions (see [14, 22, 23]). For a meromorphic function f in the whole complex plane $\mathbb{C}$, $S(r,f)$ denotes any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{dt}{t}<\infty$. A meromorphic function $a(z)$ is called a small function with respect to f if $T(r,a(z))=S(r,f)$. We use $\rho(f),\mu(f)$ to denote the order and the lower order of f, which are defined by

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r},\quad \mu(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}.$$

In 2007, Barnett, Halburd, Korhonen and Morgan [2] established an analogue of the Logarithmic Derivative Lemma on q-difference operators. Applying their results, a number of papers focused on the growth of meromorphic solutions to complex q-difference equations, and on the value distribution of difference products and q-differences in the complex plane $\mathbb{C}$, analogous to the Nevanlinna's theory [4, 13, 16, 19, 25].

In 2010, Zheng and Chen [26] further considered the growth of meromorphic solutions to q-difference equations and obtained some results which extended the theorems by Heittokangas et al [15].

Theorem 1.1 ([26, Theorem 2])**.** *Suppose that f is a transcendental meromorphic solution of*

$$\sum_{j=1}^{n}a_j(z)f(q^jz)=R(z,f(z))=\frac{P(z,f(z))}{Q(z,f(z))}.$$

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where $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j(z), b_i(z)$ are rational functions, and P, Q are relatively prime polynomials in f over the field of rational functions satisfying $p = \deg_f P, t = \deg_f Q, d = p - t \geq 2$. If f has infinitely many poles, then for sufficiently large r, $n(r,f) \geq K d^{\frac{\log r}{n \log |q|}}$ holds for some constant $K > 0$. Thus, the lower order of f, which has infinitely many poles, satisfies $\mu(f) \geq \frac{\log d}{n \log |q|}$.

Recently, Gao [6, 7, 8] and Xu [20, 21] investigated the growth and existence of meromorphic solutions to systems of complex difference equations, and obtained some existence theorems and estimates on the proximity functions and the counting functions of solutions of some systems.

In 2013, Wang, Huang and Xu [18] investigated the growth and poles of meromorphic solutions to systems of complex q-difference equations and obtained the following result.

Theorem 1.2 ([18, Theorem 1.5])**.** *Suppose that (f_1, f_2) is a pair of transcendental meromorphic functions that satisfy the system of q-shift difference equations*

$$\sum_{j=1}^{n_1} a_j^1(z) f_1(q^j z + c_j) = \frac{P_2(z, f_2(z))}{Q_2(z, f_2(z))},$$
$$\sum_{j=1}^{n_2} a_j^2(z) f_2(q^j z + c_j) = \frac{P_1(z, f_1(z))}{Q_1(z, f_1(z))},$$

where $c_j \in \mathbb{C} \setminus \{0\}$, $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j^t(z)$, $t = 1, 2$ are rational functions, and P_t, Q_t are relatively prime polynomials in f_t over the field of rational functions satisfying $p_t = \deg_{f_t} P_t, l_t = \deg_{f_t} Q_t$, $d_t = p_t - l_t \geq 2$, $t = 1, 2$. If f_t $(t = 1, 2)$ have infinitely many poles, then for sufficiently large r,

$$n(r, f_t) \geq K_t (d_1 d_2)^{\frac{\log r}{(n_1 + n_2) \log |q|}}, \quad t = 1, 2,$$

and

$$\mu(f_1) + \mu(f_2) \geq \frac{2(\log d_1 + \log d_2)}{(n_1 + n_2) \log |q|}.$$

In 2012, Beardon [3] studied entire solutions of the generalized functional equation

$$f(qz) = q f(z) f'(z), \quad f(0) = 0, \tag{1.1}$$

where q is a non-zero complex number. To state the results of Beardon [3], we firstly introduce some notation as follows.

Let the formal series $\mathcal{O}$ and $\mathcal{I}$ be defined by

$$\mathcal{O} := 0 + 0z + 0z^2 + \dots, \quad \mathcal{I} := 0 + 1z + 0z^2 + 0z^3 + \dots.$$

Let $\mathcal{K}_p = \{z : z^p = p + 2\}$, $(p = 1, 2, \dots)$, and $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \dots$. Based on the above definitions, Beardon obtained two main theorems as follows.

Theorem 1.3 ([3])**.** *Any transcendental solution f of (1.1) is of the form*

$$f(z) = z + z(bz^p + \dots),$$

where p is a positive integer, $b \not\equiv 0$ and $q \in \mathcal{K}_p$. In particular, if $q \notin \mathcal{K}$, then the only formal solutions of (1.1) are $\mathcal{O}$ and $\mathcal{I}$.

Theorem 1.4 ([3]). *For each positive integer p, there is a unique real entire function*
$$F_p(z) = z(1 + z^p + b_2 z^{2p} + b_3 z^{3p} + \dots),$$
which is a solution of (1.1) for each $q \in \mathcal{K}_p$. Further, if $q \in \mathcal{K}_p$, then the only transcendental solutions of (1.1) are the linear conjugates of F_p.

In 2013, Zhang [24] further studied the growth of solutions of (1.1) and prove the following theorem.

Theorem 1.5 ([24, Theorem 1.1]). *Suppose that f is a transcendental solution of (1.1) for $q \in \mathcal{K}$, then the order of f satisfies*
$$\rho(f) \le \frac{\log 2}{\log |q|}.$$

Inspired by the ideas of Gao [6, 7, 8], Xu [20, 21] and Beardon [3], we investigate the growth of solutions of some systems of q-difference-differential equations and obtain the following results.

