Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 111, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

THE POISSON EQUATION ON KLEIN SURFACES

MONICA ROŞIU

Abstract. We obtain a formula for the solution of the Poisson equation with Dirichlet boundary condition on a region of a Klein surface. This formula reveals the symmetric character of the solution.

1. Introduction

In this article we solve a boundary value problem involving the Poisson equation on Klein surfaces. Our technique is based on the fact that according to a classical result due to Klein, the boundary value problems on Klein surfaces can be reduced to similar problems on symmetric Riemann surfaces. On Klein surfaces the formula for the solution is expressed in terms of an analogue of the Green function, which has the symmetry in argument and parameter. The extensive study of the Klein surfaces is due to Schiffer and Spencer [10]. Other useful results about this topic are the formulas for the Green function on the Möbius strip expressed in [4] and the Dirichlet problem for harmonic functions treated in [3].

2. Preliminaries

A compact Klein surface is a pair (X, A), consisting of a compact surface X and a maximal dianalytic atlas A on X, such that A does not contain any analytic subatlas.

It is known, see [10], that given a compact Klein surface (X, A), its orientable double covering O_2 admits a fixed point free symmetry k, such that X is dianalytically equivalent with O_2/H, where H is the group generated by k, with respect to the usual composition of functions. We denote the canonical projection of O_2 onto O_2/H by π. By Klein's definition, the pair (O_2, k) is a k-symmetric compact Riemann surface. Forwards, we identify X with the orbit space O_2/H.

A set G of O_2 is called k-symmetric if $k(G) = G$. Thus, given D a subset of X, then $q^{-1}(D) = G$ is a k-symmetric subset of O_2.

A function f defined on a k-symmetric set is called a k-symmetric function if $f = f \circ k$.

Let $\widetilde{\gamma} : [0, 1] \to D$ be a piecewise smooth Jordan curve of D. Then the arc $\widetilde{\gamma}$ has exactly two liftings at $\pi^{-1}(D)$. If $\widetilde{\gamma}(0) = \widetilde{z}_0 = \{z_0, k(z_0)\}$ and if γ is the lifting of $\widetilde{\gamma}$ at z_0, then $k \circ \gamma$ is the lifting of $\widetilde{\gamma}$ at $k(z_0)$.Then $\pi^{-1}(\widetilde{\gamma}) = \gamma \cup k(\gamma)$ is a k-symmetric

2010 *Mathematics Subject Classification.* 30F50, 35J05, 35J08.
Key words and phrases. Klein surface; Poisson equation; Green's function.

Submitted January 27, 2016. Published April 28, 2016.

curve of O_2. For any real valued function F defined on $\widetilde{\gamma}$, the function $f = F \circ \pi$ is a k-symmetric function on $\gamma \cup k(\gamma)$, see [9].

The Euclidean lengths of the two curves γ and its symmetric, $k \circ \gamma$, that is their lengths with respect to the metric $ds = |dz|$ may be different. We modify this metric and get a new metric $d\sigma$ on O_2, such that the lengths of γ and $k \circ \gamma$, with respect to the metric $d\sigma$, will be the same. We define a $k-$symmetric metric $d\sigma = \frac{1}{2}(ds + ds \circ k)$. Then the $d\sigma$ -lengths of γ and $k \circ \gamma$ are equal. By definition, the length of $\widetilde{\gamma}$ is the common $d\sigma$ -length of γ and $k \circ \gamma$.

Let X be a compact Klein surface and let D be a region bounded by a finite number of σ-rectifiable Jordan curves. Given F a continuous real-valued function on D and H a continuous real-valued function on ∂D, we study the problem defined by Poisson's equation

$$\Delta U = F \quad \text{on } D \tag{2.1}$$

and the Dirichlet boundary condition

$$U = H \quad \text{on } \partial D. \tag{2.2}$$

Because the Klein surface X is dianalytically equivalent to O_2/H, problem (2.1)–(2.2) on a region D of the Klein surface X, can be replaced by an equivalent problem on a k-symmetric region G of its double O_2 as follows.

