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SUPERLINEAR SINGULAR FRACTIONAL BOUNDARY-VALUE PROBLEMS

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ABSTRACT. In this article, we study the superlinear fractional boundary-value problem

$$D^\alpha u(x) = u(x)g(x,u(x)), \quad 0<x<1,$$
$$u(0)=0, \quad \lim_{x\to 0^+} D^{\alpha-3}u(x)=0, \quad \lim_{x\to 0^+} D^{\alpha-2}u(x)=\xi, \quad u''(1)=\zeta,$$

where $3<\alpha\le 4$, D^α is the Riemann-Liouville fractional derivative and $\xi,\zeta\ge 0$ are such that $\xi+\zeta>0$. The function $g(x,u)\in C((0,1)\times[0,\infty),[0,\infty))$ that may be singular at $x=0$ and $x=1$ is required to satisfy convenient hypotheses to be stated later.

By means of a perturbation argument, we establish the existence, uniqueness and global asymptotic behavior of a positive continuous solution to the above problem.An example is given to illustrate our main results.

1. INTRODUCTION

Fractional differential equations have been of great interest recently. Many phenomena in viscoelasticity, porous structures, fluid flows, electrical networks can be modeled by these fractional boundary-value problems (see, for instance, [6, 7, 11, 14, 17] and references therein) for discussions of various applications.

Fractional boundary-value problems of the form

$$D^\alpha u(x) + f(x,u(x)) = 0, \quad 0<x<1,\ 3<\alpha\le 4, \tag{1.1}$$

subject to various boundary-value conditions have been considered by many authors, see for example, [1, 2, 3, 4, 5, 8, 10, 12, 13, 15, 16, 19, 20, 21] and the references therein.

Here D^α is the Riemann-Liouville fractional derivative of order α ($3<\alpha\le 4$) defined by [7, 14, 17],

$$D^\alpha u(x) = \begin{cases} (\frac{d}{dx})^4 I^{4-\alpha}u(x), & \text{if } 3<\alpha<4 \\ (\frac{d}{dx})^4 u(x), & \text{if } \alpha=4, \end{cases}$$

where for $\beta>0$,

$$I^\beta u(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-y)^{\beta-1}u(y)\,dy.$$

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Liang and Zhang [8] established the existence of positive solutions to problem (1.1) subject to

$$u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $f(x,u) \in C([0,1] \times [0,\infty), [0,\infty))$ is nondecreasing with respect to u,

$$f\Big(t, \frac{1}{\Gamma(\alpha)}\big(\frac{t^{\alpha-1}}{\alpha-2} - \frac{t^{\alpha}}{\alpha}\big)\Big) \neq 0$$

for $t \in (0,1)$ and there exists a positive constant $\gamma < 1$ such that f is γ-concave with respect to u, that is, for all $\lambda \in [0,1]$,

$$\lambda^{\gamma} f(x,u) \leq f(x, \lambda u).$$

Their approach is based on lower and upper solution method.

Recently Zhai et al [20], by means of fixed point theorem for a sum operator proved the existence and uniqueness of a positive solution to problem (1.1)-(1.2) with $f(x,u) = \varphi(x,u) + \psi(x,u)$, where $\varphi, \psi \in C([0,1] \times [0,\infty), [0,\infty))$ increasing with respect to the second variable. The function φ is γ-concave with respect to u for some $\gamma \in (0,1)$, $\varphi \geq \delta_0 \psi$ for some positive constant δ_0, $\psi(x,0) \neq 0$ and $\psi(x, \lambda u) \geq \lambda \psi(x,u)$ for $\lambda \in (0,1)$.

In this article, we consider the superlinear fractional problem

$$\begin{gathered} D^{\alpha} u(x) - u(x) g(x, u(x)) = 0, \quad 0 < x < 1, \\ u(0) = 0, \quad \lim_{x \to 0^+} D^{\alpha-3} u(x) = 0, \quad \lim_{x \to 0^+} D^{\alpha-2} u(x) = \xi, \quad u''(1) = \zeta, \end{gathered} \tag{1.3}$$

where $3 < \alpha \leq 4$ and $\xi, \zeta \geq 0$ with $\xi + \zeta > 0$.

The function $g(x,u) \in C((0,1) \times [0,\infty), [0,\infty))$, which may be singular at $x = 0$ and $x = 1$ is required to satisfy some convenient hypotheses to be stated later. We emphasize that the condition $\xi + \zeta > 0$ on the boundary data is essential to obtain positive solutions.

To simplify our statements, we use the following notation:

(i) $\mathcal{B}^+((0,1))$ denotes the set of nonnegative measurable functions on $(0,1)$.

(ii) $C(X)$ (resp. $C^+(X)$) denotes the set of continuous (resp. nonnegative continuous) functions on a metric space X.

(iii) We denote by $G(x,y)$ the Green's function of the operator $u \to -D^{\alpha} u$, with boundary conditions

$$u(0) = \lim_{x \to 0^+} D^{\alpha-3} u(x) = \lim_{x \to 0^+} D^{\alpha-2} u(x) = u''(1) = 0.$$

(iv) For $\alpha \in (3,4]$, we let

$$\mathcal{J}_{\alpha} = \{p \in \mathcal{B}^+((0,1)) : \int_0^1 t^{\alpha-1}(1-t)^{\alpha-3} p(t) dt < \infty\}. \tag{1.4}$$

(v) For $p \in \mathcal{B}^+((0,1))$, we denote

$$\tau_p := \sup_{x,y \in (0,1)} \int_0^1 \frac{G(x,t)G(t,y)}{G(x,y)} p(t) dt. \tag{1.5}$$

and we will prove that if $p \in \mathcal{J}_{\alpha}$, then $\tau_p < \infty$.

(vi) For $3 < \alpha \leq 4$ and $\xi, \zeta \geq 0$ with $\xi + \zeta > 0$, we define the function h on $[0,1]$ by

$$\begin{aligned} h(x) &= \frac{\xi}{\Gamma(\alpha)} x^{\alpha-2}(\alpha - 1 - (\alpha-3)x) + \frac{\zeta}{(\alpha-1)(\alpha-2)} x^{\alpha-1} \\ &= h_1(x) + h_2(x). \end{aligned} \tag{1.6}$$

It is easy to show that h is the unique solution of the problem

$$\begin{gathered} D^{\alpha}u(x)=0, \quad 0<x<1, \\ u(0)=0, \quad \lim_{x\to 0^+} D^{\alpha-3}u(x)=0, \quad \lim_{x\to 0^+} D^{\alpha-2}u(x)=\xi, \quad u''(1)=\zeta. \end{gathered} \tag{1.7}$$

Note also that, there exists a constant $M>0$, such that

$$\frac{1}{M}\phi(x)\le h(x)\le M\phi(x), \text{ for all } x\in[0,1] \tag{1.8}$$

where

$$\phi(x)=\begin{cases} x^{\alpha-1}, & \text{if } \xi=0,\\ x^{\alpha-2}, & \text{if } \xi>0. \end{cases}$$

To state our main results, we require a combination of the following conditions.

