Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 109, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LIMIT CYCLES BIFURCATED FROM A CENTER IN A THREE DIMENSIONAL SYSTEM

BO SANG, BRIGITA FERČEC, QIN-LONG WANG

Abstract. Based on the pseudo-division algorithm, we introduce a method for computing focal values of a class of 3-dimensional autonomous systems. Using the ϵ^1-order focal values computation, we determine the number of limit cycles bifurcating from each component of the center variety (obtained by Mahdi et al). It is shown that at most four limit cycles can be bifurcated from the center with identical quadratic perturbations and that the bound is sharp.

1. Introduction

Many real world phenomena can be modeled by autonomous systems of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^m, \tag{1.1}$$

where $\mathbf{f} : D \to \mathbb{R}^m$ is a smooth function and D an open connected subset of $\mathbb{R}^m$. A limit cycle of the system is a periodic orbit which is isolated among periodic orbits. Limit cycles may be used to model the behavior of many real-world oscillator systems of great importance (see [6, 20, 24, 26, 29]). The study of limit cycles was initiated by Poincaré [25]. Further research was perhaps motivated by Hilbert's 16th problem. A fundamental question in these studies is the determination of upper bounds, H_n, for the number of limit cycles in planar polynomial vector fields of degree n and their relative position. Although the problem was formulated more than a hundred years ago, it is not yet solved even for planar quadratic systems. Moreover, it is unknown even whether a uniform upper bound exists (see [8, 27]).

An essential part of the problem, called the local 16th Hilbert problem [7], is the investigation of the number of limit cycles bifurcated from singular points, i.e., the cyclicity of singular points. The concept of cyclicity was introduced by Bautin in [3], where he showed that in antisaddles of quadratic systems at most three small-amplitude limit cycles can bifurcate out of one equilibrium point. Bautin's work is important not only because of the bound that it provides, but also because of the approach it gives to the study of the problem of cyclicity in any polynomial system. Specifically, Bautin showed that the cyclicity problem in the case of a simple focus or center could be reduced to the problem of finding a basis for the ideal of focal values. Bautin's approach is described in detail and further developed in [16, 27].

2010 *Mathematics Subject Classification.* 34C05, 34C07.
Key words and phrases. Focal value; limit cycle; Hopf bifurcation; center.

Submitted September 25, 2015. Published April 26, 2016.

The cyclicity problem for some families of polynomial systems was treated also in [5, 7, 14, 30, 33, 37].

Higher-dimensional vector fields may not only exhibit limit cycles, but also may co-exist with chaotic dynamics. For results on limit cycles for higher-dimensional vector fields see [4, 15, 18] and more references therein. Llibre et al.[18] studied the limit cycles of polynomial vector fields in $\mathbb{R}^3$ which bifurcate from three different kinds of two dimensional centers (non-degenerate and degenerate). Buzzi et al. [4] studied the maximal number of limit cycles that can bifurcate from a periodic orbits of the linear center in $\mathbb{R}^4$. Han and Yu [15] showed that perturbing a simple quadratic system in $\mathbb{R}^3$ with a center-type equilibrium point can yield at least 10 small-amplitude limit cycles around an equilibrium point.

The computation of focal values (focus values, Lyapunov constants) plays an important role in the study of the center-focus problem and small-amplitude limit cycles arising in degenerated Hopf bifurcations (see [2, 5, 9, 10, 12, 15, 17, 19, 21, 29, 31, 35, 36] and references therein). For the definition and computation of focal values in 3-dimensional systems, see [34, 36] and the second part of this paper.

We consider the general n-dimensional system

$$\frac{\mathbf{dx}}{dt} = \mathbf{f}_1(\mathbf{x}, \mathbf{p}_{k_1}) + \epsilon \mathbf{f}_2(\mathbf{x}, \mathbf{p}_{k_2}), \tag{1.2}$$

(which is an integrable system for $\epsilon = 0$) associated with a Hopf bifurcation, where $\mathbf{p}_{k_1} = (p_1, p_2, \dots, p_{k_1})$ is the system parameter and $\mathbf{p}_{k_2} = (p_{k_1+1}, \dots, p_{k-1})$ is the perturbation parameter. Since the corresponding Hopf equilibrium point is a center for the flow on the center manifold when $\epsilon = 0$, the kth focal values of system (1.2) can be written in the form of

$$V_k = \tilde{v}_{k,1}(\mathbf{p})\epsilon + \tilde{v}_{k,2}(\mathbf{p})\epsilon^2 + \tilde{v}_{k,3}(\mathbf{p})\epsilon^3 + \dots, \quad k = 1, 2, \dots, \tag{1.3}$$

where $\tilde{v}_{k,m}$ are said to be ϵ^m-order focal values (see [35]). For $|\epsilon| > 0$ sufficiently small, we may use $\tilde{v}_{k,1}$ to determine the number of small-amplitude limit cycles of system (1.2) bifurcated from the equilibrium point.