We define $G = \pi^{-1}(D)$, $f = F \circ \pi$ on G and $h = H \circ \pi$ on ∂G. Then, G is a k-symmetric region bounded by a finite number of σ-rectifiable Jordan curves on O_2. Since $\pi \circ k = \pi$, we obtain $f = f \circ k$ on G and $h = h \circ k$ on the boundary ∂G, thus f and h are k-symmetric, continuous real-valued functions. Problem (2.1)–(2.2) is equivalent to the problem

$$\Delta u = f \quad \text{on } G$$
$$u = h \quad \text{on } \partial G.$$

3. Poisson's equation on the double cover

Let X be a compact Klein surface and let $\widehat{X}$ be the universal covering surface. Then $\widehat{X}$ has a unique (up to conjugation) analytic structure making the canonical projection dianalytic. Let $\mathcal{G}$ be the group of covering transformations and let $\mathcal{G}_1$ be the subgroup of conformal elements of $\mathcal{G}$. Then $\widehat{X}/\mathcal{G}$ is canonically identified with X and $\widehat{X}/\mathcal{G}_1$ is canonically identified with O_2, see [6].

Whence, a model of the bordered Riemann surface O_2 is a region of the complex plane having a finite number of σ-rectifiable Jordan boundary curves. The arc length parameter s is a boundary uniformizer near every boundary point. A half-neighborhood of a boundary point is mapped by s onto a half-neighborhood bounded by a segment of the real s-axis, see [10].

Let $d\sigma = \lambda(z)|dz|$ be the k-symmetric metric on G, where λ is a nonnegative continuous function on $\overline{G}$.

Let G be a k-symmetric region in the complex plane, where ∂G consists of σ-rectifiable Jordan curves Γ (exterior boundary, positively oriented) and $C_1, \dots, C_n$ (interior boundary, negatively oriented). Given a k-symmetric, continuous, real-valued function f on G and a k- symmetric, continuous, real-valued function h on ∂G, then (2.1)–(2.2) can be reduced to the problem consisting of the Poisson equation

$$\Delta u = f \quad \text{on } G \tag{3.1}$$

and the Dirichlet boundary condition

$$u = h \quad \text{on } \partial G. \tag{3.2}$$

To solve (3.1)–(3.2) we combine the solution of the Dirichlet problem for harmonic functions

$$\begin{gathered} \Delta u = 0 \quad \text{on } G \\ u = h \quad \text{on } \partial G \end{gathered}$$

and the solution of the Poisson equation with zero boundary values

$$\begin{gathered} \Delta u = f \quad \text{on } G \\ u = 0 \quad \text{on } \partial G \end{gathered} \tag{3.3}$$

In this paper we only consider solutions which are in the class $C^2(G) \cap C^1(\partial G)$.

Remark 3.1. From the maximum principle, it follows that a solution of (3.1)–(3.2), if it exists, is necessarily uniquely determined.

For the general existence proof of a solution to (3.1)–(3.2) we refer to [5]. Moreover, in this case, the solution has the following property.

Proposition 3.2. *A solution u of* (3.1)–(3.2) *is a k-symmetric function in G.*

Proof. Let u be a solution of (3.1)–(3.2). We define $u_k : \overline{G} \to \mathbb{R}$ by $u_k = \frac{1}{2}(u+u\circ k)$. The hypothesis, $f = f \circ k$, involves $\Delta u_k = \frac{1}{2}(f + f \circ k) = f$ on G and $u_k = 0$ on ∂G. Thus u_k is also a solution of (3.1)–(3.2). Uniqueness of the solution yields $u_k = u$ on G, therefore $u = u \circ k$ on G. □

The Dirichlet problem for harmonic functions on G was solved in [3]. To complete the solution we solve the Poisson equation with zero boundary values for the k-symmetric region G.

In solving (3.3) we use the Green function of a region G. For the existence of a harmonic function which vanishes on the boundary and has a finite number of isolated singularities, with given singular parts, in a relatively compact region of a Riemann surface, we refer to [1].

First we will derive two important formulas. We recall the meaning of the normal derivative with respect to the k-symmetric metric $d\sigma$. Let u be a C^1-function defined on a σ-rectifiable Jordan curve γ, parameterized in terms of the arc σ-length. Therefore, $\gamma : z = z(s) = x(s) + iy(s)$, $s \in [0, l]$, where l is the σ-length of γ.Then the normal derivative of u, denoted by $\frac{\partial u}{\partial n_\sigma}$, is the directional derivative of u in the direction of the unit normal vector n_σ, see [4]. For any point of the curve γ, it follows that

$$\frac{\partial u}{\partial n_\sigma} d\sigma = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

and, in polar coordinates

$$\frac{\partial u}{\partial n_\sigma} d\sigma = -\frac{1}{\rho}\frac{\partial u}{\partial \theta} d\rho + \rho \frac{\partial u}{\partial \rho} d\theta. \tag{3.4}$$

