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# INFINITELY MANY SOLUTIONS VIA VARIATIONAL-HEMIVARIATIONAL INEQUALITIES UNDER NEUMANN BOUNDARY CONDITIONS

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Abstract. In this article, we study the variational-hemivariational inequalities with Neumann boundary condition. Using a nonsmooth critical point theorem, we prove the existence of infinitely many solutions for boundary-value problems. Our technical approach is based on variational methods.

## 1. Introduction

In this article, we study following boundary-value problem, depending on the parameters $\lambda, \mu$ with nonsmooth Neumann boundary condition:

$$\begin{gathered} -\Delta_{p(x)} u + a(x)|u|^{p(x)-2}u = 0 \quad \text{in } \Omega \\ -|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} \in -\lambda\theta(x)\partial F(u) - \mu\partial\vartheta(x)G(u) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $p : \bar{\Omega} \to \mathbb{R}$ is a continuous function satisfying

$$1 < p^- = \min_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p^+ = \max_{x\in\bar{\Omega}} p(x) < +\infty.$$

Here $\lambda, \mu$ are real parameters, $\lambda \in ]0, \infty[, \mu \in [0, \infty[$ and $\theta, \vartheta \in L^1(\partial\Omega)$, where $\theta(x), \vartheta(x) \geq 0$ for a.e. $x \in \partial\Omega$. $F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions given by $F(\omega) = \int_0^\omega f(t)dt$, $G(\omega) = \int_0^\omega g(t)dt$, $\omega \in \mathbb{R}$ such that $f, g : \mathbb{R} \to \mathbb{R}$ are locally essentially bounded functions. $\partial F(u), \partial G(u)$ denote the generalized Clarke gradient of $F(u), G(u)$.

Let $X$ be real Banach space. We assume that it is also given a functional $\chi : X \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicontinuous, proper whose effective domain $dom(\chi) = \{x \in X : \chi(x) < +\infty\}$ is a (nonempty, closed, convex) cone in $X$. Our aim is to study the following variational-hemivariational inequalities problem: Find $u \in \mathcal{B}$ which is called a *weak solution* of problem (1.1), i.e; if for all

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$v \in \mathcal{B}$,

$$\begin{aligned} &\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(v-u)dx + \int_\Omega a(x)|u|^{p(x)-2}u(v-u)dx \\ &- \lambda \int_{\partial\Omega} \theta(x) F^0(u; u-v)d\sigma - \mu \int_{\partial\Omega} \vartheta(x) G^0(u; u-v)d\sigma \geq 0, \end{aligned} \tag{1.2}$$

where $\mathcal{B}$ is a closed convex subset of $W_0^{1,p(\cdot)}(\Omega)$. For simplicity $\mathcal{B} = W_0^{1,p(\cdot)}(\Omega)$. Recently, many researchers have paid attention to impulsive differential equations by variational method. We refer the reader to [1, 6, 17, 20, 21, 22, 23] and references cited therein. The operator $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the so-called $p(x)$-Laplacian, which becomes $p$-Laplacian when $p(x) \equiv p$ is a constant. More recently, the study of $p(x)$-Laplacian problems has attracted more and more attention [4, 24].

Variational-hemivariational inequalities have been extensively studied in recent years via variational methods: in [15], the author studied hemivariational inequalities on an unbounded strip-like domain; in [19], the authors studied variational-hemivariational inequalities for the existence of a whole sequence of solutions with non-smooth potential and non-zero Neumann boundary condition; in [5], the authors studied variational-hemivariational inequalities involving the $p-$Laplace operator and a nonlinear Neumann boundary condition via abstract critical point result; in [3], the authors studied variational-hemivariational inequality on bounded domains by using the mountain pass theorem and the critical point theory for Motreanu-Panagiotopoulos type functionals.

The aim of the present paper is find sufficient conditions to guarantee the existence of infinitely many weak solutions for a variational-hemivariational inequality depending on two parameters. Our approach is a variational method and the main tool is a general nonsmooth critical point theorem.

## 2. Preliminaries

In this section, we recall some definitions and results which are used further in this paper. The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} : \int_\Omega |u(x)|^{p(x)}dx < \infty\}$$

and is endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\{\ \lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\}.$$

Note that, when $p$ is constant, the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ coincides with the standard norm $\|\cdot\|_p$ of the Lebesgue space $L^p(\Omega)$. $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space.

The generalized Lebesgue-Sobolev space $W^{L,p(\cdot)}(\Omega)$ for $L = 1, 2, \dots$ is defined by

$$W^{L,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq L\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}$ with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \Sigma_{i=1}^N \alpha_i$. The space $W^{L,p(\cdot)}(\Omega)$ with the norm

$$\|u\|_{W^{L,p(\cdot)}}(\Omega) = \sum_{|\alpha| \leq L} \|D^\alpha u\|_{p(\cdot)},$$

is a separable reflexive Banach space [9].