(H1) $g:(0,1)\times[0,\infty)\to[0,\infty)$, continuous,

(H2) There exists a function $p\in C((0,1))\cap\mathcal{J}_\alpha$ with $\tau_p\le\frac{1}{2}$ such that, for all $x\in(0,1)$, the map $y\to y(p(x)-g(x,yh(x)))$ is nondecreasing on $[0,1]$.

(H3) For all $x\in(0,1)$, the function $y\to yg(x,y)$ is nondecreasing on $[0,\infty)$.

Using a perturbation method, we establish the following result.

Theorem 1.1. *Under assumptions* (H1)–(H2)*, problem* (1.3) *admits a solution* $u\in C([0,1])$ *such that, for all* $x\in[0,1]$*,*

$$c_0h(x)\le u(x)\le h(x), \tag{1.9}$$

where $c_0\in[0,1]$*. Furthermore, if assumption (H3) is also fulfilled, then this solution is unique.*

Corollary 1.2. *Let* $\psi\in C^1([0,\infty))$*,* $\psi\ge 0$ *such that the map* $y\to\varphi(y)=y\psi(y)$ *is nondecreasing on* $[0,\infty)$*. Let* $q\in C^+((0,1))$ *such that the function* $x\to\widetilde{q}(x):=q(x)\max_{0\le t\le h(x)}\varphi'(t)\in\mathcal{J}_\alpha$*. Then for* $\lambda\in[0,\frac{1}{2\tau_{\widetilde{q}}})$*, the problem*

$$\begin{gathered} D^{\alpha}u(x)=\lambda q(x)u(x)\psi(u(x)), \quad x\in(0,1),\\ u(0)=0, \quad \lim_{x\to 0^+} D^{\alpha-3}u(x)=0, \quad \lim_{x\to 0^+} D^{\alpha-2}u(x)=\xi, \quad u''(1)=\zeta, \end{gathered}$$

admits a unique positive solution $u\in C([0,1])$ *such that*

$$(1-\lambda\tau_{\widetilde{q}})h(x)\le u(x)\le h(x), \quad \textit{for all } x\in[0,1].$$

Our paper is organized as follows. In section 2, we give the explicit expression of the Green's function $G(x,y)$ and we establish some sharp estimates on it. In section 3, first for a convenient nonnegative given function p, we construct the Green's function $\mathcal{H}(x,y)$ of the operator $u\to -D^{\alpha}u+pu$, with boundary conditions $u(0)=\lim_{x\to0^+}D^{\alpha-3}u(x)=\lim_{x\to0^+}D^{\alpha-2}u(x)=u''(1)=0$ and we derive some of its properties. In particular, we prove the following statements:

(i) There exists a constant $c\in(0,1]$ such that for $(x,y)\in[0,1]\times[0,1]$,

$$cG(x,y)\le\mathcal{H}(x,y)\le G(x,y).$$

(ii) The equation holds

$$U\psi=U_p\psi+U_p(pU\psi)=U_p\psi+U(pU_p\psi), \quad \text{for all } \psi\in\mathcal{B}^+((0,1)).$$

where the kernels U and U_p are defined on $\mathcal{B}^+((0,1))$ by

$$U\psi(x):=\int_0^1 G(x,y)\psi(y)dy, \quad U_p\psi(x):=\int_0^1\mathcal{H}(x,y)\psi(y)dy, \ x\in[0,1]. \tag{1.10}$$

By exploiting these properties, we prove our main results.

2. On the Green function

We recall the following known properties.

Lemma 2.1 ([7, 14, 17]). *Let $\alpha \in (3,4)$ and $u \in C((0,1)) \cap L^1((0,1))$. Then we have*

(i) *For $0<\gamma<\alpha$, $D^\gamma I^\alpha u = I^{\alpha-\gamma}u$ and $D^\alpha I^\alpha u = u$.*
(ii) *$D^\alpha u(x) = 0$ if and only if $u(x) = c_1x^{\alpha-1} + c_2x^{\alpha-2} + c_3x^{\alpha-3} + c_4x^{\alpha-4}$, where $c_i \in \mathbb{R}$, for $i \in \{1,2,3,4\}$.*
(iii) *Assume that $D^\alpha u \in C((0,1)) \cap L^1((0,1))$, then*
$$I^\alpha D^\alpha u(x) = u(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2} + c_3x^{\alpha-3} + c_4x^{\alpha-4},$$
where $c_i \in \mathbb{R}$, for $i \in \{1,2,3,4\}$.

Next we give the explicit expression of the Green's function $G(x,y)$.

Lemma 2.2. *Let $\alpha \in (3,4]$ and $\psi \in C^+([0,1])$. Then the problem*
$$\begin{gathered} -D^\alpha u(x) = \psi(x), \quad 0<x<1, \\ u(0)=0, \quad \lim_{x\to 0^+} D^{\alpha-3}u(x) = 0, \quad \lim_{x\to 0^+} D^{\alpha-2}u(x) = 0, \quad u''(1) = 0, \end{gathered} \tag{2.1}$$
has a unique nonnegative solution
$$u(x) = \int_0^1 G(x,y)\psi(y)dy, \tag{2.2}$$
where for $x,y \in [0,1]$,
$$G(x,y) = \frac{1}{\Gamma(\alpha)} \begin{cases} x^{\alpha-1}(1-y)^{\alpha-3} - (x-y)^{\alpha-1}, & 0 \le y \le x \le 1; \\ x^{\alpha-1}(1-y)^{\alpha-3}, & 0 \le x \le y \le 1. \end{cases} \tag{2.3}$$

Proof. Since $\psi \in C([0,1])$, by Lemma 2.1, we have
$$u(x) = c_1x^{\alpha-1} + c_2x^{\alpha-2} + c_3x^{\alpha-3} + c_4x^{\alpha-4} - I^\alpha\psi(x).$$
Using the fact that $u(0)=0$, $\lim_{x\to 0^+} D^{\alpha-3}u(x) = 0$ and $\lim_{x\to 0^+} D^{\alpha-2}u(x) = 0$, $u''(1)=0$, we obtain $c_2=c_3=c_4=0$ and $c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-y)^{\alpha-3}\psi(y)dy$. Then, the unique solution of (2.1) is
$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\alpha)}\int_0^1 x^{\alpha-1}(1-y)^{\alpha-3}\psi(y)dy - \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\psi(y)dy \\ &= \int_0^1 G(x,y)\psi(y)dy. \end{aligned}$$
This completes the proof. □

Proposition 2.3. *The Green function $G(x,y)$ in Lemma 2.2 has the following properties:*

(i) *For $y \in [0,1]$, the function $x \to G(x,y)$ belongs to $C^2([0,1])$.*
(ii) *For $x,y \in [0,1]$,*
$$\frac{1}{\Gamma(\alpha)}H_0(x,y) \le G(x,y) \le \frac{2(\alpha-1)}{\Gamma(\alpha)}H_0(x,y),$$
where $H_0(x,y) = x^{\alpha-2}(1-y)^{\alpha-3}\min(x,y)$.