Theorem 1.6. *Suppose that (f_1, f_2) are a pair of solutions of system*
$$\begin{aligned} f_1(q_1 z) &= c_1 f_2(z) f_2'(z),\\ f_2(q_2 z) &= c_2 f_1(z) f_1'(z), \end{aligned} \tag{1.2}$$
where $q_1, q_2, c_1(\neq 0), c_2(\neq 0) \in \mathbb{C}$ and $|q_1| > 1, |q_2| > 1$. If $f_i (i = 1, 2)$ are transcendental entire functions. Then
$$\rho(f_i) \le \frac{2 \log 2}{\log |q_1| + \log |q_2|}, \quad i = 1, 2.$$

We easily see that Theorem 1.6 is an extension of Theorem 1.5. The following example shows that system (1.2) has a pair of non-transcendental entire solutions.

Example 1.7. Let $q_1 = q_2 = 2$ and $c_1 = 8, c_2 = -1$. Then $(f_1(z), f_2(z)) = (z, -\frac{1}{2}z)$ satisfies the system
$$\begin{aligned} f_1(2z) &= 8 f_2(z) f_2'(z),\\ f_2(2z) &= -f_1(z) f_1'(z). \end{aligned}$$

Remark 1.8. In fact, if f_1 and f_2 are polynomials, by a simple computation, we obtain that f_1 and f_2 are all polynomials of degree 1; that is, $f_1(z) = a_1 z + a_0$ and $f_2(z) = b_1 z + b_0$. Thus, we can obtain the forms of f_1 and f_2 easily.

The following example shows that system (1.2) has a pair of transcendental entire function solutions.

Example 1.9. Let $q_1 = q_2 = 2$ and $c_1 = 2, c_2 = -2$. Then $(f_1(z), f_2(z)) = (\sin z, -\sin z)$ satisfy the system
$$\begin{aligned} f_1(2z) &= 2 f_2(z) f_2'(z),\\ f_2(2z) &= -2 f_1(z) f_1'(z), \end{aligned}$$
and
$$\rho(f_i) = 1 = \frac{2 \log 2}{2 \log 2}, \quad i = 1, 2.$$

Remark 1.10. By contrasting the forms of (1.1) and (1.2), we pose the following question: Does system (1.2) have a pair of transcendental entire (or meromorphic) solutions with $c_1 = q_1$ and $c_2 = q_2$ or $c_1 = q_2$ and $c_2 = q_1$?

The following results show that system (1.2) has a pair of transcendental meromorphic solutions when the constants c_1, c_2 of the right of system (1.2) are replaced by two functions.

Theorem 1.11. *Let (f_1, f_2) be a pair of transcendental solutions of system*

$$\begin{aligned} f_1^{n_1}(q_1 z) &= R_1(z) f_2(z)[f_2^{(j)}(z)]^{s_1}, \\ f_2^{n_2}(q_2 z) &= R_2(z) f_1(z)[f_1^{(j)}(z)]^{s_2}, \end{aligned} \tag{1.3}$$

where $q_1, q_2 \in \mathbb{C}$ and $|q_1| > 1$, $|q_2| > 1$, n_1, n_2, j, s_1, s_2 are positive integers and $R_1(z), R_2(z)$ are rational functions in z. If f_i $(i = 1, 2)$ are entire functions, then $n_1 n_2 \le (s_1 + 1)(s_2 + 1)$ and

$$\rho(f_i) \le \frac{\log[(s_1+1)(s_2+1)] - \log(n_1 n_2)}{\log|q_1| + \log|q_2|}, \quad i = 1, 2.$$

Furthermore, if $n_1 = n_2 = 1$ and $f_i (i = 1, 2)$ are meromorphic functions with infinitely many poles, then

$$\frac{\log[(s_1+1)(s_2+1)]}{\log|q_1| + \log|q_2|} \le \mu(f_i) \le \rho(f_i) \le \frac{\log[(s_1 j + s_1 + 1)(s_2 j + s_2 + 1)]}{\log|q_1| + \log|q_2|},$$

for $i = 1, 2$.

The following example shows that system (1.3) has pairs of transcendental entire and meromorphic solutions.

Example 1.12. Let $q_1 = q_2 = 2$, $n_1 = n_2 = 1$ and $s_1 = s_2 = 1$, then $(f_1, f_2) = (e^z, ze^z)$ satisfies

$$\begin{gathered} f_1(2z) = \frac{1}{z(z+1)} f_2(z) f_2'(z), \\ f_2(2z) = 2z f_1(z) f_1'(z), \end{gathered}$$

and

$$\rho(f_i) = 1 \le \frac{2\log 2}{2\log 2}.$$

Example 1.13. Let $q_1 = q_2 = 2$, $n_1 = n_2 = 1$ and $s_1 = s_2 = 1$, then $(f_1, f_2) = (\frac{e^z}{z}, \frac{e^z}{z^2})$ satisfies

$$\begin{aligned} f_1(2z) &= \frac{2z^6}{z-2} f_2(z) f_2'(z), \\ f_2(2z) &= \frac{4z^5}{z-1} f_1(z) f_1'(z), \end{aligned}$$

and

$$\frac{2\log 2}{2\log 2} = 1 \le \mu(f_i) = \rho(f_i) = 1 \le \frac{2\log 3}{2\log 2}, \; i = 1, 2.$$