$W_0^{L,p(\cdot)}(\Omega)$ denotes the closure in $W^{L,p(\cdot)}(\Omega)$ of the set of functions in $W^{L,p(\cdot)}(\Omega)$ with compact support.

For every $u \in W_0^{L,p(\cdot)}(\Omega)$ the Poincaré inequality holds, where $C_p > 0$ is a constant

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C_p \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

(see [12]). Hence, an equivalent norm for the space $W_0^{L,p(\cdot)}(\Omega)$ is given by

$$\|u\|_{W_0^{L,p(\cdot)}(\Omega)} = \sum_{|\alpha|=L} \|D^\alpha u\|_{p(\cdot)}.$$

Given $p(x)$, let $p_L^*$ denote the critical variable exponent related to $p$, defined for all $x \in \bar{\Omega}$ by the pointwise relation

$$p_L^*(x) = \begin{cases} \frac{Np(x)}{N-Lp(x)} & Lp(x) < N, \\ +\infty & Lp(x) \ge N, \end{cases} \tag{2.1}$$

is the critical exponent related to $p$. Let

$$\mathcal{K} = \sup_{u \in X\setminus\{0\}} \frac{\max_{x\in\bar{\Omega}} |u(x)|^p}{\|u\|^p}, \quad \mathcal{M} = \inf_{u \in X\setminus\{0\}} \frac{\min_{x\in\bar{\Omega}} |u(x)|^p}{\|u\|^p}\,. \tag{2.2}$$

Since $p > N$, $X$ are compactly embedded in $C^0(\bar{\Omega})$, it follows that $\mathcal{K}, \mathcal{M} < \infty$.

**Proposition 2.1.** *For $\Phi(u) = \int_\Omega [|\nabla u|^{p(x)} + a(x)|u(x)|^{p(x)}]dx$, and $u, u_n \in X$, we have*

(i) $\|u\| < (=,>) 1 \Leftrightarrow \Phi(u) < (=,>) 1$,
(ii) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le \Phi(u) \le \|u\|^{p^-}$,
(iii) $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le \Phi(u) \le \|u\|^{p^+}$,
(iv) $\|u_n\| \to 0 \Leftrightarrow \Phi(u_n) \to 0$,
(v) $\|u_n\| \to \infty \Leftrightarrow \Phi(u_n) \to \infty$.

The proof of the above proposition is similar to that in [11].

**Proposition 2.2** ([11, 14])**.** *For $p, q \in C_+(\overline{\Omega})$ in which $q(x) \le p_L^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{L,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

*If we replace $\le$ with $<$, the embedding is compact.*

**Remark 2.3.** (i) By the proposition 2.2 there is a continuous and compact embedding of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}$ where $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$. $W_0^{1,p(\cdot)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and since $p^- > N$, we deduce that $W_0^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$, So, there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\|u\|, \quad \forall u \in X, \tag{2.3}$$

where $\|u\|_\infty := \sup_{x\in\bar{\Omega}} |u(x)|$.

(ii) Denote

$$\|u\| = \inf\{\lambda > 0 : \int_\Omega [|\frac{\nabla u}{\lambda}|^{p(x)} + a(x)|\frac{u}{\lambda}|^{p(x)}]dx \le 1\},$$

which is a norm on $W_0^{1,p(\cdot)}(\Omega)$.

Let $\eta : \partial\Omega \to \mathbb{R}$ be a measurable. Define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{\eta(x)}(\partial\Omega) = \{u : \partial\Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial\Omega} |\eta(x)||u|^{p(x)} d\sigma < \infty\},$$

with the norm

$$|u|_{(p(x),\eta(x))} = \inf\{\tau > 0; \int_{\partial\Omega} |\eta(x)|\,|\frac{u}{\tau}|^{p(x)} d\sigma \le 1\},$$

where $d\sigma$ is the measure on the boundary.

**Lemma 2.4** ([8]). *Let $\rho(x) = \int_{\partial\Omega} |\eta(x)||u|^{p(x)} d\sigma$ for $u \in L^{p(x)}_{\eta(x)}(\partial\Omega)$ we have*

$$|u|_{(p(x),\eta(x))} \ge 1 \Rightarrow |u|^{p^-}_{(p(x),\eta(x))} \le \rho(u) \le |u|^{p^+}_{(p(x),\eta(x))},$$
$$|u|_{(p(x),\eta(x))} \le 1 \Rightarrow |u|^{p^+}_{(p(x),\eta(x))} \le \rho(u) \le |u|^{p^-}_{(p(x),\eta(x))}.$$

For $A \subseteq \bar{\Omega}$ denote by $\inf_{x\in A} p(x) = p^-$, $\sup_{x\in A} p(x) = p^+$. Define

$$p^\partial(x) = (p(x))^\partial := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & p(x) < N, \\ +\infty & p(x) \ge N, \end{cases} \tag{2.4}$$
$$p^\partial(x)_{r(x)} := \frac{r(x)-1}{r(x)} p^\partial(x),$$

where $x \in \partial\Omega, r \in C(\partial\Omega, \mathbb{R})$ and $r(x) > 1$.