Let G be a k-symmetric region bounded by a finite number of σ -rectifiable Jordan curves. Suppose that p and q are continuous with continuous partial derivatives functions on $\overline{G}$. By Green's theorem

$$\int_{\partial G} p(x,y)dx + q(x,y)dy = \iint_G \Big(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\Big)\, dx\, dy.$$

Given u and v two functions in the class $C^2(G)\cap C^1(\partial G)$, parameterized by $x(s)$ and $y(s)$, where s is the arc σ-length, applying Green's theorem, with $p=-u\frac{\partial v}{\partial y}$, $q=u\frac{\partial v}{\partial x}$ we obtain Green's first identity for the symmetric region G,

$$\iint_G (u\Delta v + (\operatorname{grad} u)\cdot(\operatorname{grad} v))\,dx\,dy = \int_{\partial G} u\frac{\partial v}{\partial n_\sigma}d\sigma. \tag{3.5}$$

Reversing the roles of u and v in Green's first identity and subtracting the new identity from (3.5) we arrive at Green's second identity for the symmetric region G,

$$\iint_G (u\ \Delta v - v\Delta u)\,dx\,dy = \int_{\partial G}\Big(u\frac{\partial v}{\partial n_\sigma} - v\frac{\partial u}{\partial n_\sigma}\Big)d\sigma.$$

Let ζ be a point inside G. The function $\Phi(z,\zeta)=\ln|z-\zeta|$ is harmonic at all points $z\neq\zeta$. Let w be the solution of the Dirichlet boundary-value problem on G, with the boundary condition $w(z)=\Phi(z,\zeta)$ on ∂G. The unique function $g_G(z;\zeta)=-\Phi(z,\zeta)+w(z)$ defined on $\overline{G}\backslash\{\zeta\}$ is called the Green's function for the region G, with respect to the point ζ, see [1].

Let $g_G^{(k)}(z,\widetilde{\zeta})$ be the k-invariant Green's function for the region G, with singularities at ζ and $k(\zeta)$, defined by

$$g_G^{(k)}(z,\widetilde{\zeta}) = \frac{1}{2}[g_G(z,\zeta)+g_G(z,k(\zeta))]$$

on $\overline{G}\backslash\{\zeta,k(\zeta)\}$. For additional information on this topic we refer to [4] and to the original source [10].

Let w_s be the solution of the Dirichlet boundary-value problem on G, with the boundary condition $w_s(z)=\frac{1}{2}[\Phi(z,\zeta)+\Phi(z,k(\zeta)]$ on ∂G. From the definition of the Green function and the definition of $g_G^{(k)}(z,\widetilde{\zeta})$, it follows that

$$g_G^{(k)}(z,\widetilde{\zeta}) = -\frac{1}{2}[\Phi(z,\zeta)+\Phi(z,k(\zeta)]+w_s(z).$$

Therefore, $g_G^{(k)}(z,\widetilde{\zeta})$ is a harmonic function of z in $G\backslash\{\zeta,k(\zeta)\}$, with singularities $-\frac{1}{2}\ln|z-\zeta|$ and $-\frac{1}{2}\ln|z-k(\zeta)|$ at ζ and $k(\zeta)$, respectively. Also, $g_G^{(k)}(z,\widetilde{\zeta})=0$ for all z on ∂G.

Theorem 3.3. *Let G be a k-symmetric region in the complex plane, where ∂G consists of a finite number of σ-rectifiable Jordan curves. Let u be a C^2-function on G, such that $u=0$ on ∂G. Then, for all ζ in G,*

$$u(\zeta)=\frac{1}{2\pi}\iint_G \Delta u(z)g_G(z;\zeta)\,dx\,dy. \tag{3.6}$$

Proof. Let C_ε be a negatively oriented circle of radius ε, centered at ζ and let G_ε be G minus D_ε, the closed disk bounded by C_ε. The boundary of G_ε is the union of ∂G and C_ε. Applying Green's second identity for G_ε, with $v=g_G(\cdot,\zeta)$ and using that $u=0$ on ∂G, g_G is harmonic in G_ε and $g_G=0$ on ∂G, we get

$$-\iint_{G_\varepsilon}\Delta u(z)g_G(z,\zeta)\,dx\,dy = \int_{C_\varepsilon} u(z)\frac{\partial g_G}{\partial n_\sigma}(z,\zeta)d\sigma - \int_{C_\varepsilon} g_G(z,\zeta)\frac{\partial u}{\partial n_\sigma}(z)d\sigma.$$

Next, we let ε tends to zero, taking into account that the outward normal derivative (with respect to the region G_ε) on C_ε is the inner radial derivative pointing towards the pole ζ.