Theorem 1.14. *Let (f_1, f_2) be a pair of transcendental solutions of the system*

$$\begin{aligned} f_1^{n_1}(q_1 z) &= \varphi_1(z) f_2(z)[f_2^{(j)}(z)]^{s_1}, \\ f_2^{n_2}(q_2 z) &= \varphi_2(z) f_1(z)[f_1^{(j)}(z)]^{s_2}, \end{aligned} \tag{1.4}$$

where $q_1, q_2 \in \mathbb{C}$ and $|q_1| > 1$, $|q_2| > 1$, n_1, n_2, j, s_1, s_2 are positive integers and $\varphi_t(z)$ $(t = 1, 2)$ are small functions with respect of f_i $(i = 1, 2)$. If $f_i (i = 1, 2)$ are meromorphic functions satisfying $\overline{N}(r, f_i) = S(r, f_i)$, $(i = 1, 2)$, then $n_1 n_2 \leq (s_1 + 1)(s_2 + 1)$ and f_i $(i = 1, 2)$ satisfy

$$\rho(f_i) \leq \frac{\log[(s_1 + 1)(s_2 + 1)] - \log(n_1 n_2)}{\log|q_1| + \log|q_2|}, \quad i = 1, 2.$$

Furthermore, if $n_1 = n_2 = 1$ and $f_i (i = 1, 2)$ have infinitely many poles, and the number of distinct common poles of $f_i (i = 1, 2)$ and $\frac{1}{\varphi_t}$, $(t = 1, 2)$ are finite, then we have

$$\mu(f_i) = \rho(f_i) = \frac{\log[(s_1 + 1)(s_2 + 1)]}{\log|q_1| + \log|q_2|}, \quad i = 1, 2. \tag{1.5}$$

The following example shows a case where equality in (1.5) holds.

Example 1.15. Let $q_1 = q_2 = 2$ and $s_1 = s_2 = 3$, then $(f_1, f_2) = (\frac{e^{z^2}}{z-1}, \frac{e^{z^2}}{z})$ satisfy the system

$$\begin{aligned} f_1(2z) &= \varphi_1(z) f_2(z) [f_2'(z)]^3, \\ f_2(2z) &= \varphi_2(z) f_1(z) [f_1'(z)]^3, \end{aligned}$$

where

$$\varphi_1(z) = \frac{z^7 (2z - 1)}{(2z^2 - 1)^3}, \quad \varphi_2(z) = \frac{2z(z - 1)^4}{(2z^2 - 2z - 1)^3}.$$

Thus, we have $T(r, \varphi_t) = O(\log r) = S(r, f_i)$ and

$$\rho(f_i) = 2 = \frac{\log[(3 + 1)(3 + 1)]}{2 \log 2}, \quad i = 1, 2.$$

Let $p(z) = p_k z^k + p_{k-1} z^{k-1} + \dots + p_1 z + p_0$, where $p_k (\not\equiv 0), \dots, p_0$ are complex constants. Now, we will investigate the growth of solutions of such systems, which qz is replaced by $p(z)$ in systems (1.2)-(1.4), and obtain the following results.

Theorem 1.16. *Let (f_1, f_2) be a pair of transcendental solutions to the system*

$$\begin{aligned} f_1(p(z))^{n_1} &= \varphi_1(z) f_2(z) [f_2^{(j)}(z)]^{s_1}, \\ f_2(p(z))^{n_2} &= \varphi_2(z) f_1(z) [f_1^{(j)}(z)]^{s_2}, \end{aligned} \tag{1.6}$$

where $k \geq 2$, n_1, n_2, j, s_1, s_2 are positive integers and $\varphi_t(z)$ $(t = 1, 2)$ are small functions with respect of $f_i (i = 1, 2)$. If f_i $(i = 1, 2)$ are transcendental meromorphic functions and $n_1 n_2 < (s_1 j + s_1 + 1)(s_2 j + s_2 + 1)$, then f_i $(i = 1, 2)$ satisfy

$$T(r, f_i) = O((\log r)^\alpha), \quad i = 1, 2,$$

where

$$\alpha = \frac{\log(s_1 j + s_1 + 1)(s_2 j + s_2 + 1) - \log(n_1 n_2)}{2 \log k}.$$

Theorem 1.17. *Suppose that (f_1, f_2) are a pair of transcendental meromorphic solutions of system*

$$\begin{aligned} f_1(q_1 z) f_2'(z) &= R_2(z, f_2(z)) = \frac{P_2(z, f_2(z))}{Q_2(z, f_2(z))}, \\ f_2(q_2 z) f_1'(z) &= R_1(z, f_1(z)) = \frac{P_1(z, f_1(z))}{Q_1(z, f_1(z))}, \end{aligned} \tag{1.7}$$

where $q_1, q_2 \in \mathbb{C}$, $|q_1| > 1$, $|q_2| > 1$, and P_i, Q_i $(i = 1, 2)$ are relatively prime polynomials in f_i over the field of rational functions satisfying $p_i = \deg_f P_i$, $t_i = \deg_f Q_i, d_i = p_i - t_i \geq 4, (i = 1, 2)$, where the coefficients of P_i, Q_i, $(i = 1, 2)$ are rational functions in z. If $f_i(i = 1, 2)$ have infinitely many poles, then for sufficiently large r, we get that

$$n(r, f_i) \geq K_i[(d_1 - 1)(d_2 - 1)]^{\frac{\log r}{\log |q_1| + \log |q_2|}}, \quad i = 1, 2$$

hold for some constant $K_i > 0$. Thus, the order and the lower order of $f_i(i = 1, 2)$, which has infinitely many poles, satisfy

$$\rho(f_i) \geq \mu(f_i) \geq \frac{\log(d_1 - 1) + \log(d_2 - 1)}{\log |q_1| + \log |q_2|}, \ i = 1, 2.$$