**Proposition 2.5** ([10, 14]). *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^\partial(x)$ for any $x \in \overline{\Omega}$, then the embedding $W^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\partial\Omega)$ is compact and continuous.*

In this part we introduce the definitions and basic properties from the theory of generalized differentiation for locally Lipschitz functions. Let $X$ be a Banach space and $X^\star$ its topological dual. By $\|\cdot\|$ we will denote the norm in $X$ and by $\langle\cdot,\cdot\rangle_X$ the duality brackets for the pair $(X, X^\star)$. A function $h : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighbourhood $U$ of $x$ and a constant $K > 0$ depending on $U$ such that $|h(y)-h(z)| \le K\|y-z\|$ for all $y, z \in U$. For a locally Lipschitz function $h : X \to \mathbb{R}$ is defined by the generalized directional derivative of $h$ at $u \in X$ in the direction $\gamma \in X$ by

$$h^0(u;\gamma) = \limsup_{w\to u, t\to 0^+} \frac{h(w+t\gamma) - h(w)}{t}.$$

The generalized gradient of $h$ at $u \in X$ is defined by

$$\partial h(u) = \{x^\star \in X^\star : \langle x^\star, \gamma\rangle_X \le h^0(u;\gamma),\ \forall\gamma \in X\},$$

which is non-empty, convex and $w^\star-$compact subset of $X^\star$, where $< \cdot,\cdot >_X$ is the duality pairing between $X^\star$ and $X$.

**Proposition 2.6** ([7]). *Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functions. Then:*

(i) *$h^0(u;\cdot)$ is subadditive, positively homogeneous.*
(ii) *$(-h)^0(u;v) = h^0(u;-v)$ for all $u, v \in X$.*
(iii) *$h^0(u;v) = \max\{< \xi, v >: \xi \in \partial h(u)\}$ for all $u, v \in X$.*
(iv) *$(h+g)^0(u;v) \le h^0(u;v) + g^0(u;v)$ for all $u, v \in X$.*

**Definition 2.7** ([18]). Let $X$ be a Banach space, $\mathcal{I} : X \to (-\infty, +\infty]$ is called a Motreanu-Panagiotopoulos-type functional, if $\mathcal{I} = h + \chi$, where $h : X \to \mathbb{R}$ is locally Lipschitz and $\chi : X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous.

**Definition 2.8** ([13]). An element $u \in X$ is said to be a critical point of $\mathcal{I} = h + \chi$ if

$$h^0(u; v-u) + \chi(v) - \chi(u) \geq 0, \quad \forall v \in X.$$

Let $X$ is a reflexive real Banach space, $\phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous and coercive, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous, $\lambda$ is a positive real parameter, $\chi : X \to (-\infty, +\infty]$ is a convex, proper, lower semicontinuous functional and $D(\chi)$ is the effective domain of $\chi$. Assuming also that $\phi$ and $\Upsilon$ are locally Lipschitz continuous functionals. Set

$$\mathcal{E} := \Upsilon - \chi, \quad \mathcal{L}_\lambda := \phi - \lambda\mathcal{E} = (\phi - \lambda\Upsilon) + \lambda\chi.$$

We assume that

$$\phi^{-1}(]-\infty, r[) \cap D(\chi) \neq \emptyset, \quad \forall r > \inf_X \phi,$$

and define for every $r > \inf_X \phi$,

$$\varphi(r) = \inf_{u \in \phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\phi^{-1}(]-\infty,r[)} \mathcal{E}(v)\right) - \mathcal{E}(u)}{r - \phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \phi)^+} \varphi(r).$$

We recall the following nonsmooth version of a critical point result.

**Theorem 2.9** ([16]). *Under the above assumptions on $X$, $\phi$ and $\mathcal{E}$, we have*

(a) *For every $r > \inf_X \phi$, and every $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional*

$$\mathcal{L}_\lambda = \phi - \lambda\mathcal{E}$$

*to $\phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of $\mathcal{L}_\lambda$ in $X$.*

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either (b1) $\mathcal{L}_\lambda$ possesses a global minimum, or (b2) there is a sequence $\{u_n\}$ of critical points (local minima) of $\mathcal{L}_\lambda$ such that*

$$\lim_{n\to+\infty} \phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: either (c1) there is a global minimum of $\phi$ which is a local minimum of $\mathcal{L}_\lambda$, or (c2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $\mathcal{L}_\lambda$ that converges weakly to a global minimum of $\phi$.*

Consider $\phi, \mathcal{F}, \mathcal{G} : X \to \mathbb{R}$, as follows

$$\phi(u) = \int_\Omega \frac{1}{p(x)}[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}]dx, \quad u \in W_0^{1,p(\cdot)}(\Omega),$$

$$\mathcal{F}(u) = \int_{\partial\Omega} F(u(x))d\sigma, \quad u \in W_0^{1,p(\cdot)}(\Omega),$$

$$\mathcal{G}(u) = \int_{\partial\Omega} G(u(x))d\sigma, \quad u \in W_0^{1,p(\cdot)}(\Omega).$$

The next lemma characterizes some properties of $\phi$ [2].