(iii) *For* $x, y \in [0,1]$,

$$\frac{1}{\Gamma(\alpha)} x^{\alpha-1} y(1-y)^{\alpha-3} \le G(x,y) \le \frac{2(\alpha-1)}{\Gamma(\alpha)} x^{\alpha-2} y(1-y)^{\alpha-3}.$$

(iv) *For* $x \in (0,1]$ *and* $y \in [0,1)$,

$$\frac{(\alpha-1)}{\Gamma(\alpha)} H(x,y) \le \frac{\partial}{\partial x} G(x,y) \le \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} H(x,y),$$

where $H(x,y) = x^{\alpha-3}(1-y)^{\alpha-3}\min(x,y)$.

(v) *For* $x \in (0,1]$ *and* $y \in [0,1)$,

$$\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{\Gamma(\alpha)} \widetilde{H}(x,y) \le \frac{\partial^2}{\partial x^2} G(x,y) \le \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \widetilde{H}(x,y),$$

where $\widetilde{H}(x,y) = x^{\alpha-4}(1-y)^{\alpha-4}\min(x,y)(1-\max(x,y))$.

Proof. (i) From Lemma 2.2, for $x, y \in [0,1]$, we have

$$\begin{aligned} G(x,y) &= \frac{1}{\Gamma(\alpha)} \begin{cases} x^{\alpha-1}(1-y)^{\alpha-3} - (x-y)^{\alpha-1}, & 0 \le y \le x \le 1; \\ x^{\alpha-1}(1-y)^{\alpha-3}, & 0 \le x \le y \le 1, \end{cases} \\ &= \frac{1}{\Gamma(\alpha)} \Big[x^{\alpha-1}(1-y)^{\alpha-3} - (\max(x-y,0))^{\alpha-1} \Big]. \end{aligned}$$

Since $\alpha > 3$, if follows that the function $x \to (\max(x-y,0))^{\alpha-1}$ belongs to $C^2([0,1])$. This implies the result.

(ii) Observe that for $a, b > 0$ and $c, y \in [0,1]$, we have

$$\min(1, \frac{b}{a})(1-cy^a) \le 1 - cy^b \le \max(1, \frac{b}{a})(1-cy^a). \tag{2.4}$$

Now, since for $x, y \in [0,1]$, we have

$$G(x,y) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}(1-y)^{\alpha-3} \Big[1 - (1-y)^2 (\frac{\max(x-y,0)}{x(1-y)})^{\alpha-1} \Big],$$

and $\frac{\max(x-y,0)}{x(1-y)} \in [0,1]$, for $x \in (0,1]$ and $y \in [0,1)$, then the required result follows from (2.4) with $b = \alpha - 1$, $a = 1$ and $c = (1-y)^2$.

(iii) The inequalities follows from (i) and the fact that

$$xy \le \min(x,y) \le y, \text{ for } x, y \in [0,1].$$

(iv) Since for $x, y \in [0,1]$,

$$\begin{aligned} \frac{\partial}{\partial x} G(x,y) &= \frac{\alpha-1}{\Gamma(\alpha)} \begin{cases} x^{\alpha-2}(1-y)^{\alpha-3} - (x-y)^{\alpha-2}, & 0 \le y \le x \le 1; \\ x^{\alpha-2}(1-y)^{\alpha-3}, & 0 \le x \le y \le 1, \end{cases} \\ &= \frac{\alpha-1}{\Gamma(\alpha)} x^{\alpha-2}(1-y)^{\alpha-3} \Big[1 - (1-y) \Big(\frac{\max(x-y,0)}{x(1-y)} \Big)^{\alpha-2} \Big], \end{aligned}$$

the required result follows from (2.4) with $b = \alpha - 2$, $a = 1$ and $c = (1-y)$.

(v) Since for $x \in (0,1]$ and $y \in [0,1)$,

$$\begin{aligned} \frac{\partial^2}{\partial x^2} G(x,y) &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \begin{cases} x^{\alpha-3}(1-y)^{\alpha-3} - (x-y)^{\alpha-3}, & 0 \le y \le x \le 1; \\ x^{\alpha-3}(1-y)^{\alpha-3}, & 0 \le x \le y \le 1, \end{cases} \\ &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} x^{\alpha-3}(1-y)^{\alpha-3} [1 - (\frac{\max(x-y,0)}{x(1-y)})^{\alpha-3}], \end{aligned}$$

the required result follows again from (2.4) with $b = \alpha - 3$, $a = 1$ and $c = 1$. This completes the proof. □

From Proposition 2.3 (iii), we deduce the following result.

Corollary 2.4. *Let $\psi \in \mathcal{B}^+((0,1))$, then*

$$U\psi \in C([0,1]) \Longleftrightarrow \int_0^1 y(1-y)^{\alpha-3}\psi(y)dy < \infty.$$

Proposition 2.5. *Let $3 < \alpha < 4$ and $\psi \in C((0,1))$. Assume that the function $y \to y(1-y)^{\alpha-3}\psi(y) \in C((0,1)) \cap L^1((0,1)$, then $U\psi$ is the unique solution in $C([0,1])$ of*

$$\begin{gathered} -D^\alpha u(x) = \psi(x), \quad 0 < x < 1, \\ u(0) = 0, \quad \lim_{x\to 0^+} D^{\alpha-3}u(x) = 0, \quad \lim_{x\to 0^+} D^{\alpha-2}u(x) = 0, \quad u''(1) = 0. \end{gathered} \tag{2.5}$$