Remark 1.18. Under the conditions of Theorem 1.17, by using the same argument as in Theorem 1.16, we can get that the lower order, order of f_i $(i = 1, 2)$, which has infinitely many poles, satisfy

$$\frac{\log(d_1 - 1) + \log(d_2 - 1)}{\log |q_1| + \log |q_2|} \leq \mu(f_i) \leq \rho(f_i) \leq \frac{\log(d_1 + 2) + \log(d_2 + 2)}{\log |q_1| + \log |q_2|},$$

for $i = 1, 2$.

The following examples show that (1.7) has a pair of non-transcendental solutions.

Example 1.19. Let $q_1 = q_2 = 2$ and $d_1 = 3, d_2 = 4$, then $(f_1, f_2) = (\frac{1}{z}, \frac{1}{z^2})$ satisfies

$$f_1(2z)f_2'(z) = -zf_2(z)^3,$$
$$f_2(2z)f_1'(z) = -\frac{1}{4}f_1(z)^4.$$

The following examples show that (1.7) has a pair of transcendental solutions.

Example 1.20. Let $q_1 = q_2 = 2$ and $d_1 = d_2 = 3$, then $(f_1, f_2) = (\sin z, \cos z)$ satisfies

$$f_1(2z)f_2'(z) = 2f_2(z)^3 - 2f_2(z),$$
$$f_2(2z)f_1'(z) = f_1(z) - 2f_1(z)^3.$$

Then we have $\mu(f_i) = \rho(f_i) = 1 = \frac{\log(3-1)}{\log 2}$, $i = 1, 2$.

Example 1.21. Let $q_1 = q_2 = 2$ and $d_1 = d_2 = 5$, then $(f_1, f_2) = (\frac{e^{z^2}}{z}, \frac{e^{z^2}}{z-1})$ satisfies the system

$$f_1(2z)f_2'(z) = \frac{1}{2z}(z - 1)^3(2z^2 - 2z - 1)f_2(z)^5,$$
$$f_2(2z)f_1'(z) = \frac{1}{2z - 1}z^3(2z^2 - 1)f_1(z)^5.$$

Then, we have $\mu(f_i) = \rho(f_i) = 2 = \frac{\log(5-1)}{\log 2}$, $i = 1, 2$.

Example 1.22. *Let $q_1 = q_2 = 2$ and $d_1 = d_2 = 3$, then $(f_1, f_2) = (\frac{1}{\sin z}, -\frac{1}{\sin z})$ satisfy system*

$$f_1(2z)f_2'(z) = \frac{1}{2}f_2(z)^3,$$

$$f_2(2z)f_1'(z) = -\frac{1}{2}f_1(z)^3.$$

Thus, we have that $f_i(z)$, $(i = 1, 2)$ *have infinitely many poles and* $\mu(f_i) = \rho(f_1) = 1 = \frac{\log(3-1)}{\log 2}$, $i = 1, 2$.

Comparing with Example 1.22 with Theorem 1.17, there remains open question whether or not the condition $d_i = p_i - t_i \geq 4$ may be relaxed to $d_i \geq 3$ or $d_i \geq 2$ in Theorem 1.16.

2. Some Lemmas

Lemma 2.1 (Valiron-Mohon'ko [17]). *Let* $f(z)$ *be a meromorphic function. Then for all irreducible rational functions in* f,

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z)f(z)^i}{\sum_{j=0}^{n} b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, *the characteristic function of* $R(z, f(z))$ *satisfies*

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ *and* $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Lemma 2.2 ([23, p. 37] or [22]). *Let* $f(z)$ *be a nonconstant meromorphic function in the complex plane and* l *be a positive integer. Then*

$$N(r, f^{(l)}) = N(r, f) + l\overline{N}(r, f), \quad T(r, f^{(l)}) \leq T(r, f) + l\overline{N}(r, f) + S(r, f).$$

Lemma 2.3 ([12]). *Let* $\Phi : (1, \infty) \to (0, \infty)$ *be a monotone increasing function, and let* f *be a nonconstant meromorphic function. If for some real constant* $\alpha \in (0, 1)$, *there exist real constants* $K_1 > 0$ *and* $K_2 \geq 1$ *such that*

$$T(r, f) \leq K_1\Phi(\alpha r) + K_2T(\alpha r, f) + S(\alpha r, f),$$

then the order of growth of f *satisfies*

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \limsup_{r\to\infty} \frac{\log \Phi(r)}{\log r}.$$

Lemma 2.4 ([9]). *Let* $f(z)$ *be a transcendental meromorphic function and* $p(z) = p_kz^k + p_{k-1}z^{k-1} + \dots + p_1z + p_0$ *be a complex polynomial of degree* $k > 0$. *For given* $0 < \delta < |p_k|$, *let* $\lambda = |p_k| + \delta, \mu = |p_k| - \delta$, *then for given* $\varepsilon > 0$ *and for* r *large enough,*

$$(1 - \varepsilon)T(\mu r^k, f) \leq T(r, f \circ p) \leq (1 + \varepsilon)T(\lambda r^k, f).$$

Lemma 2.5 ([1, 5, 10]). *Let* $g : (0, +\infty) \to R, h : (0, +\infty) \to R$ *be monotone increasing functions such that* $g(r) \leq h(r)$ *outside of an exceptional set* E *with finite linear measure, or* $g(r) \leq h(r)$, $r \notin H \cup (0, 1]$, *where* $H \subset (1, \infty)$ *is a set of finite logarithmic measure. Then, for any* $\alpha > 1$, *there exists* r_0 *such that* $g(r) \leq h(\alpha r)$ *for all* $r \geq r_0$.