Lemma 1.1 ([15]). *Assume that at $\mathbf{p} = \mathbf{p}_c = (p_{1c}, p_{2c}, \dots, p_{(k-1)c})$, the ϵ^1-order focal values of system* (1.2) *satisfy*

$$\tilde{v}_{j,1}(\mathbf{p}_c) = 0, \quad j = 1, 2, \dots, k-1; \ \tilde{v}_{k,1}(\mathbf{p}_c) \neq 0$$

and

$$\operatorname{Rank}\Big[\frac{D(\tilde{v}_{1,1}, \tilde{v}_{2,1}, \dots, \tilde{v}_{k-1,1})}{D(p_1, p_2, \dots, p_{k-1})}\Big]_{\mathbf{p}=\mathbf{p}_c} = k - 1. \tag{1.4}$$

Then, proper perturbations can be made to the parameters $p_1, p_2, \dots, p_{k-1}$ around the critical point $\mathbf{p}_c$ to generate $k-1$ small-amplitude limit cycles in the vicinity of the Hopf equilibrium point.

In fact, this Lemma was first presented in [15] without proof. For completeness and convenience, we give a proof using the basic idea found in [13].

Proof. Proving the existence of $k - 1$ small-amplitude limit cycles near the equilibrium point for system (1.2) is equivalent to proving that the amplitude equation of the normal form (expressed in polar coordinates) of the system up to kth order, given by

$$\frac{dr}{dt} = r^3(V_1 + V_2 r^2 + \dots + V_k r^{2k-2})$$

$$
\begin{aligned}
&= r^3 \sum_{j=1}^{k} \Big(\sum_{s=1}^{\infty} \tilde{v}_{j,s}(\mathbf{p}) \epsilon^s \Big) (r^2)^{j-1} \\
&= \epsilon r^3 \sum_{j=1}^{k} \tilde{v}_{j,1}(\mathbf{p}) (r^2)^{j-1} + o(\epsilon) \\
&= \epsilon r^3 \Big(\sum_{j=1}^{k} \tilde{v}_{j,1}(\mathbf{p}) (r^2)^{j-1} + o(1) \Big), \quad \epsilon \to 0,
\end{aligned}
$$

has $k-1$ small positive zeros for r^2, where $\mathbf{p} = (\mathbf{p}_{k_1}, \mathbf{p}_{k_2}) = (p_1, p_2, \dots, p_{k-1})$. Let $\rho = r^2$. By the implicit function theorem, one can reduce the problem to the existence of $k-1$ small positive simple zeros for ρ in the algebraic equation

$$
\tilde{v}_{k,1}(\mathbf{p}) \rho^{k-1} + \tilde{v}_{k-1,1}(\mathbf{p}) \rho^{k-2} + \dots + \tilde{v}_{1,1}(\mathbf{p}) = 0. \tag{1.5}
$$

By the conditions given in (1.4), there exists a $\mathbf{p} = (p_1, p_2, \dots, p_{k-1})$ in a neighborhood of $\mathbf{p}_c$, such that

$$
|\tilde{v}_{1,1}(\mathbf{p})| \ll |\tilde{v}_{2,1}(\mathbf{p})| \ll |\tilde{v}_{3,1}(\mathbf{p})| \ll \dots \ll |\tilde{v}_{k,1}(\mathbf{p})| \ll 1, \quad \tilde{v}_{j,1}(\mathbf{p}) \tilde{v}_{j+1,1}(\mathbf{p}) < 0,
$$

for $j = 1, 2, \dots k-1$, which ensures the existence of $k-1$ positive simple zeros for (1.5). Hence, $k-1$ small-amplitude limit cycles can bifurcate from the equilibrium point. This completes the proof. □

Mulholland [24] studied the behavior of the solutions of the third-order non-linear differential equation

$$
\frac{d^3x}{dt^3} + F(r) \frac{d^2x}{dt^2} + F(r) \frac{dx}{dt} + x = 0, \tag{1.6}
$$

in which

$$
F(r) = 1 - \epsilon f(r), \quad f(r) = 1 - r^2, \quad r^2 = x^2 + (\frac{dx}{dt})^2 + (\frac{d^2x}{dt^2})^2. \tag{1.7}
$$

Here $F(r)$ represents a central restoring force, which has important applications in modern control theory. For this equation with small non-linearities, the existence of a limit cycle is established by a fixed point technique, the approach to the limit cycle is approximated by averaging methods, and the periodic solution is harmonically represented by perturbation.