Proof. From Corollary 2.4, we deduce that the function $U\psi \in C([0,1])$. This implies that $I^{4-\alpha}(U|\psi|)$ is finite on $[0,1]$. Hence, we obtain

$$\begin{aligned} I^{4-\alpha}(U\psi)(x) &= \frac{1}{\Gamma(4-\alpha)}\int_0^x (x-y)^{3-\alpha}U\psi(y)dy \\ &= \frac{1}{\Gamma(4-\alpha)}\int_0^1 \Big(\int_0^x (x-y)^{3-\alpha}G(y,z)dy\Big)\psi(z)dz \\ &= \int_0^1 \mathcal{K}(x,z)\psi(z)dz, \end{aligned}$$

where

$$\mathcal{K}(x,z) := \frac{1}{\Gamma(4-\alpha)}\int_0^x (x-y)^{3-\alpha}G(y,z)dy.$$

Next we will express explicitly $\mathcal{K}(x,z)$. Using (2.3), we obtain

$$\begin{aligned} &\mathcal{K}(x,z) \\ &= \frac{(1-z)^{\alpha-3}}{\Gamma(4-\alpha)\Gamma(\alpha)}\int_0^x (x-y)^{3-\alpha}y^{\alpha-1}dy \\ &\quad - \frac{1}{\Gamma(4-\alpha)\Gamma(\alpha)}\int_0^x (x-y)^{3-\alpha}(\max(y-z,0))^{\alpha-1}dy \\ &= \frac{1}{6}x^3(1-z)^{\alpha-3} - \frac{1}{\Gamma(4-\alpha)\Gamma(\alpha)}\int_0^x (x-y)^{3-\alpha}(\max(y-z,0))^{\alpha-1}dy \end{aligned}$$

If $z \le x$, then we have

$$\begin{aligned} \int_0^x (x-y)^{3-\alpha}((y-z)^+)^{\alpha-1}dy &= \int_z^x (x-y)^{3-\alpha}(y-z)^{\alpha-1}dy \\ &= \frac{\Gamma(\alpha)\Gamma(4-\alpha)}{6}(x-z)^3. \end{aligned} \tag{2.6}$$

On the other hand, if $x \le z$ and $y \in (0,x)$, we have

$$\int_0^x (x-y)^{3-\alpha}(\max(y-z,0))^{\alpha-1}dy = 0. \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\mathcal{K}(x,z) = \frac{1}{6}x^3(1-z)^{\alpha-3} - \frac{1}{6}(\max(x-z,0))^3.$$

Hence for $x \in (0,1)$, we have

$$\begin{aligned} 6I^{4-\alpha}(U\psi)(x) &= 6\int_0^1 \mathcal{K}(x,z)\psi(z)dz \\ &= x^3\int_0^x [(1-z)^{\alpha-3}-1]\psi(z)dz + 3x^2\int_0^x z\psi(z)dz \\ &\quad - 3x\int_0^x z^2\psi(z)dz + \int_0^x z^3\psi(z)dz + x^3\int_x^1 (1-z)^{\alpha-3}\psi(z)dz \\ &:= J_1(x)+J_2(x)+J_3(x)+J_4(x)+J_5(x). \end{aligned}$$

We claim that

$$D^\alpha(U\psi)(x) := \frac{d^4}{dx^4}(I^{4-\alpha}(U\psi))(x) = -\psi(x), \text{ for } x \in (0,1).$$

Indeed, since the function $z \mapsto z\psi(z)$ is continuous and integrable in a neighborhood of 0 and the function $z \mapsto (1-z)^{\alpha-3}\psi(z)$ is continuous and integrable in a neighborhood of 1, we deduce that $J_2(x), J_3(x), J_4(x)$ and $J_5(x)$ are differentiable.

On the other hand, since $(1-z)^{\alpha-3}-1 = O(z)$ near 0, it follows that $J_1(x)$ is differentiable. So, we have

$$\begin{aligned} \frac{d}{dx}(6I^{4-\alpha}(U\psi))(x) &= 3x^2\int_0^x [(1-z)^{\alpha-3}-1]\psi(z)dz + 6x\int_0^x z\psi(z)dz \\ &\quad - 3x\int_0^x z^2\psi(z)dz + 3x^2\int_x^1 (1-z)^{\alpha-3}\psi(z)dz, \\ &= K_1(x)+K_2(x)+K_3(x)+K_4(x). \end{aligned}$$

Similarly, we obtain

$$\frac{d^4}{dx^4}(I^{4-\alpha}(U\psi))(x) = -\psi(x), \quad \text{for } x \in (0,1).$$

It remains to verify the boundary conditions. Since $U\psi \in C([0,1])$, we deduce that $U\psi(0)=0$. On the other hand, clearly we have

$$\lim_{x\to 0^+} K_1(x) = \lim_{x\to 0^+} K_2(x) = \lim_{x\to 0^+} K_3(x) = 0$$

and by [9, Lemma 2.2], we have $\lim_{x\to 0^+} K_4(x) = 0$. Now, since $D^{\alpha-3}(U\psi)(x) = \frac{d}{dx}(I^{4-\alpha}(U\psi))(x)$, we deduce that

$$\lim_{x\to 0^+} D^{\alpha-3}(U\psi)(x) = 0.$$

Similarly, we show that $\lim_{x\to 0^+} D^{\alpha-2}(U\psi)(x) = 0$, by using the fact that

$$D^{\alpha-2}(U\psi)(x) = \frac{d^2}{dx^2}(I^{4-\alpha}(U\psi))(x).$$

Let $\eta > 0$. By Proposition 2.3 (v), there exists a constant $c > 0$, such that for $x \in (\eta, 1]$ and $y \in (0,1)$, we have

$$\left|\frac{\partial^2}{\partial x^2}G(x,y)\right| \le c\eta^{\alpha-4}y(1-y)^{\alpha-4}(1-\max(x,y)) \le c\eta^{\alpha-4}y(1-y)^{\alpha-3}.$$

So by the Lebesgue theorem, we deduce that $(U\psi)''(1) = 0$.

Finally, the uniqueness follows immediately from Lemma 2.1. The proof is complete. □

Same properties in Proposition 2.5 remain true for $\alpha = 4$.

Proposition 2.6. *For each $x, t, y \in (0,1)$, we have*

$$\frac{G(x,t)G(t,y)}{G(x,y)} \leq \frac{4(\alpha-1)^2}{\Gamma(\alpha)} t^{\alpha-1}(1-t)^{\alpha-3}. \tag{2.8}$$

Proof. Using Proposition 2.3 (ii), for each $x, t, y \in (0,1)$, we have

$$\frac{G(x,t)G(t,y)}{G(x,y)} \leq \frac{4(\alpha-1)^2}{\Gamma(\alpha)} t^{\alpha-2}(1-t)^{\alpha-3} \frac{\min(x,t)\min(t,y)}{\min(x,y)}.$$

So the result follows from the fact that

$$\frac{\min(x,t)\min(t,y)}{\min(x,y)} \leq t.$$

This completes the proof. □

Proposition 2.7. *Let $p \in \mathcal{J}_\alpha$. We have: (i)*

$$\tau_p \leq \frac{4(\alpha-1)^2}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\alpha-3} p(t)dt < \infty, \tag{2.9}$$

where τ_p is given by (1.5).