Lemma 2.6 ([10]). *Let* $\psi(r)$ *be a function of* $r(r \geq r_0)$, *positive and bounded in every finite interval.*

(i) Suppose that $\psi(\mu r^m) \leq A\psi(r) + B(r \geq r_0)$, *where* $\mu(\mu > 0)$, $m(m > 1)$, $A(A \geq 1)$, B *are constants. Then* $\psi(r) = O((\log r)^\alpha)$ *with* $\alpha = \frac{\log A}{\log m}$, *unless* $A = 1$ *and* $B > 0$; *and if* $A = 1$ *and* $B > 0$, *then for any* $\varepsilon > 0$, $\psi(r) = O((\log r)^\varepsilon)$.

(ii) Suppose that (with the notation of (i)) $\psi(\mu r^m) \geq A\psi(r)(r \geq r_0)$. Then for all sufficiently large values of r, $\psi(r) \geq K(\log r)^\alpha$ with $\alpha = \frac{\log A}{\log m}$, for some positive constant K.

Lemma 2.7 ([4])**.**
$$T(r, f(qz)) = T(|q|r, f) + O(1)$$
holds for any meromorphic function f and any non-zero constant q.

3. Proofs of Theorems 1.6–1.14

Proof of Theorem 1.6. From (1.2), we have
$$\begin{aligned} T(r, f_1(q_1 z)) &\leq T(r, f_2) + T(r, f_2^{(j)}(z)) + O(1), \\ T(r, f_2(q_2 z)) &\leq T(r, f_1) + T(r, f_1^{(j)}(z)) + O(1). \end{aligned}$$
Since $f_i (i = 1, 2)$ are transcendental entire functions, then it follows by Lemma 2.2 and Lemma 2.7 that
$$\begin{aligned} T(|q_1|r, f_1(z)) &\leq 2T(r, f_2) + S(r, f_2), \\ T(|q_2|r, f_2(z)) &\leq 2T(r, f_1) + S(r, f_1). \end{aligned} \tag{3.1}$$
Thus, from (3.1), we have
$$\begin{aligned} T(|q_1 q_2|r, f_1(z)) &\leq 4T(r, f_1) + S(r, f_1), \\ T(|q_1 q_2|r, f_2(z)) &\leq 4T(r, f_2) + S(r, f_2). \end{aligned} \tag{3.2}$$
Since $|q_1| > 1$, $|q_2| > 1$ and f_i $(i = 1, 2)$ are transcendental, set $\alpha = \frac{1}{|q_1 q_2|}$, it follows from (3.2) that
$$T(r, f_i(z)) \leq 4T(\alpha r, f_i) + S(\alpha r, f_i), \quad i = 1, 2.$$
Since $0 < \alpha < 1$, it follows by Lemma 2.3 that $\rho(f_i) \leq \frac{2\log 2}{\log |q_1 q_2|}$ for $i = 1, 2$.

Proof of Theorem 1.11. Suppose that f_i $(i = 1, 2)$ are transcendental meromorphic solutions of (1.3). Since $R_i(z)$, $(i = 1, 2)$ are rational functions, then we have $T(r, R_i(z)) = O(\log r)$, $(i = 1, 2)$. By Lemma 2.1 and Lemma 2.2, it follows from (1.3) that
$$\begin{aligned} T(r, f_1(q_1 z)) &\leq \frac{1}{n_1} T(r, f_2) + \frac{s_1}{n_1} T(r, f_2^{(j)}(z)) + O(\log r) \\ &\leq \frac{s_1 + 1}{n_1} T(r, f_2) + \frac{j s_1}{n_1} \overline{N}(r, f_2) + S(r, f_2), \\ T(r, f_2(q_2 z)) &\leq \frac{1}{n_2} T(r, f_1) + \frac{s_2}{n_2} T(r, f_1^{(j)}(z)) + O(\log r) \\ &\leq \frac{s_2 + 1}{n_2} T(r, f_1) + \frac{j s_2}{n_2} \overline{N}(r, f_1) + S(r, f_1). \end{aligned}$$
By Lemma 2.7, we obtain
$$\begin{aligned} T(|q_1|r, f_1(z)) &\leq \frac{s_1 j + s_1 + 1}{n_1} T(r, f_2) + S(r, f_2), \\ T(|q_2|r, f_2(z)) &\leq \frac{s_2 j + s_2 + 1}{n_2} T(r, f_1) + S(r, f_1). \end{aligned}$$

Then we have

$$T(|q_1q_2|r, f_i) \le \frac{s_1j+s_1+1}{n_1}\frac{s_2j+s_2+1}{n_2}T(r, f_i) + S(r, f_i), \quad i = 1, 2. \tag{3.3}$$

Since $|q_1| > 1$ and $|q_2| > 1$, set $\alpha = \frac{1}{|q_1q_2|}$, then $0 < \alpha < 1$. From (3.3), we have

$$T(r, f_i) \le \frac{s_1j+s_1+1}{n_1}\frac{s_2j+s_2+1}{n_2}T(\alpha r, f_i) + S(\alpha r, f_i), \quad i = 1, 2.$$