Mahdi et al [22, 23] investigated the center-focus problem of a third-order differential equation of the form

$$
\frac{d^3u}{dt^3} = \frac{d^2u}{dt^2} + \frac{du}{dt} + u + f(u, \frac{du}{dt}, \frac{d^2u}{dt^2}), \tag{1.8}
$$

where $f(u, \frac{du}{dt}, \frac{d^2u}{dt^2})$ is an analytic function starting with quadratic terms. It can be reduced to a system of first order differential equations

$$
\begin{aligned}
\frac{du}{dt} &= -v + h(u, v, w), \\
\frac{dv}{dt} &= u + h(u, v, w), \\
\frac{dw}{dt} &= -w + h(u, v, w),
\end{aligned} \tag{1.9}
$$

where $h(u, v, w) = f(-u + w, v - w, u + w)/2$, which is equivalent to (1.8), see [22]. The center conditions on the local center manifold for system (1.9) with

$$h(u, v, w) = a_1 u^2 + a_2 v^2 + a_3 w^2 + a_4 uv + a_5 uw + a_6 vw \tag{1.10}$$

were obtained in [23]. For center conditions of some other polynomial differential systems in $\mathbb{R}^3$ we refer to [1, 11, 32].

Lemma 1.2 ([23]). *The system* (1.9) *with* $h(u, v, w)$ *as in* (1.10) *admits a center on the local center manifold (for the equilibrium point at the origin) if and only if one of the following holds:*

(1) $a_1 = a_2 = a_4 = 0$;
(2) $a_1 - a_2 = a_3 = a_5 = a_6 = 0$;
(3) $a_1 + a_2 = a_3 = a_5 = a_6 = 0$;
(4) $a_1 + a_2 = 2a_2 - a_3 + a_6 = a_3 - a_4 - 2a_5 = 2a_4 + 3a_5 + a_6 = 0$;
(5) $2a_1 - a_6 = 2a_2 + a_5 = 2a_3 - a_5 + a_6 = a_4 + a_5 + a_6 = 0$;
(6) $a_1 - a_2 = 2a_2 + a_6 = a_4 = a_5 + a_6 = 0$;
(7) $2a_1 + a_2 = 2a_2 + a_6 = 4a_3 + 5a_6 = a_4 = 2a_5 - a_6 = 0$.

For system (1.9), we consider the perturbed system (with identical nonlinearities for the three components) of the form

$$\begin{aligned} \frac{du}{dt} &= -v + h(u, v, w) + \epsilon h_1(u, v, w), \\ \frac{dv}{dt} &= u + h(u, v, w) + \epsilon h_1(u, v, w), \\ \frac{dw}{dt} &= -w + h(u, v, w) + \epsilon h_1(u, v, w), \end{aligned} \tag{1.11}$$

where

$$h_1(u, v, w) = b_1 u^2 + b_2 v^2 + b_3 w^2 + b_4 uv + b_5 uw + b_6 vw. \tag{1.12}$$

The purpose of this paper is prove that four is an upper bound for small amplitude limit cycles that bifurcate from the origin of the system (1.11) when one of conditions in Lemma 1.2 holds.

The remainder of the paper is organized as follows. Based on a previously developed algorithm of Sang [28] for 2-dimensional systems, in Section 2, we introduced a new algorithm for computing focal values of 3-dimensional systems. In Section 3, we prove that at most four small-amplitude limit cycles can bifurcate out of the center based on the analysis of ϵ^1-order focal values.

2. Algorithm for computing focal values

In this section, we present an algorithm for computing focal values of a class of 3-dimensional differential systems

$$\begin{aligned} \frac{du}{dt} &= -v + \sum_{j+k+s=2}^{\infty} \tilde{a}_{jks} u^j v^k w^s, \\ \frac{dv}{dt} &= u + \sum_{j+k+s=2}^{\infty} \tilde{b}_{jks} u^j v^k w^s, \\ \frac{dw}{dt} &= -dw + \sum_{j+k+s=2}^{\infty} \tilde{c}_{jks} u^j v^k w^s, \end{aligned} \tag{2.1}$$

which is analytic in the neighborhood of the origin and $u, v, w, t \in \mathbb{R}$, $d > 0$, $\tilde{a}_{jks}, \tilde{b}_{jks}, \tilde{c}_{jks} \in \mathbb{R}$, $j, k, s \in \mathbb{N} \cup \{0\}$.