(ii)

$$U(ph)(x) \leq \tau_p h(x), \text{ for } x \in [0,1]. \tag{2.10}$$

Proof. Let $p \in \mathcal{J}_\alpha$. (i) Using (1.5) and (2.8), we obtain (2.9). (ii) Since $h = h_1 + h_2$, we need to prove (2.10) for h_1 and h_2.

To this end, observe that for each $x, y \in (0,1]$, we have $\lim_{z\to 1} \frac{G(y,z)}{G(x,z)} = \frac{h_2(y)}{h_2(x)}$. Therefore, by applying Fatou lemma and (1.5), we obtain

$$\begin{aligned} \frac{1}{h_2(x)} U(ph_2)(x) &= \int_0^1 G(x,y) \frac{h_2(y)}{h_2(x)} p(y) dy \\ &\leq \liminf_{z\to 1} \int_0^1 G(x,y) \frac{G(y,z)}{G(x,z)} p(y) dy \leq \tau_p. \end{aligned}$$

Similarly, we prove $U(ph_1)(x) \leq \tau_p h_1(x)$, by observing that $\lim_{z\to 0} \frac{G(y,z)}{G(x,z)} = \frac{h_1(y)}{h_1(x)}$. This completes the proof. □

3. Proofs of main results

3.1. **On the Green's function of the perturbed operator.** In this subsection, our goal is to determine the positive solution to the linear fractional problem

$$\begin{gathered} -D^\alpha u(x) + p(x)u(x) = \psi(x), \quad 0 < x < 1, \\ u(0) = \lim_{x\to 0^+} D^{\alpha-3}u(x) = \lim_{x\to 0^+} D^{\alpha-2}u(x) = u''(1) = 0. \end{gathered} \tag{3.1}$$

To this end, we need to construct the Green's function to the homogeneous problem associated with (3.1).

Let $p \in \mathcal{J}_\alpha$. For $(x,y) \in [0,1]\times[0,1]$, put $G_0(x,y) = G(x,y)$ and

$$G_n(x,y) = \int_0^1 G(x,t)G_{n-1}(t,y)p(t)dt, \quad n \geq 1. \tag{3.2}$$

Let $\mathcal{H} : [0,1]\times[0,1] \to \mathbb{R}$, be defined by

$$\mathcal{H}(x,y) = \sum_{n=0}^{\infty} (-1)^n G_n(x,y), \tag{3.3}$$

provided that the series converges.

Lemma 3.1. *Let $p \in \mathcal{J}_\alpha$ with $\tau_p < 1$, then for all $(x,y) \in [0,1]\times[0,1]$, we have*

(i) $G_n(x,y) \le \tau_p^n G(x,y)$ *for each* $n \in \mathbb{N}$. *So,* $\mathcal{H}(x,y)$ *is well defined in* $[0,1]\times[0,1]$.

(ii) *For each* $n \in \mathbb{N}$,
$$L_n x^{\alpha-1} y(1-y)^{\alpha-3} \le G_n(x,y) \le R_n x^{\alpha-2} y(1-y)^{\alpha-3}, \tag{3.4}$$
where
$$L_n = \frac{1}{(\Gamma(\alpha))^{n+1}}\Big(\int_0^1 t^\alpha (1-t)^{\alpha-3} p(t)dt\Big)^n,$$
$$R_n = (\frac{2\alpha-2}{\Gamma(\alpha)})^{n+1}\Big(\int_0^1 t^{\alpha-1}(1-t)^{\alpha-3}p(t)dt\Big)^n.$$

(iii) $G_{n+1}(x,y) = \int_0^1 G_n(x,t)G(t,y)p(t)dt$ *for each* $n \in \mathbb{N}$.

(iv) $\int_0^1 \mathcal{H}(x,t)G(t,y)p(t)dt = \int_0^1 G(x,t)\mathcal{H}(t,y)p(t)dt$.

Proof. (i) Obviously the inequality is valid for $n = 0$. Assume that $G_n(x,y) \le \tau_p^n G(x,y)$, then by using (3.2) and (1.5), we obtain
$$G_{n+1}(x,y) \le \tau_p^n \int_0^1 G(x,t)G(t,y)p(t)dt \le \tau_p^{n+1} G(x,y).$$
So, $\mathcal{H}(x,y)$ is well defined in $[0,1]\times[0,1]$.

(ii) The inequality in (3.4), follows by induction and Proposition 2.3 (iii).

(iii) We will proceed by induction. Obviously the equality is valid for $n = 0$. Assume that
$$G_n(x,y) = \int_0^1 G_{n-1}(x,t)G(t,y)p(t)dt. \tag{3.5}$$
Then by using (3.2) and the Fubini-Tonelli theorem, we obtain
$$\begin{aligned} G_{n+1}(x,y) &= \int_0^1 G(x,t)\Big(\int_0^1 G_{n-1}(t,z)G(z,y)p(z)dz\Big)p(t)dt \\ &= \int_0^1 \Big(\int_0^1 G(x,t)G_{n-1}(t,z)p(t)dt\Big)G(z,y)p(z)dz \\ &= \int_0^1 G_n(x,z)G(z,y)p(z)dz. \end{aligned}$$
(iv) Let $n \in \mathbb{N}$ and $x,t,y \in [0,1]$. From Lemma 3.1 (i) we deduce that
$$0 \le G_n(x,t)G(t,y)p(t) \le \tau_p^n G(x,t)G(t,y)p(t).$$
So, the series $\sum_{n\ge 0}\int_0^1 G_n(x,t)G(t,y)p(t)dt$ is convergent. By the dominated convergence theorem and Lemma 3.1 (iii), we deduce that
$$\begin{aligned} \int_0^1 \mathcal{H}(x,t)G(t,y)p(t)dt &= \sum_{n=0}^\infty \int_0^1 (-1)^n G_n(x,t)G(t,y)p(t)dt \\ &= \sum_{n=0}^\infty \int_0^1 (-1)^n G(x,t)G_n(t,y)p(t)dt \\ &= \int_0^1 G(x,t)\mathcal{H}(t,y)p(t)dt. \end{aligned}$$

□

Proposition 3.2. *For $p \in \mathcal{J}_\alpha$ with $\tau_p < 1$, the function $(x,y) \to \mathcal{H}(x,y)$ belongs to $C([0,1]\times[0,1])$.*

Proof. The function $(x,y) \to G_n(x,y) \in C([0,1]\times[0,1])$, for all $n \in \mathbb{N}$. Clearly $G_0 = G \in C([0,1]\times[0,1])$,

Assume that the function $(x,y) \to G_{n-1}(x,y) \in C([0,1]\times[0,1])$. Using Lemma 3.1 (i) and Proposition 2.3 (iii), we obtain

$$\begin{aligned} G(x,t)G_{n-1}(t,y)p(t) &\le \tau_p^{n-1} G(x,t)G(t,y)p(t) \\ &\le 4\big(\frac{\alpha-1}{\Gamma(\alpha)}\big)^2 t^{\alpha-1}(1-t)^{\alpha-3}p(t). \end{aligned}$$

Therefore by (3.2) and the dominated convergence theorem, we deduce that the function $(x,y) \to G_n(x,y) \in C([0,1]\times[0,1])$.