Since $f_i (i = 1, 2)$ are transcendental functions, then $n_1n_2 \le (s_1j + s_1 + 1)(s_2j + s_2 + 1)$, and by Lemma 2.3, we have

$$\rho(f_i) \le \frac{\log[(s_1j+s_1+1)(s_2j+s_2+1)]}{\log|q_1| + \log|q_2|}, \quad i = 1, 2.$$

If f_i $(i = 1, 2)$ are transcendental entire functions, similar to above argument, we can easily obtain

$$\rho(f_i) \le \frac{\log[(s_1+1)(s_2+1)] - \log(n_1n_2)}{\log|q_1| + \log|q_2|}, \quad i = 1, 2.$$

Since $R_1(z), R_2(z)$ are rational functions, we can choose a sufficiently large constant $R > 0$ such that $R_1(z), R_2(z)$ have no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$. Since f_1 has infinitely many poles, we can choose a pole z_0 of f_1 of multiplicity $\tau \ge 1$ satisfying $|z_0| > R$. Then the right side of the second equation in system (1.3) has a pole of multiplicity $\tau_1' = (s_2 + 1)\tau + s_2j$ at z_0. Then f_2 has a pole of multiplicity τ_1' at q_2z_0. Replacing z by q_2z_0 in the first equation in system (1.3), we have that f_1 has a pole of multiplicity $\tau_1 = (s_1 + 1)(s_2 + 1)\tau + s_2(s_1 + 1)j + s_1j$ at $q_1q_2z_0$. We proceed to follow the step above. Since $R_1(z), R_2(z)$ have no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$ and f_1, f_2 have infinitely many poles, we may construct poles $\zeta_k = |q_1q_2|^kz_0$ $k \in N_+$ of f of multiplicity τ_k satisfying

$$\begin{aligned}
\tau_k &= (s_1 + 1)(s_2 + 1)\tau_{k-1} + s_2(s_1 + 1)j + s_1j \\
&= [(s_1 + 1)(s_2 + 1)]^k\tau + j[s_2(s_1 + 1) + s_1]\left\{[(s_1 + 1)(s_2 + 1)]^{k-1} + \dots + 1\right\} \\
&= [(s_1 + 1)(s_2 + 1)]^k\tau + j[s_2(s_1 + 1) + s_1]\frac{[(s_1 + 1)(s_2 + 1)]^k - 1}{(s_1 + 1)(s_2 + 1) - 1},
\end{aligned}$$

as $k \to \infty, k \in \mathbb{N}$. Since $|q| > 1$, it follows that $|\zeta_k| \to \infty$ as $k \to \infty$, for sufficiently large k, we have

$$\begin{aligned}
{}^k\tau &\le \tau_k \le \tau + \tau_1 + \dots + \tau_k \\
&\le n(|\zeta_k|, f_1) \le n(|q_1q_2|^k|z_0|, f_1).
\end{aligned} \tag{3.4}$$

Thus, for each sufficiently large r, there exists a $k \in \mathbb{N}$ such that

$$r \in [|q_1q_2|^k|z_0|, |q_1q_2|^{(k+1)}|z_0|), \quad \text{i.e. } k > \frac{\log r - \log r_0 - \log|q_1q_2|}{\log|q_1q_2|}. \tag{3.5}$$

Thus, it follows from (3.4) and (3.5) that

$$\begin{aligned}
n(r, f_1) &\ge [(s_1 + 1)(s_2 + 1)]^k\tau \ge \tau[(s_1 + 1)(s_2 + 1)]^{\frac{\log r - \log r_0 - \log|q_1q_2|}{\log|q_1q_2|}} \\
&\ge K_1[(s_1 + 1)(s_2 + 1)]^{\frac{\log r}{\log|q_1| + \log|q_2|}},
\end{aligned}$$

where

$$K_1 = \tau[(s_1 + 1)(s_2 + 1)]^{\frac{-\log r_0 - \log|q_1| - \log|q_2|}{\log|q_1| + \log|q_2|}}.$$

Since for all $r \geq r_0$,

$$K_1[(s_1+1)(s_2+1)]^{\frac{\log r}{\log|q_1|+\log|q_2}} \leq n(r,f_1) \leq \frac{1}{\log 2}N(2r,f_1) \leq \frac{1}{\log 2}T(2r,f_1),$$

we obtain

$$\rho(f_1) \geq \mu(f_1) \geq \frac{\log[(s_1+1)(s_2+1)]}{\log|q_1|+\log|q_2|}.$$

Similar to the above argument, we can also obtain

$$\rho(f_2) \geq \mu(f_2) \geq \frac{\log[(s_1+1)(s_2+1)]}{\log|q_1|+\log|q_2|}.$$

This completes the proof of Theorem 1.11.