**Lemma 2.10.** *Let*

$$\phi(u) = \int_\Omega \frac{1}{p(x)}[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}]dx.$$

*Then*

(i) $\phi : X \to \mathbb{R}$ *is sequentially weakly lower semicontinuous;*
(ii) $\phi'$ *is of* $(S_+)$ *type;*
(iii) $\phi'$ *is a homeomorphism.*

**Proposition 2.11** ([15])**.** *Let* $F, G : \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz functions. Then* $\mathcal{F}$ *and* $\mathcal{G}$ *are well-defined and*

$$\mathcal{F}^0(u; v) \le \int_{\partial\Omega} F^0(u(x); v(x))d\sigma, \quad \forall u, v \in W_0^{1,p(\cdot)}(\Omega),$$
$$\mathcal{G}^0(u; v) \le \int_{\partial\Omega} G^0(u(x); v(x))d\sigma, \quad \forall u, v \in W_0^{1,p(\cdot)}(\Omega).$$

## 3. Main results

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function whose potential $F(t) = \int_0^t f(\omega)d\omega$ for all $t \in \mathbb{R}$. Set

$$\alpha := \liminf_{\omega\to+\infty} \frac{\max_{|t|\le\omega} F(t)}{|\omega|^{p^-}}, \quad \beta := \limsup_{\omega\to+\infty} \frac{F(\omega)}{|\omega|^{p^+}}.$$

**Theorem 3.1.** *Let* $\theta, \vartheta \in L^1(\partial\Omega)$ *be non-negative and non-zero identically zero functions. Assume that*

$$\alpha < \frac{p^-\mathcal{M}\theta^*\beta}{p^+\mathcal{K}} \tag{3.1}$$

*for each* $\lambda \in (\lambda_1, \lambda_2)$*, where*

$$\lambda_1 = \frac{1}{p^-\mathcal{M}\theta^*\beta}, \quad \lambda_2 = \frac{1}{p^+\mathcal{K}\alpha},$$

*and* $\theta^* = \int_{\partial\Omega} \theta(x)d\sigma$*. Also assume that for each locally essentially bounded function* $g : \mathbb{R} \to \mathbb{R}$ *with potential* $G(t) = \int_0^t g(\omega)d\omega$*, for all* $t \in \mathbb{R}$*, satisfies*

$$G_\infty = \limsup_{\omega\to+\infty} \frac{\max_{|t|\le\omega} G(t)}{|\omega|^{p^-}} < +\infty, \tag{3.2}$$

*for every* $\mu \in [0, \mu_{G,\lambda})$*, where*

$$\mu_{G,\lambda} = \frac{1}{p^+\mathcal{K}G_\infty}(1 - p^+\mathcal{K}\lambda\alpha).$$

*Then* (1.1) *has a sequence of weak solutions for every* $\mu \in [0, \mu_{G,\lambda})$ *in* $X$ *such that*

$$\int_\Omega \frac{1}{p(x)}[|\nabla u_n|^{p(x)} + a(x)|u_n|^{p(x)}]dx \to +\infty.$$

*Proof.* Our strategy is to apply Theorem 2.9 (b).

**Case 1.** Assume that $\|u\| \ge 1$. Let $\bar\lambda \in (\lambda_1, \lambda_2)$ and $G$ satisfy our assumptions. Since $\bar\lambda < \lambda_2$, it follows that

$$\mu_{G,\bar\lambda} = \frac{1}{p^+\mathcal{K}G_\infty}(1 - p^+\mathcal{K}\bar\lambda\alpha).$$

Fix $\bar{\mu} \in (0, \mu_{G,\bar{\lambda}})$ and define the functionals $\phi, \mathcal{E} : X \to \mathbb{R}$ for each $u \in X$ as follows:

$$\begin{gathered} \phi(u) = \int_{\Omega} \frac{1}{p(x)}[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}]dx, \\ \Upsilon(u) = \int_{\partial\Omega} \theta(x)[F(u(x))]d\sigma + \frac{\bar{\mu}}{\bar{\lambda}} \int_{\partial\Omega} \vartheta(x)[G(u(x))]d\sigma, \\ \chi(u) = \begin{cases} 0 & u \in \mathcal{B}, \\ +\infty & u \notin \mathcal{B}, \end{cases} \\ \mathcal{E}(u) = \Upsilon(u) - \chi(u)\,. \end{gathered} \tag{3.3}$$

Then define the functional

$$\mathcal{L}_{\bar{\lambda}}(u) := \phi(u) - \bar{\lambda}\mathcal{E}(u)$$

whose critical points are the weak solutions of (1.1).