By means of the transformation

$$u = \frac{1}{2}(x+y), \quad v = \frac{\mathtt{i}}{2}(-x+y), \quad w = z, \quad t = -\mathtt{i}t_1, \tag{2.2}$$

where $\mathtt{i} = \sqrt{-1}$, system (2.1) can be transformed into the following complex system

$$\begin{aligned} \frac{dx}{dt_1} &= x + \sum_{j+k+s=2}^{\infty} a_{jks} x^j y^k z^s = X(x,y,z), \\ \frac{dy}{dt_1} &= -y + \sum_{j+k+s=2}^{\infty} b_{jks} x^j y^k z^s = Y(x,y,z), \\ \frac{dz}{dt_1} &= \mathtt{i}dz + \sum_{j+k+s=2}^{\infty} c_{jks} x^j y^k z^s = Z(x,y,z). \end{aligned} \tag{2.3}$$

Lemma 2.1 ([36]). *For system* (2.3)*, there exists a formal power series*

$$F(x,y,z) = xy + \sum_{s=3}^{\infty} \sum_{k=0}^{s} \sum_{j=0}^{s-k} B_{s,k,j} x^{s-k-j} y^k z^j, \tag{2.4}$$

such that

$$\frac{dF}{dt_1}\Big|_{(2.3)} = \frac{\partial F}{\partial x} X + \frac{\partial F}{\partial y} Y + \frac{\partial F}{\partial z} Z = \sum_{n=1}^{\infty} W_n (xy)^{n+1}, \tag{2.5}$$

where $B_{s,k,j}$ are determined by the recursive formula (see [36]*) with $B_{2k,k,0} = 0$. The terms W_n are called the nth singular point values of system* (2.3) *at the origin.*

Lemma 2.2 ([36]). *For any positive natural number n, the following assertion holds:*

$$V_n = i\pi W_n \mod \langle W_1, W_2, \dots, W_{n-1} \rangle, \tag{2.6}$$

where V_n is the n-th focal value of system (2.1)*, and W_j is the jth singular point value of system* (2.3)*, $j = 1, 2, \dots, n$. More precisely, when $W_1 = W_2 = \dots = W_{n-1} = 0$, the following assertion holds:*

$$V_n = i\pi W_n. \tag{2.7}$$

Note that the equilibrium point of system (2.1) at the origin is either a center or a fine focus for the flow on the local center manifold (see [36]). The problem of distinguishing between these two cases is called the center problem. The origin is said to be a fine focus of order k ($k \in \mathbb{N}$) if V_k is the first non-zero focal value. In this case at most k limit cycles can be bifurcated from the fine focus; these limit cycles are called small-amplitude limit cycles. The origin is a center when all the focal values are zero.

Grouping like terms in the second expression of (2.5), we obtain

$$\begin{aligned} &\frac{\partial F}{\partial x}X+\frac{\partial F}{\partial y}Y+\frac{\partial F}{\partial z}Z\\ &=\sum_{s=3}^{2n+1}\sum_{k=0}^{s}\sum_{j=0}^{s-k}f_{s,k,j}x^{s-k-j}y^kz^j\\ &\quad+\sum_{\substack{k,j\geq 0\\ k+j\leq 2n+2\\ (k,j)\neq(n+1,0)}}f_{2n+2,k,j}x^{2n+2-k-j}y^kz^j+D_n(xy)^{n+1}+\dots, \end{aligned} \tag{2.8}$$

where $D_n, f_{s,k,j}, f_{2n+2,k,j}$ can be considered as linear polynomials of variables $B_{s,k,j}$ with coefficients formed from $a_{jks}, b_{jks}, c_{jks}$.

Suppose that $W_1 = W_2 = \dots = W_{n-1} = 0$. In this case, we are in a position to develop the algorithm for computing the nth singular point value W_n of system (2.3) based on pseudo-divisions. It is remarkable that the idea behind it is similar to the situation of 2-dimensional systems described by Sang in [28]. When computing the nth singular point value W_n, the coefficients $f_{s,k,j}, f_{2n+2,k,j}$ have to be zero. Thus in order to eliminate variables $B_{s,k,j}$ from D_n, we use successive pseudo-divisions: first choosing an adequate variable order of $B_{s,k,j}$; then rearranging some polynomials $f_{s,k,j}, f_{2n+2,k,j}$ to get a triangular set TS_n, next performing successive pseudo-division of $D_n+\xi$ by TS_n to get the pseudo-remainder R_n, and finally expressing the nth singular point value W_n as $\frac{R_n}{\operatorname{coeff}(R_n,\xi)}-\xi$, where $\operatorname{coeff}(R_n,\xi)$ is the coefficient of ξ in the polynomial R_n, and ξ is a dummy variable. The termination of the algorithm is trivial because the number of variables $B_{s,k,j}$ is finite when n is fixed.

Recalling Lemma 2.2, once W_n is returned, the nth focal value V_n of system (2.1) can be obtained from relation (2.7). Thus, the algorithm can be modified for computing the nth focal value V_n of system (2.1).