Proof of Theorem 1.14. Since $\varphi_1(z), \varphi_2(z)$ are small functions, and $\overline{N}(r,f_i) = S(r,f_i)$, similar to argument as in Theorem 1.6, we have

$$\begin{aligned} T(|q_1|r, f_1(z)) &\leq \frac{1+s_1}{n_1}T(r,f_2)+S(r,f_2),\\ T(|q_2|r, f_2(z)) &\leq \frac{1+s_2}{n_2}T(r,f_1)+S(r,f_1). \end{aligned} \tag{3.6}$$

Thus, it follows from (3.6) that

$$T(|q_1q_2|r, f_i(z)) \leq \frac{1+s_1}{n_1}\frac{1+s_2}{n_2}T(r,f_i)+S(r,f_i), \quad i=1,2. \tag{3.7}$$

Since f_i, $(i=1,2)$ are transcendental functions, it follows from (3.7) that $n_1n_2 \leq (s_1+1)(s_2+1)$, and by Lemma 2.3, we can also obtain

$$\rho(f_i) \leq \frac{\log[(s_1+1)(s_2+1)]-\log(n_1n_2)}{\log|q_1|+\log|q_2|}, \quad i=1,2. \tag{3.8}$$

Suppose that $n_1=n_2=1$ and $f_i(i=1,2)$ has infinitely many poles. Since the number of distinct common poles of $f_1, f_2, \frac{1}{\varphi_1}$, and $\frac{1}{\varphi_2}$ is finite, we can choose a sufficiently large constant $R>0$ such that $f_1, f_2, \frac{1}{\varphi_1}$, and $\frac{1}{\varphi_2}$ have no common poles in $\{z \in \mathbb{C} : |z| > R\}$. Thus, we can take a pole z_0 of f_1 of multiplicity $\tau \geq 1$ satisfying $|z_0| > R$. By using the same argument as in Theorem 1.6, we obtain

$$\rho(f_i) \geq \mu(f_i) \geq \frac{\log[(s_1+1)(s_2+1)]}{\log|q_1|+\log|q_2|}, \quad i=1,2. \tag{3.9}$$

Hence, from (3.8) and (3.9), we have the conclusions of Theorem 1.14.

4. Proof of Theorem 1.16

Since (f_1,f_2) are a pair of transcendental meromorphic solutions of (1.6), and $\varphi_1(z), \varphi_2(z)$ are small functions with respect to f_1, f_2, similar to the proof of Theorem 1.14, and by applying Lemma 2.2, we have

$$\begin{aligned} T(r, f_1(p(z))) &\leq \frac{s_1+s_1j+1}{n_1}T(r,f_2(z))+S(r,f_2)\\ &= \Big(\frac{s_1+s_1j+1}{n_1}+o(1)\Big)T(r,f_2),\\ T(r, f_2(p(z))) &\leq \frac{s_2+s_2j+1}{n_2}T(r,f_1(z))+S(r,f_1) \end{aligned}$$

$$= \Big(\frac{s_2+s_2j+1}{n_1}+o(1)\Big)T(r,f_1).$$

Then, by Lemma 2.5, for any $\beta_1>1,\beta_2>1$ and for all $r>r_0$, we have

$$\begin{aligned} T(r,f_1(p(z))) &\le \Big(\frac{s_1+s_1j+1}{n_1}+o(1)\Big)T(\beta_2 r,f_2),\\ T(r,f_2(p(z))) &\le \Big(\frac{s_2+s_2j+1}{n_1}+o(1)\Big)T(\beta_1 r,f_1). \end{aligned} \tag{4.1}$$

Since $p(z)$ is a polynomial with $\deg_z p(z)=k\ge 2$, by Lemma 2.4, for given $0<\delta_i<|p_k|$, we let $\mu_i=|p_k|-\delta_i$, $i=1,2$. For a given $\varepsilon>0$ and for r large enough, from (4.1) we have

$$\begin{aligned} (1-\varepsilon)T(\mu_1 r^k,f_1) &\le \Big(\frac{s_1+s_1j+1}{n_1}+o(1)\Big)T(\beta_2 r,f_2),\\ (1-\varepsilon)T(\mu_2 r^k,f_2) &\le \Big(\frac{s_2+s_2j+1}{n_2}+o(1)\Big)T(\beta_1 r,f_1). \end{aligned}$$

Then, we have

$$\begin{aligned} (1-\varepsilon)^2T(\mu_1 r^{k^2},f_1) &\le \Big(\frac{s_1+s_1j+1}{n_1}\frac{s_2+s_2j+1}{n_2}+o(1)\Big)T\Big(\beta_1(\frac{\beta_2}{\mu_2})^{1/k}r,f_1\Big),\\ (1-\varepsilon)^2T(\mu_2 r^{k^2},f_2) &\le \Big(\frac{s_1+s_1j+1}{n_1}\frac{s_2+s_2j+1}{n_2}+o(1)\Big)T\Big(\beta_2(\frac{\beta_1}{\mu_1})^{1/k}r,f_2\Big), \end{aligned}$$

Set $R_1=\beta_1(\frac{\beta_2}{\mu_2})^{1/k}r$ and $R_2=\beta_2(\frac{\beta_1}{\mu_1})^{1/k}r$, then we have

$$\begin{aligned} &(1-\varepsilon)^2T(\mu_1(\mu_2)^k(\beta_1^k\beta_2)^{-k}R_1^{k^2},f_1)\\ &\le \Big(\frac{s_1+s_1j+1}{n_1}\frac{s_2+s_2j+1}{n_2}+o(1)\Big)T(R_1,f_1),\\ &(1-\varepsilon)^2T(\mu_2(\mu_1)^k(\beta_1\beta_2^k)^{-k}R_2^{k^2},f_2)\\ &\le \Big(\frac{s_1+s_1j+1}{n_1}\frac{s_2+s_2j+1}{n_2}+o(1)\Big)T(R_2,f_2), \end{aligned}$$