To apply Lemma 2.10, we assume that $\phi$ satisfies the regularity assumptions of Theorem 2.9. By standard argument, $\Upsilon$ is sequentially weakly continuous. First, we claim that $\bar{\lambda} < 1/\gamma$. Note that $\phi(0) = \mathcal{E}(0) = 0$, then for every $n$ large enough, one has

$$\begin{aligned} \varphi(r) &= \inf_{u\in\phi^{-1}(]-\infty,r[)} \frac{\big(\sup_{v\in\phi^{-1}(]-\infty,r[)} \mathcal{E}(v)\big) - \mathcal{E}(u)}{r - \phi(u)} \\ &\le \frac{\sup_{v\in\phi^{-1}(]-\infty,r[)} \mathcal{E}(v)}{r}. \end{aligned}$$

Coercivity of $\phi$ implies that $\inf_X \phi = \phi(0) = 0$. Since $\mathcal{B}$ contains constant functions, $0 \in \mathcal{B} = D(\chi)$, thus

$$0 \in \phi^{-1}(]-\infty, r[) \cap D(\chi), \quad \forall r > \inf_X \phi.$$

For $v \in X$ with $\phi(v) < r$ and in view of (2.2),

$$\begin{aligned} \phi^{-1}(]-\infty, r[) : &= \{v \in X : \phi(v) < r\} = \{v \in X : \frac{1}{p^+}\|v\|^{p^-} < r\} \\ &\subseteq \{v \in X : |v(x)| < (p^+\mathcal{K}r)^{\frac{1}{p^-}}\}. \end{aligned} \tag{3.4}$$

Then

$$\varphi(r) \le \frac{\big(\sup_{\{v\in X:|v(x)|<(p^+\mathcal{K}r)^{\frac{1}{p^-}}\}} \mathcal{E}(v) - \chi(v)\big)}{r}.$$

Let $\{\omega_n\}$ be a sequence of positive numbers in $X$ such that $\lim_{n\to+\infty} \omega_n = +\infty$ and

$$\alpha = \lim_{n\to+\infty} \frac{\max_{|t|\le\omega_n} F(t)}{|\omega_n|^{p^-}}.$$

Set

$$r_n = \frac{|\omega_n|^{p^-}}{\mathcal{K}p^+},\ n \in \mathbb{N}.$$

Take $v \in \phi^{-1}(]-\infty, r_n[)$, from (3.4), we have $|v(x)| < (p^+\mathcal{K}r)^{\frac{1}{p^-}}$. Hence,

$$\varphi(r_n) \le \frac{\sup_{\{v\in X:|v(x)|<\omega_n,\ \forall x\in\partial\Omega\}} \int_{\partial\Omega}[\theta(x)F(v) + \frac{\bar{\mu}}{\bar{\lambda}}\vartheta(x)G(v)]d\sigma}{\frac{|\omega_n|^{p^-}}{\mathcal{K}p^+}}$$

$$
\begin{aligned}
&\le \mathcal{K}p^+ \frac{\int_{\partial\Omega} \max_{|t|\le \omega_n}[\theta(x)F(t) + \frac{\bar{\mu}}{\bar{\lambda}}\vartheta(x)G(t)]d\sigma}{|\omega_n|^{p^-}} \\
&\le \mathcal{K}p^+ \Big[\frac{\theta^* \max_{|t|\le \omega_n} F(t)}{|\omega_n|^{p^-}} + \frac{\bar{\mu}}{\bar{\lambda}}\vartheta^* \frac{\max_{|t|\le\omega_n} G(t)}{|\omega_n|^{p^-}}\Big],
\end{aligned}
$$

where $\vartheta^* = \int_{\partial\Omega}\vartheta(x)d\sigma$. Moreover, from (3.1) and (3.2),

$$
\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le \mathcal{K}p^+(\theta^*\alpha + \frac{\bar{\mu}}{\bar{\lambda}}\vartheta^* G_\infty) < +\infty.
$$

It is clear that, for every $\bar{\mu} \in (0, \mu_{G,\bar{\lambda}})$,

$$
\gamma \le \mathcal{K}p^+\theta^*\alpha + \vartheta^* \frac{(1-\bar{\lambda}\mathcal{K}p^+\alpha)}{\bar{\lambda}}.
$$

Then

$$
\bar{\lambda} = \frac{1}{\mathcal{K}p^+\theta^*\alpha + \vartheta^*(1-\bar{\lambda}\mathcal{K}p^+\alpha)/\bar{\lambda}} < \frac{1}{\gamma}.
$$