(1) First we prove that

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} g_G(z,\zeta)\frac{\partial u}{\partial n_\sigma}(z)d\sigma = 0.$$

The curve $-C_\varepsilon$ is parameterized by $z = z(\theta) = \zeta + \varepsilon e^{i\theta}$, $0 \le \theta \le 2\pi$ and using (3.4), we obtain

$$-\int_{C_\varepsilon} g_G(z;\zeta)\frac{\partial u}{\partial n_\sigma}(z)d\sigma = \int_{-C_\varepsilon} g_G(z;\zeta)\frac{\partial u}{\partial n_\sigma}(z)d\sigma = \varepsilon\int_0^{2\pi} g_G(z(\theta);\zeta)\frac{\partial u}{\partial \rho}(z(\theta))d\theta.$$

By definition, $g_G(z;\zeta) = -\ln|z-\zeta| + w(z)$, where w is harmonic in G. Then w is continuous in D_ε and thus w is bounded on C_ε. As the function u has continuous partial derivatives in G, $\frac{\partial u}{\partial \rho}$ is continuous on D_ε, hence is bounded on C_ε. Therefore there is a constant m, such that $|w| \le m$ and $|\frac{\partial u}{\partial \rho}| \le m$ on C_ε. Since on C_ε, $\ln|z-\zeta| = \ln\varepsilon$, we arrive at $|g_G(z,\zeta)| \le m + |\ln\varepsilon|$, for $z \in C_\varepsilon$. Therefore

$$\Big|\int_{C_\varepsilon} g_G(z,\zeta)\frac{\partial u}{\partial n_\sigma}(z)d\sigma\Big| \le 2\pi\varepsilon(m + |\ln\varepsilon|)m$$

and the right side of the last inequality tends to zero as ε tends to zero. Therefore,

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} g_G(z,\zeta)\frac{\partial u}{\partial n_\sigma}(z)d\sigma = 0.$$

(2) We prove that

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} u(z)\frac{\partial g_G}{\partial n_\sigma}(z,\zeta)d\sigma = -2\pi u(\zeta).$$

The definition of the Green function yields

$$\int_{C_\varepsilon} u(z)\frac{\partial g_G}{\partial n_\sigma}(z,\zeta)d\sigma = -\int_{C_\varepsilon} u(z)\frac{\partial \Phi}{\partial n_\sigma}(z)d\sigma + \int_{C_\varepsilon} u(z)\frac{\partial w}{\partial n_\sigma}(z)d\sigma.$$

Using (3.4) and the mean value property, we have

$$-\int_{C_\varepsilon} u(z)\frac{\partial \Phi}{\partial n_\sigma}(z)d\sigma = -\int_0^{2\pi} u(z(\theta))\frac{1}{\varepsilon}\varepsilon d\theta = -2\pi u(\zeta).$$

Using again (3.4), we get

$$\int_{C_\varepsilon} u(z)\frac{\partial w}{\partial n_\sigma}(z)d\sigma = \varepsilon\int_0^{2\pi} u(z(\theta))\frac{\partial w}{\partial \rho}(z(\theta))d\theta.$$

As the function w is harmonic, w has continuous partial derivatives in G, then $\frac{\partial w}{\partial \rho}$ is continuous on D_ε, hence is bounded on C_ε. The function u is also bounded on C_ε. Therefore there is a constant M, such that $|u| \le M$ and $|\frac{\partial w}{\partial \rho}| \le M$ on C_ε. So

$$\Big|\int_{C_\varepsilon} u(z)\frac{\partial w}{\partial n_\sigma}(z)d\sigma\Big| \le 2\pi\varepsilon M^2,$$

which tends to zero as ε tends to zero. Thus

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} u(z)\frac{\partial w}{\partial n_\sigma}(z)d\sigma = 0,$$