Since $n_1n_2<(s_1+s_1j+1)(s_2+s_2j+1)$ and $\beta_i>1,\mu_i>0$, $i=1,2$, we have $\frac{(s_1+s_1j+1)(s_2+s_2j+1)}{n_1n_2}>1$ and $\mu_1(\mu_2)^k(\beta_1^k\beta_2)^{-k}>0$, $\mu_2(\mu_1)^k(\beta_1\beta_2^k)^{-k}>0$. Thus, by Lemma 2.6, letting $\varepsilon\to 0$ and $\beta_i\to 1$, $i=1,2$, we obtain

$$T(r,f_i)=O((\log r)^{\alpha}),\quad i=1,2,$$

where

$$\alpha=\frac{\log[(s_1+s_1j+1)(s_2+s_2j+1)]-\log(n_1n_2)}{2\log k}.$$

This completes the proof of Theorem 1.16.

5. Proof of Theorem 1.17

Suppose that (f_1,f_2) is a pair of transcendental solutions to (1.7). From the assumption of the coefficients of $P_i(z,f_i(z)),Q_i(z,f_i(z))$, $(i=1,2)$ being rational functions, we can choose a sufficiently large constant $R(>0)$ such that the coefficients of $P_i(z,f_i(z)),Q_i(z,f_i(z)),(i=1,2)$ have no zeros or poles in $\{z\in\mathbb{C}:|z|>R\}$. Since f_i $(i=1,2)$ have infinitely many poles, we can choose a pole z_0 of f_1 of multiplicity $\tau\ge 1$ satisfying $|z_0|>R$. From the second equation of (1.7), we get that f_2 has a pole of multiplicity $\tau_1' = d_1\tau - \tau - 1$ at $q_2 z_0$. Replacing z by $q_2 z_0$ in the first equation of (1.7), then it follows that $q_1 q_2 z_0$ is a pole of f_1 of multiplicity

$$\tau_1 = d_2\tau_1' - \tau_1' - 1 = (d_1-1)(d_2-1)\tau - (d_2-1) - 1.$$

Set $H = (d_1-1)(d_2-1)$. We follow the step above. Since f has infinitely many poles, we may construct poles $\zeta_k = (q_1q_2)^k z_0$ $k \in N_+$ of f_1 of multiplicity τ_k satisfying

$$\begin{aligned}\tau_k &= H^k\tau - d_2(H^{k-1} + h^{k-2} + \dots + 1)\\ &= H^k\tau - d_2\frac{H^k-1}{H-1} = H^k\Big(\tau - \frac{d_2}{H-1}\Big) + \frac{d_2}{H-1}.\end{aligned}$$

Since $d_i \geq 4$, $i = 1, 2$, then $\frac{d_2}{H-1} < 1$. Thus, it follows from $\tau \geq 1$ that $\tau - \frac{d_2}{H-1} > 0$. Since $|\zeta_k| \to \infty$ as $k \to \infty$, for sufficiently large k, we have

$$\begin{aligned}H^k\Big(\tau - \frac{d_2}{H-1}\Big) &< \tau_k \leq \tau_1 + \tau_2 + \dots + \tau_k)\\ &\leq n(|\zeta_k|, f_1) \leq n(|q_1q_2|^k|z_0|, f_1).\end{aligned} \tag{5.1}$$

Thus, for each sufficiently large r, there exists a $k \in \mathbb{N}_+$ such that $r \in [|q_1q_2|^k|z_0|, |q_1q_2|^{k+1}|z_0|)$, by using the same argument as in the proof of Theorem 1.11, from (5.1), we have

$$\begin{aligned}n(r, f_1) &\geq H^k\Big(\tau - \frac{d_2}{H-1}\Big)\\ &\geq H^{\frac{\log r - \log|z_0| - \log|q_1q_2|}{\log|q_1q_2|}}\Big(\tau - \frac{d_2}{H-1}\Big)\\ &\geq K_1 H^{\frac{\log r}{\log|q_1| + \log|q_2|}},\end{aligned} \tag{5.2}$$

where

$$K_1 = \Big(\tau - \frac{d_2}{H-1}\Big)H^{\frac{-\log|z_0| - \log|q_1q_2|}{\log|q_1q_2|}}.$$

As in the above argument, we can obtain

$$\begin{aligned}n(r, f_2) &\geq H^k\Big(\tau - \frac{d_1}{H-1}\Big)\\ &\geq H^{\frac{\log r - \log|z_0| - \log|q_1q_2|}{\log|q_1q_2|}}\Big(\tau - \frac{d_1}{H-1}\Big)\\ &\geq K_2 H^{\frac{\log r}{\log|q_1| + \log|q_2|}},\end{aligned} \tag{5.3}$$

where

$$K_2 = \Big(\tau - \frac{d_1}{H-1}\Big)H^{\frac{-\log|z_0| - \log|q_1q_2|}{\log|q_1q_2|}}.$$

Since for all $r \geq r_0$, we have

$$K_i H^{\frac{\log r}{\log|q_1q_2|}} \leq n(r, f_i) \leq \frac{1}{\log 2}N(2r, f_i) \leq \frac{1}{\log 2}T(2r, f_i), \quad i = 1, 2. \tag{5.4}$$

Hence, it follows from (5.2)–(5.4) that

$$\rho(f_i) \geq \mu(f_i) \geq \frac{\log[(d_1-1)(d_2-1)]}{\log|q_1| + \log|q_2|}.$$

Thus, we complete the proof of Theorem 1.17.

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