3. Four limit cycles obtained from ϵ-order focal values

Suppose that the condition (1) in Lemma 1.2 holds. It is easy to obtain the ϵ^1-order focal values of system (1.11) (up to a positive constant multiple):

$$\begin{aligned} \tilde{v}_{1,1}&=\frac{13a_5b_1}{20}+\frac{7a_5b_2}{20}-\frac{1}{20}a_5b_4+\frac{11a_6b_1}{20}+\frac{9a_6b_2}{20}+\frac{3a_6b_4}{20},\\ \tilde{v}_{2,1}&=\frac{473{a_5}^3b_1}{1080}+\frac{371{a_5}^3b_2}{1080}-\frac{2{a_5}^3b_4}{27}+\frac{409a_6{a_5}^2b_1}{1080}+\frac{71{a_5}^2a_6b_2}{360}\\ &\quad+\frac{31a_5{a_6}^2b_1}{360}-\frac{a_5{a_6}^2b_2}{120}+\frac{{a_6}^3b_1}{120}-\frac{1}{40}{a_6}^3b_2,\\ \tilde{v}_{3,1}&=-\frac{49{a_5}^5b_1}{2720}+\frac{2661a_6{a_5}^4b_1}{19040}-\frac{3981{a_5}^3{a_6}^2b_1}{3332}-\frac{156509{a_5}^2{a_6}^3b_1}{13328}\\ &\quad-\frac{667381{a_5}^2{a_6}^3b_2}{93296}-\frac{5379919a_5{a_6}^4b_1}{133280}-\frac{142871a_5{a_6}^4b_2}{5831}-\frac{25080959{a_6}^5b_1}{932960}\\ &\quad-\frac{2066875{a_6}^5b_2}{93296}-\frac{342555{a_6}^5b_4}{46648},\\ \tilde{v}_{m,1}&=0,\quad m\geq 4, \end{aligned}$$

where the quantity $\tilde{v}_{k,1}$ is reduced with respect to the Gröbner basis of $\{\tilde{v}_{j,1} : j < k\}$.

Theorem 3.1. *Based on the analysis of ϵ-order focal values for the case* (1)*, the perturbed system* (1.11) *can have at most two small-amplitude limit cycles bifurcated from the center, and the bound is sharp.*

Proof. For this case, an appropriate selection of (a_5, b_1, b_2) for system (1.11) is:

$$a_5 = 0, \quad b_1 = -\frac{3}{14}b_4, \quad b_2 = -\frac{1}{14}b_4, \quad b_4, a_6 \neq 0, \tag{3.1}$$

which implies

$$\tilde{v}_{1,1} = \tilde{v}_{2,1} = 0, \quad \tilde{v}_{3,1} = -\frac{11b_4{a_6}^5}{38080} \neq 0,$$

and the rank of Jacobian matrix (evaluated at the critical point) of $\tilde{v}_{1,1}, \tilde{v}_{2,1}$ with respect to b_1, b_2 is two, hence by Lemma 1.1, the perturbed system (1.11) can have at most two small-amplitude limit cycles bifurcated from the center, and the bound is sharp. $\square$

Now, we assume that condition (2) in Lemma 1.2 holds. It is easy to obtain the ϵ^1-order focal values of system (1.11) (up to a positive constant multiple):