On the other hand, from Lemma 3.1 (i) and Proposition 2.3 (iii), we have

$$G_n(x,y) \le \tau_p^n G(x,y) \le \frac{2(\alpha-1)}{\Gamma(\alpha)}\tau_p^n.$$

So, the series $\sum_{n\ge0}(-1)^n G_n(x,y)$ is uniformly convergent on $[0,1]\times[0,1]$ and therefore the function $(x,y) \to \mathcal{H}(x,y)$ belongs to $C([0,1]\times[0,1])$. □

Lemma 3.3. *Let $p \in \mathcal{J}_\alpha$ such that $\tau_p \le 1/2$. On $[0,1]\times[0,1]$, one has*

$$(1-\tau_p)G(x,y) \le \mathcal{H}(x,y) \le G(x,y). \tag{3.6}$$

Proof. By using Lemma 3.1 (i), we obtain

$$|\mathcal{H}(x,y)| \le \sum_{n=0}^{\infty}(\tau_p)^n G(x,y) = \frac{1}{1-\tau_p}G(x,y). \tag{3.7}$$

On the other hand, we have

$$\mathcal{H}(x,y) = G(x,y) - \sum_{n=0}^{\infty}(-1)^n G_{n+1}(x,y). \tag{3.8}$$

Since the series $\sum_{n\ge0}\int_0^1 G(x,z)G_n(z,y)p(z)dz$ is convergent, we deduce by (3.8) and (3.2) that

$$\begin{aligned} \mathcal{H}(x,y) &= G(x,y) - \sum_{n=0}^{\infty}(-1)^n \int_0^1 G(x,z)G_n(z,y)p(z)dz \\ &= G(x,y) - \int_0^1 G(x,z)\Big(\sum_{n=0}^{\infty}(-1)^n G_n(z,y)\Big)p(z)dz; \end{aligned}$$

that is,

$$\mathcal{H}(x,y) = G(x,y) - U(p\mathcal{H}(.,y))(x). \tag{3.9}$$

Using (3.7) and Lemma 3.1 (i), we obtain

$$U(p\mathcal{H}(.,y))(x) \le \frac{1}{1-\tau_p}U(pG(.,y))(x) = \frac{1}{1-\tau_p}G_1(x,y) \le \frac{\tau_p}{1-\tau_p}G(x,y).$$

So, by (3.9), we obtain

$$\mathcal{H}(x,y) \ge G(x,y) - \frac{\tau_p}{1-\tau_p}G(x,y) = \frac{1-2\tau_p}{1-\tau_p}G(x,y) \ge 0.$$

So, $\mathcal{H}(x,y) \le G(x,y)$ and by (3.9) and Lemma 3.1 (i), we obtain

$$\mathcal{H}(x,y) \ge G(x,y) - U(pG(.,y))(x) \ge (1-\tau_p)G(x,y).$$

□

Corollary 3.4. *Let $p \in \mathcal{J}_\alpha$ with $\tau_p \le \frac{1}{2}$ and $\psi \in \mathcal{B}^+((0,1))$. Then*

$$U_p\psi \in C([0,1]) \Longleftrightarrow \int_0^1 y(1-y)^{\alpha-3}\psi(y)dy < \infty.$$

Proof. The assertion follows from Proposition 3.2, (3.6) and Proposition 2.3 (iii). □

Lemma 3.5. *Let $p \in \mathcal{J}_\alpha$ with $\tau_p \le \frac{1}{2}$ and $\psi \in \mathcal{B}^+((0,1))$. Then we have*

$$U\psi = U_p\psi + U_p(pU\psi) = U_p\psi + U(pU_p\psi). \tag{3.10}$$

In particular, if $U(p\psi) < \infty$, then

$$(I - U_p(p.))(I + U(p.))\psi = (I + U(p.))(I - U_p(p.))\psi = \psi. \tag{3.11}$$

Here $U(p.)(\psi) := U(p\psi)$.

Proof. From (3.9), we have

$$G(x,y) = \mathcal{H}(x,y) + U(p\mathcal{H}(\cdot,y))(x), \quad \text{for } (x,y) \in [0,1]\times[0,1].$$

So by the Fubini-Tonelli theorem we deduce that

$$\begin{aligned} U\psi(x) &= \int_0^1 (\mathcal{H}(x,y) + U(p\mathcal{H}(.,y))(x))\psi(y)dy \\ &= U_p\psi(x) + U(pU_p\psi)(x). \end{aligned}$$

Now, using Lemma 3.1 (iv) and again the Fubini theorem, we obtain

$$\int_0^1\int_0^1 \mathcal{H}(x,t)G(t,y)p(t)\psi(y)\,dt\,dy = \int_0^1\int_0^1 G(x,t)\mathcal{H}(t,y)p(t)\psi(y)\,dt\,dy;$$

that is,

$$U_p(pU\psi)(x) = U(pU_p\psi)(x).$$

Therefore

$$U\psi = U_p\psi + U(pU_p\psi) = U_p\psi + U_p(pU\psi)(x).$$

□

Proposition 3.6. *Let $\psi \in \mathcal{B}^+((0,1))$ such that $y \mapsto y(1-y)^{\alpha-3}\psi(y)$ belongs to $C((0,1))\cap L^1((0,1))$ and $p \in C((0,1))\cap\mathcal{J}_\alpha$ with $\tau_p \le \frac{1}{2}$. Then $u = U_p\psi$ is the unique nonnegative solution in $C([0,1])$ to problem (3.1) satisfying*

$$(1-\tau_p)U\psi \le u \le U\psi. \tag{3.12}$$

Proof. By Corollary 3.4, we conclude that the function $x \to p(x)U_p\psi(x) \in C((0,1))$.