We claim that the functional $\mathcal{L}_{\bar{\lambda}}$ is unbounded from below. Indeed, since $\frac{1}{\bar{\lambda}} < \mathcal{M}p^-\theta^*\beta$, we can consider a sequence $\{\tau_n\}$ of positive numbers and $\eta > 0$ such that $\tau_n \to +\infty$ and

$$
\frac{1}{\bar{\lambda}} < \eta < \lim_{n\to+\infty} \frac{\mathcal{M}p^-\theta^* F(\tau_n)}{|\tau_n|^{p^+}}, \tag{3.5}
$$

for every $n \in \mathbb{N}$ large enough. Let $\xi_n(x) = \tau_n$ be a sequence in $X$ for all $n \in \mathbb{N}$, $x \in \bar{\Omega}$. Fix $n \in \mathbb{N}$, by proposition 2.1,

$$
\phi(\xi_n) = \int_\Omega \frac{1}{p(x)}[|\nabla\xi_n|^{p(x)} + a(x)|\xi_n|^{p(x)}]dx \le \frac{1}{p^-}\|\tau_n\|^{p^+} \le \frac{1}{\mathcal{M}p^-}|\tau_n|^{p^+}. \tag{3.6}
$$

Since $G$ is non-negative and from the definition of $\mathcal{E}$

$$
\begin{aligned}
\mathcal{E}(\xi_n) &= \int_{\partial\Omega} [\theta(x)F(\xi_n) + \frac{\bar{\mu}}{\bar{\lambda}}\vartheta(x)G(\xi_n)]d\sigma - \chi(\xi_n) \\
&\ge \int_{\partial\Omega} \theta(x)F(\xi_n)d\sigma = \theta^* F(\tau_n).
\end{aligned} \tag{3.7}
$$

According to (3.5), (2.1) and (3.7),

$$
L_\lambda(\xi_n) \le \frac{1}{p^-}\|\tau_n\|^{p^+} - \bar{\lambda}\int_{\partial\Omega}\theta(x)F(\tau_n)d\sigma < \frac{1}{\mathcal{M}p^-}|\tau_n|^{p^+} - \frac{1}{\mathcal{M}p^-}\bar{\lambda}|\tau_n|^{p^+}\eta,
$$

for every enough large $n \in \mathbb{N}$. Since $\bar{\lambda}\eta > 1$ and $\lim_{n\to+\infty}\tau_n = +\infty$, it results that

$$
\lim_{n\to+\infty} \mathcal{L}_{\bar{\lambda}}(\xi_n) = -\infty.
$$

Hence, the functional $\mathcal{L}_{\bar{\lambda}}$ is unbounded from below, and it follows that $\mathcal{L}_{\bar{\lambda}}$ has no global minimum. Therefore, applying 3.5 we deduce that there is a sequence $u_n \in X$ of critical points of $\mathcal{L}_{\bar{\lambda}}$ such that

$$
\int_\Omega \frac{1}{p(x)}[|\Delta u_n|^{p(x)} + a(x)|u_n|^{p(x)}]dx \to +\infty.
$$

**Case 2.** If $\|u\| \le 1$ the proof is similar to the first case and the proof of theorem is complete. □

**Lemma 3.2.** *Every critical point of the functional $\mathcal{L}_\lambda$ is a solution of* (1.1).

*Proof.* By definition 2.7, $\mathcal{L}_\lambda = (\phi - \lambda\Upsilon) + \lambda\chi$ is a Motreanu-Panagiotopoulos type functional. Let $\{u_n\} \subset X$ be a critical sequence of $\mathcal{L}_\lambda = \phi - \lambda\mathcal{F} - \mu\mathcal{G} + \lambda\chi$ then $u_n \in \mathcal{B}$, definition 2.8 and proposition 2.6 imply that

$$(\phi - \lambda\Upsilon)^0(u_n; v - u_n) \geq 0, \quad \forall v \in \mathcal{B}.$$

Using proposition 2.11,

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(v-u_n)dx + \int_\Omega a(x)|u_n|^{p(x)-2}u_n(v-u_n)dx \\ &- \lambda\int_{\partial\Omega}\theta(x)F^0(u_n; v-u_n)d\sigma - \mu\int_{\partial\Omega}\vartheta(x)G^0(u_n; v-u_n)d\sigma \geq 0. \end{aligned} \tag{3.8}$$

for every $v \in \mathcal{B}$. This completes the proof. □

Now, we give a concrete application of Theorem 3.1.