$$\begin{aligned}
\tilde{v}_{1,1} &= -a_2b_1 + a_2b_2 + b_5a_2 + b_6a_2 - \frac{1}{20}b_5a_4 + \frac{3b_6a_4}{20},\\
\tilde{v}_{2,1} &= \frac{326b_1{a_2}^3}{405} - \frac{326b_2{a_2}^3}{405} + \frac{86b_5{a_2}^3}{81} - \frac{704b_6{a_2}^3}{405} - 4b_3{a_2}^3 + \frac{107b_1{a_2}^2a_4}{135}\\
&\quad - \frac{107b_2{a_2}^2a_4}{135} - \frac{302a_4{a_2}^2b_5}{405} + \frac{2}{5}b_3a_4{a_2}^2 - \frac{4b_1{a_4}^2a_2}{45} + \frac{4b_2{a_4}^2a_2}{45}\\
&\quad + \frac{43a_2{a_4}^2b_5}{135} - \frac{53b_3{a_4}^2a_2}{150} + \frac{{a_4}^3b_5}{180} + \frac{{a_4}^3b_3}{300},\\
\tilde{v}_{3,1} &= -\frac{1376747429{a_2}^5b_1}{1275} + \frac{1376747429{a_2}^5b_2}{1275} - \frac{1917866{a_2}^5b_3}{85}\\
&\quad + \frac{185356263b_5{a_2}^5}{170} + \frac{913355709b_6{a_2}^5}{850} - \frac{68663387{a_2}^4a_4b_1}{1275}\\
&\quad + \frac{68663387{a_2}^4a_4b_2}{1275} + \frac{5211337{a_2}^4a_4b_3}{2550} + \frac{33139013{a_2}^4a_4b_6}{150}\\
&\quad - \frac{55023{a_2}^3{a_4}^2b_1}{34} + \frac{55023{a_2}^3{a_4}^2b_2}{34} - \frac{50399083{a_2}^3{a_4}^2b_3}{25500}\\
&\quad + \frac{50559593{a_2}^3{a_4}^2b_6}{5100} - \frac{24129{a_2}^2{a_4}^3b_1}{1700}\\
&\quad + \frac{24129{a_2}^2{a_4}^3b_2}{1700} + \frac{1877003{a_2}^2{a_4}^3b_6}{10200} - \frac{269a_2{a_4}^4b_1}{3400}\\
&\quad + \frac{269a_2{a_4}^4b_2}{3400} + \frac{31057a_2{a_4}^4b_6}{20400} + \frac{87{a_4}^5b_6}{13600},\\
\tilde{v}_{4,1} &= -\frac{153971816454597031246{a_2}^7b_1}{491160925125} + \frac{153971816454597031246{a_2}^7b_2}{491160925125}\\
&\quad - \frac{16610753188310104{a_2}^7b_3}{2518773975} + \frac{691036797753634751b_5{a_2}^7}{2182937445}\\
&\quad + \frac{1309536133953556187b_6{a_2}^7}{4197956625} - \frac{264798540296377058{a_2}^6a_4b_1}{16936583625}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{264798540296377058a_2{}^6a_4b_2}{16936583625}+\frac{293602854116874307a_2{}^6a_4b_3}{491160925125}\\
&+\frac{3150760909024688186 2a_2{}^6a_4b_6}{491160925125}-\frac{25504331681017132a_2{}^5a_4{}^2b_1}{54573436125}\\
&+\frac{25504331681017132a_2{}^5a_4{}^2b_2}{54573436125}-\frac{218556152766434059a_2{}^5a_4{}^2b_3}{377816096250}\\
&+\frac{2825095303169559761a_2{}^5a_4{}^2b_6}{982321850250}-\frac{27926323546531a_2{}^4a_4{}^3b_1}{7276458150}\\
&+\frac{27926323546531a_2{}^4a_4{}^3b_2}{7276458150}+\frac{103518014332944689a_2{}^4a_4{}^3b_6}{1964643700500}\\
&-\frac{1454958217433a_2{}^3a_4{}^4b_1}{163720308375}+\frac{1454958217433a_2{}^3a_4{}^4b_2}{163720308375}\\
&+\frac{755866567299851a_2{}^3a_4{}^4b_6}{1964643700500}+\frac{265910137a_2{}^2a_4{}^5b_1}{2509123500}\\
&-\frac{265910137a_2{}^2a_4{}^5b_2}{2509123500}-\frac{529a_2a_4{}^6b_1}{4806750}+\frac{529a_2a_4{}^6b_2}{4806750},\\
\tilde v_{5,1}&=\tilde v_{5,1}(a_2,a_4,b_5,b_6),\\
\tilde v_{m,1}&=0,\quad m\ge 6,
\end{aligned}
$$

where the quantity $\tilde v_{k,1}$ is reduced with respect to the Gröbner basis of $\{\tilde v_{j,1}: j<k\}$ and the expression of $\tilde v_{5,1}$ is too lengthy to be presented here.

Theorem 3.2. *Based on the analysis of ϵ-order focal values for the case* (2)*, the perturbed system* (1.11) *can have at most four small-amplitude limit cycles bifurcated from the center, and the bound is sharp.*

Proof. For this case, an appropriate selection of (a_2,b_1,b_3,b_5) for system (1.11) is that

$$
\begin{aligned}
a_2 &= -0.124090617911090057250206646846125321120939094367768\text{(cont.)}\\
&\quad 2051487342750a_4,\\
b_1 &= 1.000000000000000000000000000000000000000000000000000000000000001b_2\\
&\quad + 6.264551641463457593818886043698212586619541984593273\text{(cont.)}\\
&\quad 2576435655588b_6,\\
b_3 &= 1.309260028633156162644799671413719205654316212923457750123345163b_6,\\
b_5 &= 4.614157131800682898459705227984840546895131617055149564347672012b_6
\end{aligned}
$$

with $a_4,b_6\neq 0$, which implies

$$
\begin{aligned}
\tilde v_{1,1}&=1.0\times 10^{-64}a_4b_2-1.00\times 10^{-63}b_6a_4,\\
\tilde v_{2,1}&=2.0\times 10^{-65}b_2a_4{}^3-3.0\times 10^{-64}b_6a_4{}^3,\\
\tilde v_{3,1}&=2.0\times 10^{-62}a_4{}^5b_2-1.0\times 10^{-61}a_4{}^5b_6,\\
\tilde v_{4,1}&=1.00\times 10^{-61}a_4{}^7b_2-1.00\times 10^{-60}a_4{}^7b_6,\\
\tilde v_{5,1}&=-0.000251781050212589233448605740001622578389034887 76\text{(cont.)}\\
&\qquad 44746869970435709a_4{}^9b_6\neq 0.
\end{aligned}
$$