On the other hand, from (3.10) and Proposition 2.3 (iii), we have that there exists $m \ge 0$ such that

$$U_p\psi(x) \le U\psi(x) \le \frac{2(\alpha-1)}{\Gamma(\alpha)}\int_0^1 x^{\alpha-2}y(1-y)^{\alpha-3}\psi(y)dy \equiv mx^{\alpha-2}. \tag{3.13}$$

Therefore

$$\int_0^1 y(1-y)^{\alpha-3}p(y)U_p\psi(y)dy \le m\int_0^1 y^{\alpha-1}(1-y)^{\alpha-3}p(y)dy < \infty.$$

By applying Proposition 2.5, the function $u = U_p\psi = U\psi - U(pU_p\psi)$ satisfies the equation

$$\begin{gathered} D^\alpha u(x) = -\psi(x) + p(x)u(x), \quad x \in (0,1), \\ u(0) = \lim_{x\to 0^+} D^{\alpha-3}u(x) = \lim_{x\to 0^+} D^{\alpha-2}u(x) = u''(1) = 0. \end{gathered}$$

Integrating the inequalities (3.6), we obtain (3.12).

Next, we prove the uniqueness. Let $w \in C^+([0,1])$ be another solution to problem (3.1) satisfying $w \le U\psi$. Since by (3.12) and (3.13) the function $y \to y(1-y)^{\alpha-3}p(y)w(y) \in C((0,1))\cap L^1((0,1))$, by Proposition 2.5 the function $\widetilde{w} := w + U(pw)$ satisfies

$$\begin{gathered} D^\alpha \widetilde{w}(x) + \psi(x) = 0, \quad x \in (0,1), \\ \widetilde{w}(0) = \lim_{x\to 0^+} D^{\alpha-3}\widetilde{w}(x) = \lim_{x\to 0^+} D^{\alpha-2}\widetilde{w}(x) = \widetilde{w}''(1) = 0. \end{gathered}$$

From Proposition 2.5 we deduce that

$$\widetilde{w} := w + U(pw) = U\psi.$$

Therefore,

$$(I + U(p.))((w-u)^+) = (I + U(p.))((w-u)^-),$$

where $(w-u)^+ = \max(w-u, 0)$ and $(w-u)^- = \max(u-w, 0)$.

Since $|w(y) - u(y)| \le 2U\psi(y) \le 2my^{\alpha-2}$, we deduce by Proposition 2.3 (ii) that

$$\begin{aligned} U(p|w-u|)(x) &\le \frac{4m(\alpha-1)}{\Gamma(\alpha)} \int_0^1 y^{\alpha-2}(1-y)^{\alpha-3} \min(x,y)p(y)dy \\ &\le \frac{4m(\alpha-1)}{\Gamma(\alpha)} \int_0^1 y^{\alpha-1}(1-y)^{\alpha-3}p(y)dy < \infty. \end{aligned}$$

So by (3.11), we obtain that $u = w$. □

3.2. **Proofs of main results.**

Proof of Theorem 1.1. Let $3 < \alpha \le 4$ and $\xi, \zeta \ge 0$ with $\xi + \zeta > 0$. We recall that

$$h(x) := \frac{\xi}{\Gamma(\alpha)} x^{\alpha-2}(\alpha - 1 - (\alpha-3)x) + \frac{\zeta}{(\alpha-1)(\alpha-2)} x^{\alpha-1}, \quad \text{for } x \in [0,1].$$

Let $p \in C((0,1))\cap \mathcal{J}_\alpha$ with $\tau_p \le \frac{1}{2}$ such that assumption (H2) is satisfied. Let

$$F := \{u \in \mathcal{B}^+((0,1)) : (1-\tau_p)h \le u \le h\}.$$

Consider the operator A defined on F by

$$Au = h - U_p(ph) + U_p((p - g(.,u))u).$$

By (3.10) and (2.10) we have

$$U_p(ph) \le U(ph) \le \tau_p h \le h, \tag{3.14}$$

and by (H2), we obtain

$$0 \le g(.,u) \le p \quad \text{for all } u \in F. \tag{3.15}$$

Next, we prove that $A(F) \subset F$. Form (3.15) and (3.14), we obtain

$$\begin{gathered} Au \le h - U_p(ph) + U_p(pu) \le h, \\ Au \ge h - U_p(ph) \ge (1-\tau_p)h. \end{gathered}$$

Since the map $y \mapsto y(p(x) - g(x, yh(x)))$ is nondecreasing on $[0,1]$, for $x \in (0,1)$, the operator A becomes nondecreasing on F.

Define the sequence $\{u_k\}$ by $u_0 = (1-\tau_p)h$ and $u_{k+1} = Au_k$ for $k \in \mathbb{N}$. Since F is invariant under A, we have $u_1 = Au_0 \geq u_0$ and by the monotonicity of A, we obtain

$$(1-\tau_p)h = u_0 \leq u_1 \leq \dots \leq u_k \leq u_{k+1} \leq h.$$

Therefore, the sequence $\{u_k\}$ converges to a function $u \in F$ satisfying

$$u = (I - U_p(p.))h + U_p((p - g(., u))u).$$

Namely

$$(I - U_p(p.))u = (I - U_p(p.))h - U_p(ug(., u)). \tag{3.16}$$

Now, since $U(pu) \leq U(ph) \leq h < \infty$, by applying the operator $(I + U(p.))$ on (3.16) and using (3.10) and (3.11), we conclude that u satisfies

$$u = h - U(ug(., u)). \tag{3.17}$$

We claim that u is a solution of (1.3). From (3.15) and (1.8), we have

$$u(y)g(y, u(y)) \leq p(y)h(y) \leq Mp(y)\phi(y) \leq My^{\alpha-2}p(y). \tag{3.18}$$

This implies that $\int_0^1 y(1-y)^{\alpha-3}u(y)g(y,u(y))dy < \infty$. Hence from Corollary 2.4, we deduce that the function $x \mapsto U(ug(., u))(x) \in C([0,1])$ and from (3.17), we conclude that $u \in C([0,1])$.

Using (H1) and (3.18), we obtain that the function $y \mapsto y(1-y)^{\alpha-3}u(y)g(y,u(y))$ belongs to $C((0,1)) \cap L^1((0,1)))$, which implies by Proposition 2.5 that u is a solution of (1.3).