**Theorem 3.3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, continuous function and set $F(\omega) = \int_0^\omega f(t)dt$ for $\omega \in \mathbb{R}$. Assume that*

$$\liminf_{\omega\to+\infty} \frac{F(\omega)}{\omega} < \frac{\mathcal{M}(\theta(1)+\theta(0))}{2\mathcal{K}} \limsup_{\omega\to+\infty} \frac{F(\omega)}{\omega^2}. \tag{3.9}$$

*Then, for each*

$$\lambda \in \Big]\frac{1}{\mu(\theta(1)+\theta(0))\limsup_{\omega\to+\infty}\frac{F(\omega)}{\omega^2}}, \frac{1}{2\mathcal{K}\liminf_{\omega\to+\infty}\frac{F(\omega)}{\omega}}\Big[,$$

*for each non-negative, continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\omega) = \int_0^\omega g(t)dt$ satisfies*

$$\limsup_{\omega\to+\infty} \frac{G(\omega)}{\omega} < +\infty$$

*and for every $\mu \in [0, \mu_{G,\lambda}[$, where*

$$\mu_{G,\lambda} := \frac{1}{2\mathcal{K}G_\infty}\big(1 - 2\mathcal{K}\lambda \liminf_{\xi\to+\infty}\frac{F(\omega)}{\omega}\big),$$

*there is a sequence of pairwise distinct functions $\{u_n\} \subset W_0^{1,2-x}]0,1[$ such that for all $n \in \mathbb{N}$ one has*

$$\begin{gathered} -(|u'(x)|^{-x}u'(x))' + |u(x)|^{-x}u(x) = 0 \quad x \in ]0,1[, \\ |u_n'(1)|^{-1}u_n'(1) = \bar{\lambda}\theta(1)f(u_n(1)) + \bar{\mu}\vartheta(1)g(u_n(1)), \\ |u_n'(0)|^{-1}u_n'(0) = \bar{\lambda}\theta(0)f(u_n(0)) + \bar{\mu}\vartheta(0)g(u_n(0)). \end{gathered} \tag{3.10}$$

*Proof.* The first step is the inequality

$$\begin{aligned} &\int_0^1 \theta(x)[F(u(x))] + \vartheta(x)[G(u(x))]d\sigma \\ &\leq (\theta(1)+\theta(0))\max_{|\omega|\leq\omega_n} F(\omega) + (\vartheta(1)+\vartheta(0))\max_{|\omega|\leq\omega_n} G(\omega). \end{aligned}$$

It results that

$$\gamma \leq \liminf_{n\to+\infty}\varphi(r_n) \leq \mathcal{K}p^+\alpha(\theta(1)+\theta(0)) + \mathcal{K}p^+(\vartheta(1)+\vartheta(0))\frac{\bar{\mu}}{\bar{\lambda}}G_\infty) < +\infty.$$

The second step is the inequality

$$\int_0^1 \vartheta(x)[G(\xi_n(x))]d\sigma = (\vartheta(1)+\vartheta(0))G(\tau_n) \geq (\vartheta(1)+\vartheta(0))\liminf_{\omega\to+\infty} G(\omega) \geq 0,$$

which implies that $\lim_{n\to+\infty}\mathcal{L}_{\bar\lambda}(\xi_n)=-\infty$. The last one is

$$
\begin{aligned}
&\Big[\int_{\partial\Omega}\theta(x)F(u_n(x);v(x)-u_n(x))d\sigma+\int_{\partial\Omega}\vartheta(x)G(u_n(x);v(x)-u_n(x))d\sigma\Big]^\circ\\
&\le\Big[\int_{\partial\Omega}\theta(x)F(u_n(x);v(x)-u_n(x))d\sigma\Big]^\circ+\Big[\int_{\partial\Omega}\vartheta(x)G(u_n(x);v(x)-u_n(x))d\sigma\Big]^\circ\\
&\le\Big[\theta(1)F(u_n(1);v(1)-u_n(1))+\theta(0)F(u_n(0);v(0)-u_n(0))]^\circ\\
&\quad+\Big[\vartheta(1)G(u_n(1);v(1)-u_n(1))+\vartheta(0)G(u_n(0);v(0)-u_n(0))\Big]^\circ\\
&\le\Big[\theta(1)F^\circ(u_n(1);v(1)-u_n(1))+\theta(0)F^\circ(u_n(0);v(0)-u_n(0))\Big]\\
&\quad+\Big[\vartheta(1)G^\circ(u_n(1);v(1)-u_n(1))+\vartheta(0)G^\circ(u_n(0);v(0)-u_n(0))\Big].
\end{aligned}
$$

Choosing $X=W^{1,2-x}(]0,1[)$, $\Omega=]0,1[$, $p(x)=2-x$ and $a(x)=1$, then the conditions of Theorem 3.1 hold. Hence,

$$
\begin{aligned}
&\int_0^1[|u_n'(x)|^{-x}u_n'(x)(v'-u_n')+|u_n(x)|^{-x}u_n(x)(v-u_n)]dx\\
&-\bar\lambda[\theta(1)f(u_n(1))v(1)+\theta(0)f(u_n(0))v(0)]\\
&-\bar\mu[\vartheta(1)g(u_n(1))v(1)+\vartheta(0)g(u_n(0))v(0)]\ge 0.
\end{aligned}
$$