The errors on $\tilde v_{1,1},\tilde v_{2,1},\tilde v_{3,1},\tilde v_{4,1}$ are due to numerical computation in the step of solving a 12th-degree polynomial. The rank of Jacobian matrix (evaluated at the critical point) of $\tilde{v}_{1,1}, \tilde{v}_{2,1}, \tilde{v}_{3,1}, \tilde{v}_{4,1}$ with respect to a_2, b_1, b_3, b_5 is four, hence by Lemma 1.1, the perturbed system (1.11) can have at most four small-amplitude limit cycles bifurcated from the center, and the bound is sharp. $\square$

Imitating the arguments in the proof of Theorem 3.2, we obtain the following result.

Theorem 3.3. *Based on the analysis of ϵ-order focal values for the cases* $(3)-(7)$*, the perturbed system* (1.11) *can have at most four small-amplitude limit cycles bifurcated from the center respectively, and the bound is sharp.*

Acknowledgements. This work was supported by a grant from the National Natural Science Foundation of China (No. 11461021), a grant from Guangxi Colleges and Universities Key Laboratory of Symbolic Computation and Engineering Data Processing (No. FH201505), a grant from the Foundation for Research in Experimental Techniques of Liaocheng University (No. LDSY2014110), a grant from the Scientic Research Foundation of Guangxi Education Department (No. ZD2014131), and PhD research startup foundation of Liaocheng University.

The authors are very grateful to the referee for the helpful and valuable comments on this manuscript. The authors should also be grateful to Dr. A. Mahdi for providing the citation information of his preprint [23].

References

[1] A. Algaba, F. Fernández-Sánchez, M. Merino, A. J. Rodríguez-Luis; *Centers on center manifolds in the Lorenz, Chen and Lü systems*, Commun. Nonlinear Sci. Numer. Simul., **19** (2014), No. 4, 772–775.

[2] L. Barreira, C. Valls, J. Llibre; *Integrability and limit cycles of the Moon-Rand system*, Int. J. Nonlin. Mech., **69** (2015), 129–136.

[3] N. N. Bautin; *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Math. Sb.(N.S.), **30** (1952), 181–196; Amer. Math. Soc. Transl., **100** (1954), 181–196.

[4] C. A. Buzzi, J. Llibre, J. C. Medrado, J. Torregrosa; *Bifurcation of limit cycles from a centre in R^4 in resonance* $1:N$, Dynam. Syst., **24** (2009), 123–137.

[5] C. Christopher; *Estimating limit cycle bifurcations from centers*, in: D.M. Wang, Z.M. Zheng, *Differential equations with symbolic computation, Trends Math.*, Birkhäuser, Basel, 2005, 23–35.

[6] A. Fidlin; *Nonlinear oscilations in mechanical engineering*, Springer-Verlag, Berlin Heidelberg, 2006.

[7] J. P. Francoise, Y. Yomdin; *Bernstein Inequalities and Applications to Analytic Geometry and Differential Equations*, Journal of Functional Analysis, **146** (1997), 185–205.

[8] J. Giné; *On some open problems in planar differential systems and Hilbert's 16th problem*, Chaos, Solitions and Fractals, **31** (2007), 1118–1134.

[9] J. Giné; *Limit cycle bifurcations from a non-degenerate center*, Appl. Math. Comput., **218** (2012), 4703–4709.

[10] J. Giné; *Higher order limit cycle bifurcations from non-degenerate centers*, Appl. Math. Comput., **218**(2012), 8853–8860.

[11] J. Giné, C. Valls; *Center problem in the center manifold for quadratic differential systems in* $\mathbb{R}^3$, J. Symbolic Comput., **73** (2016), 250–267.

[12] M. Gyllenberg, P. Yan; *Four limit cycles for a three-dimensional competitive Lotka-Volterra system with a heteroclinic cycle*, Comput. Math. Appl., **58** (2009), 649–669.

[13] M. A. Han, Y. Lin, P. Yu; *A study on the existence of limit cycles of a planar system with third-degree polynomials*, Int. J. Bifurcation and Chaos, **14**(2004), 41–60.

[14] M. A. Han, H. Zang, T. H. Zhang; *A new proof to Bautin's theorem*, Chaos, Solitons and Fractals, **31** (2007), 218–223.