Now assume further that condition (H3) is satisfied. Let $v \in C([0,1])$ be another nonnegative solution to problem (1.3) satisfying (1.9). As above, we have

$$0 \leq v(y)g(y, v(y)) \leq p(y)h(y) \leq My^{\alpha-2}p(y).$$

So the function $y \to y(1-y)^{\alpha-3}v(y)g(y,v(y)) \in C((0,1)) \cap L^1((0,1)))$. Put $\widetilde{v} := v + U(vg(., v))$. By Proposition 2.5 we have

$$D^\alpha \widetilde{v}(x) = 0, \quad 0 < x < 1,$$
$$u(0) = 0, \quad \lim_{x \to 0^+} D^{\alpha-3}u(x) = 0, \quad \lim_{x \to 0^+} D^{\alpha-2}u(x) = \xi, \quad u''(1) = \zeta.$$

Hence

$$\widetilde{v} := v + U(vg(., v)) = h.$$

That is,

$$v = h - U(vg(., v)). \tag{3.19}$$

For $z \in (0,1)$, we let

$$\varrho(z) = \begin{cases} \frac{v(z)g(z,v(z)) - u(z)g(z,u(z))}{v(z)-u(z)}, & \text{if } v(z) \neq u(z), \\ 0, & \text{if } v(z) = u(z). \end{cases}$$

Note that, from (H3), we have $\varrho \in \mathcal{B}^+((0,1))$. Using (3.17) and (3.19) we deduce

$$(I + U(\varrho.))((v-u)^+) = (I + U(\varrho.))((v-u)^-),$$

where $(v-u)^+ = \max(v-u, 0)$ and $(v-u)^- = \max(u-v, 0)$. Since $\varrho \leq p$, we deduce by (2.10), that

$$U(\varrho|v-u|) \leq 2U(ph) \leq 2\tau_p h < \infty.$$

Hence $u = v$ by (3.11). This ends the proof. $\square$

Proof of Corollary 1.2. The conclusion follows from Theorem 1.1 with $g(x,y) = \lambda q(x)\psi(y)$ and $p(x) := \lambda\widetilde{q}(x)$. □

Example 3.7. Let $3 < \alpha \leq 4$ and $\xi, \zeta \geq 0$ with $\xi + \zeta > 0$. Let $r \geq 0$, $\nu \geq 0$ and $q \in C^+((0,1))$ such that

$$\int_0^1 t^{(\alpha-1)+(\alpha-2)(r+\nu)}(1-t)^{\alpha-3}q(t)dt < \infty.$$

Let $\varphi(s) = s^{r+1}\log(1+s^\nu)$ and $\widetilde{q}(y) := q(y)\max_{0\leq t\leq h(y)}\varphi'(t)$. Since $\widetilde{q} \in \mathcal{J}_\alpha$, then for $\lambda \in [0, \frac{1}{2\tau_{\widetilde{q}}})$, the problem

$$D^\alpha u(x) = \lambda q(x)u^{r+1}(x)\log(1+u^\nu(x)), \quad 0 < x < 1,$$
$$u(0) = 0, \quad \lim_{x\to 0^+} D^{\alpha-3}u(x) = 0, \quad \lim_{x\to 0^+} D^{\alpha-2}u(x) = \xi, \quad u''(1) = \zeta,$$

admits a unique positive solution $u \in C([0,1])$ satisfying

$$(1-\lambda\tau_{\widetilde{q}})h(x) \leq u(x) \leq h(x), \quad \text{for all } 0 \leq x \leq 1.$$

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References

[1] R. P. Agarwal, D. O'Regan, S. Staněk; *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations*, J. Math. Anal. Appl., 371 (2010), 57-68.
[2] I. Bachar, M. Mâagli, V. D. Rădulescu; *Fractional Navier boundary value problems*, Bound Value Probl 2016:79 (2016) DOI 10.1186/s13661-016-0586-7
[3] I. Bachar, M. Mâagli, F. Toumi, Z. Zine el Abidine; *Existence and Global Asymptotic Behavior of Positive Solutions for Sublinear and Superlinear Fractional Boundary Value Problems*, Chinese Annals of Mathematics, Series B, 37 (2016), no. 1, 1-28.
[4] J. Caballero Mena, J. Harjani, K. Sadarangani; *Existence and uniqueness of positive and nondecreasing solutions for a class of singular fractional boundary value problems*. Bound Value Probl 2009:421310 (2009) DOI 10.1155/2009/421310.
[5] J. Giacomoni, P. Kumar Mishra, K. Sreenadh; *Fractional elliptic equations with critical exponential nonlinearity*, Adv. Nonlinear Anal., 5 (2016), 57-74.
[6] R. Hilfer; *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[7] A. Kilbas, H. Srivastava, J. Trujillo; *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics studies, Vol. 204, Elsevier, Amsterdam, 2006.
[8] S. Liang, J. Zhang; *Positive solutions for boundary value problems of nonlinear fractional differential equation*, Nonlinear Anal., 71 (2009), 5545–5550.
[9] R. Ma, C. C. Tisdell; *Positive solutions of singular sublinear fourth order boundary value problems*, Appl. Anal., 84 (2005), no. 12, 1199–1220.
[10] H. Mâagli, N. Mhadhebi, N. Zeddini; *Existence and Estimates of Positive Solutions for Some Singular Fractional Boundary Value Problems*, Abstract and Applied Analysis Vol. 2014 (2014), Article ID 120781, 7 pages.
[11] K. Miller, B. Ross; *An introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[12] G. Molica Bisci, V. D. Rădulescu, R. Servadei; *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications, Vol. 162, Cambridge University Press, Cambridge, 2016.
[13] G. Molica Bisci, D. Repovš; *Existence and localization of solutions for nonlocal fractional equations*, Asymptot. Anal. 90 (2014), no. 3-4, 367–378.
[14] I. Podlubny; *Fractional Differential Equations*, in: Mathematics in Sciences and Engineering, vol. 198, Academic Press, San Diego, 1999.
[15] P. Pucci, M. Xiang, B. Zhang; *Existence and multiplicity of entire solutions for fractional p-Kirchhof equations*, Adv. Nonlinear Anal., 5 (2016), 27-55.

[16] M. Roşiu; *Trajectory structure near critical points.* An. Univ. Craiova Ser. Mat. Inform. 25 (1998), 35-44.

[17] S. Samko, A. Kilbas, O. Marichev; *Fractional Integrals and derivative.* Theory and applications. Gordon and Breach, Yverdon, 1993.

[18] V. Tarasov; *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer-Verlag, New York, 2011.

[19] Z. Xiangkui, C. Chengwen, G. Weigao; *Existence and nonexistence results for a class of fractional boundary value problems*, J Appl Math Comput, 41 (2013), 17-31.

[20] C. Zhai, W. Yan, C. Yang; *A sum operator method for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems*, Commun Nonlinear Sci Numer Simul, 18 (2013), 858-866.

[21] Y. Zhao, S. Sun, Z. Ha, Q. Li; *The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations*, Commun Nonlinear Sci Numer Simul, 16 (2011), 2086-2097.

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