There exists an unbounded sequence $\{u_n\}\subset W^{1,2-x}(]0,1[)$ such that

$$
\begin{aligned}
&\int_0^1[|u_n'(x)|^{-x}u_n'(x)v'(x)+|u_n(x)|^{-x}u_n(x)v(x)]dx\\
&-\Big(\bar\lambda\theta(1)f(u_n(1))+\bar\mu\vartheta(1)g(u_n(1))\\
&+\bar\lambda\theta(0)f(u_n(0))+\bar\mu\vartheta(0)g(u_n(0))\Big)\ge 0.
\end{aligned}
$$

Therefore $\{u_n\}$ is the unique solution of the problem (3.10). □

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## References


[1] G. Afrouzi, M. Mirzapour, V. Rădulescu; *Qualitative properties of anisotropic elliptic Schrödinger equations*, Adv. Nonlinear Stud. **14(3)** (2014), 747-765.

[2] A. R. El Amrouss, A. Ourraoui; *Existence of solutions for a boundary problem involving $p(x)$-biharmonic operator,* Bol. Soc. Paran. Mat. **31(1)** (2013), 179-192.

[3] M. Alimohammady, F. Fattahi; *Existence of solutions to hemivaritional inequalities involving the p(x)-biharmonic operator,* Electron. J. Diff. Equ., Vol. **2015** (2015), no. 79, 1-12.

[4] M. Allaoui; *Existence of solutions for a Robin problem involving the $p(x)$-Laplacian,* Applied Mathematics E-Notes, **14** (2014), 107-115.

[5] G. Bonannoa, P. Winkert; *Multiplicity results to a class of variational-hemivariational inequalities,* Topological Methods in Nonlinear Analysis, **43(2)** (2014), 493-516.

[6] J. Chabrowski; *Inhomogeneous Neumann problem with critical Sobolev exponent,* Adv. Nonlinear Anal. **1** (2012), no. 3, 221-255.

[7] F. H. Clarke; *Optimization and nonsmooth analysis,* Society for Industrial and Applied Mathematics, Philadelphia, (1990).

[8] S. G. Dend; *Eigenvalues of the $p(x)-$laplacian Steklov problem,* J. Math. Anal. Appl., **339** (2008), 925-937.

[9] L. Diening, P. Harjulehto, P. Hästö, M. Růžička; *Lebesgue and Sobolev spaces with variable exponents,* Vol. 2017, Springer-Verlag, Berlin, 2011.

[10] X. L. Fan; *Regularity of minimizers of variational integrals with continuous $p(x)$-growth conditions,* Chinese Ann. Math., **17A(5)** (1996), 557-564.

[11] X. L. Fan, D. Zhao; *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$,* J. Math. Anal. Appl., **263** (2001), 424-446.

[12] X. L. Fan, Q. H. Zhang; *Existence of solutions for $p(x)$-Laplacian Dirichlet problems,* Nonlinear Anal., **52**(2003), 1843-1852.

[13] A. Iannizzotto; *Three critical points for perturbed nonsmooth functionals and applications,* Nonlinear Analysis, **72** (2010), 1319-1338.

[14] O. Kováčik, J. Rákosnínk; *On spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$,* Czechoslovak Math. J., **41** (1991), 592-618.

[15] A. Kristály; *Multiplicity results for an eigenvalue problem for hemivariational inequalities in strip-like domains,* Set-Valued Analysis, **13**(2005), 85-103.

[16] S. A. Marano, D. Motreanu; *Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian,* J. Differential Equations, **182(1)** (2002), 108-120.

[17] G. Molica Bisci, D. Repovš; *Multiple solutions for elliptic equations involving a general operator in divergence form,* Ann. Acad. Sci. Fenn. Math. **39**(2014), 259-273.

[18] D. Motreanu, V. Rădulescu; *Variational and non-variational methods in nonlinear analysis and boundary value problems,* Kluwer Academic Publishers, Boston-Dordrecht-London, 2003.

[19] D. Motreanu, P. Winkert; *Variational-hemivariational inequalities with nonhomogeneous neumann boundary condition,* Le Matematiche, Vol. 2010, 109-119, doi: 10.4418/2010.65.2.12

[20] V. Rădulescu, I. Ionescu; *Nonlinear eigenvalue problems arising in earthquake initiation,* Adv. Differential Equations, **8** (2003), 769-786.

[21] V. Rădulescu, D. Repovš; *Partial differential equations with variable exponents:* Variational Methods and Qualitative Analysis, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.

[22] V. Rădulescu; *Nonlinear elliptic equations with variable exponent: old and new,* Nonlinear Analysis: Theory, Methods and Applications, **121** (2015), 336-369.

[23] M. Rosiu; *Local trajectories on Klein surfaces,* Rev. Roumaine Math. Pures Appl. **54** (2009), no. 5-6, 541-547.

[24] C. Zhou, S. Liang; *Infinitely many small solutions for the $p(x)$-Laplacian operator with critical growth,* J. Appl. Math. & Informatics, **32** (2014), 137-152.



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