[15] M. A. Han, P. Yu; *Ten limit cycles around a center-type singular point in a 3-d quadratic system with quadratic perturbation*, Appl. Math. Lett., **44** (2015), 17–20.
[16] Yu. Il'yashenko, S. Yakovenko; *Lectures on analytic differential equations*, Graduate Studies in Mathematics, Vol. **86**, American Mathematical Society, Providence, RI, 2008.
[17] Y. A. Kuznetsov; *Elements of Applied Bifurcation Theory*, Springer Verlag, New York, 2004.
[18] J. Llibre, J. Yu, X. Zhang; *Limit cycles coming from the perturbation of 2-dimensional centers of vector fields in* $\mathbb{R}^3$, Dynamic Systems and Applications, **17** (2008), 625–636.
[19] J. Llibre, C. Valls; *Hopf bifurcation of a generalized Moon-Rand system*, Commun. Nonlinear Sci. Numer. Simulat., **20** (2015), 1070–1077.
[20] F. C. Moon, R. H. Rand; *Parametric stiffness control of flexible structures*, in: Jet Propulsion Laboratory Publication 85-29, Vol. **II**, California Institute of Technology, 1985, 329–342.
[21] A. Mahdi, V. G. Romanovski, D. S. Shafer; *Stability and periodic oscillations in the Moon-Rand systems*, Nonlinear Anal.-Real, **14** (2013), 294–313.
[22] A. Mahdi; *Center problem for third-order ODEs*, Int. J. Bifurcation and Chaos, **23**(2013), 1350078 (11 pages).
[23] A. Mahdi, C. Pessoa, J. D. Hauenstein; *A hybrid symbolic-numerical approach to the center-focus problem*, arXiv preprint, arXiv:1509.00375(2015), 1–18.
[24] R. J. Mulholland; *Non-linear oscillations of a third-order differential equation*, Int. J. Nonlin. Mech., **6** (1971), 279–294.
[25] H. Poincaré; *Sur les courbes définies par une équation différentielle*, Journal de Mathématiques Pures et Appliquées, IV, **2** (1886), 151–217.
[26] M. I. Rabinovich, D. I. Trubetskov; *Oscillations and waves in linear and nonlinear systems*, in: M. Hazewinkel et al (Eds.), Mathematics and its Applications (Soviet Series), Vol. **50**, Kluwer Academic Publishers, Netherlands, 1989.
[27] R. Roussarie; *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, Progress in Mathematics, Vol. **164**, Birkhäuser Verlag, Basel, 1998.
[28] B. Sang; *Center problem for a class of degenerate quartic systems*, Electron. J. Qual. Theory Differ. Equ., Vol. **2014** (2014), No. 74, pp. 1–17.
[29] B. Sang, Q. L. Wang, W. T. Huang; *Computation of focal values and stability analysis of 4-dimensional systems*, Electron. J. Diff. Equ., Vol. **2015** (2015), No. 209, pp. 1–11.
[30] K. S. Sibirskii; *On the number of limit cycles in the neighborhood of a singular point*, Differ. Equ., **1** (1965), 36–47.
[31] Y. Tian, P. Yu; *Seven limit cycles around a focus point in a simple three-dimensional quadratic vector field*, Int. J. Bifurcation and Chaos, **24** (2014), 1450083 (10 pages).
[32] C. Valls; *Center problem in the center manifold for quadratic and cubic differential systems in* $\mathbb{R}^3$, Appl. Math. Comput., **251** (2015), 180–191.
[33] S. Yakovenko; *A geometric proof of the Bautin theorem. Concerning the Hilbert Sixteenth Problem*, Advances in Mathematical Sciences, Amer. Math. Soc. Transl., **165** (1995), 203–219.
[34] P. Yu; *Computation of normal forms via a perturbation technique*, J. Sound Vib., **211**(1998), 19–38.
[35] P. Yu, M. A. Han; *Eight limit cycles around a center in quadratic hamiltonian system with third-order perturbation*, Int. J. Bifurcation and Chaos, **23** (2013), 1350005 (18 pages).
[36] Q. L. Wang, Y. R. Liu, H. B. Chen; *Hopf bifurcation for a class of three-dimensional nonlinear dynamic systems*, Bull. Sci. Math., **134** (2010), 786–798.
[37] H. Żołądek; *On a certain generalization of Bautin's theorem*, Nonlinearity, **7** (1994), 273–279.

Bo Sang (corresponding author)
School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, China.
Guangxi Education Department Key Laboratory of Symbolic Computation and Engineering Data Processing, Hezhou University, Hezhou 542899, China
E-mail address: sangbo_76@163.com

Brigita Ferčec
Faculty of Energy Technology, University of Maribor, Hočevarjev trg 1, 8270 Krško, Slovenia
E-mail address: brigita.fercec@gmail.com

Qin-Long Wang
School of Science, Hezhou University, Hezhou 542800, China
E-mail address: wqinlong@163.com