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} u(z)\frac{\partial g_G}{\partial n_\sigma}(z,\zeta)d\sigma = -2\pi u(\zeta).$$

(3) Since u is a C^2-function on G, Δu is continuous on D_ε, then Δu is bounded on D_ε. Also, as we argued above, w is bounded on D_ε.Thus there is a constant M_0, such that $|\Delta u| \le M_0$ and $|w| \le M_0$ on D_ε. Then

$$\begin{aligned}
&|\iint_G \Delta u(z) g_G(z,\zeta)\,dx\,dy - \iint_{G_\varepsilon} \Delta u(z) g_G(z,\zeta)\,dx\,dy| \\
&= |\iint_{D_\varepsilon} \Delta u(z) g_G(z,\zeta)\,dx\,dy| \\
&\le \iint_{D_\varepsilon} |\Delta u(z)\ln|z-\zeta||\,dx\,dy + \iint_{D_\varepsilon} |\Delta u(z) w(z)|\,dx\,dy \\
&\le M_0 \iint_{D_\varepsilon} |\ln|z-\zeta||\,dx\,dy + M_0^2 \iint_{D_\varepsilon} dx\,dy \\
&= M_0^2\pi\varepsilon^2 + M_0\int_0^{2\pi}\int_0^{\varepsilon} \rho|\ln\rho| d\rho d\theta
\end{aligned}$$

which tends to zero as ε tends to zero. Thus

$$\lim_{\varepsilon\to 0}\iint_{G_\varepsilon} \Delta u(z) g_G(z,\zeta)\,dx\,dy = \iint_G \Delta u(z) g_G(z,\zeta)\,dx\,dy.$$

By (1), (2) and (3) it follows (3.6). □

The next theorem yields the formula for the solution of the Poisson problem with zero boundary values on a k-symmetric region G.

Theorem 3.4. *Let G be a k-symmetric region bounded by a finite number of σ-rectifiable Jordan curves. Let f be a k-symmetric, continuous function on G. There is a unique k-symmetric, C^2-function u on G, such that $\Delta u = f$ on G and $u = 0$ on ∂G. For all ζ in G,*

$$u(\zeta) = \frac{1}{4\pi}\iint_G f(z)[g_G(z,\zeta) + g_G(z;k(\zeta))]\,dx\,dy. \tag{3.7}$$

Proof. Since k is an involution of G, the function $\frac{u(\zeta)+u(k(\zeta))}{2}$ is a k-symmetric function on G. By Theorem 3.3,

$$\begin{gathered}
u(\zeta) = \frac{1}{2\pi}\iint_G f(z) g_G(z,\zeta)\,dx\,dy, \\
u(k(\zeta)) = \frac{1}{2\pi}\iint_G f(k(z)) g_G(z;k(\zeta))\,dx\,dy.
\end{gathered}$$

The k-symmetry of f implies

$$\frac{u(\zeta)+u(k(\zeta))}{2} = \frac{1}{2\pi}\iint_G f(z)\frac{g_G(z,\zeta)+g_G(z;k(\zeta))}{2}\,dx\,dy.$$

By Proposition 3.2, u is a k-symmetric function on G, then the left side of the last equality is $u(\zeta)$ and we conclude that

$$u(\zeta) = \frac{1}{4\pi}\iint_G f(z)[g_G(z,\zeta) + g_G(z;k(\zeta))]\,dx\,dy.$$

By Remark 3.1, it follows (3.7). □

4. Poisson's equation on the orbit space

Let X be compact Klein surface and let D be a region bounded by a finite number of σ-rectifiable Jordan curves. The Klein surface X is the factor manifold of the k-symmetric Riemann surface O_2 with respect to the group H. Then, D is obtained from the k-symmetric region G by identifying the k-symmetric points. Therefore, the k-symmetric Green's function $g_G^{(k)}(z,\widetilde{\zeta})$, where $\widetilde{\zeta}=\pi(\zeta)$, is continuous on $\overline{D}$, harmonic on $\overline{D}\backslash\{\widetilde{\zeta}\}$, $g_G^{(k)}(z,\widetilde{\zeta})=0$ for all z on ∂D and has the singularity $-\frac{1}{2}\ln|z-\widetilde{\zeta}|$ at $\widetilde{\zeta}$. Then, $g_G^{(k)}(z,\widetilde{\zeta})$ is the Green function of D with singularity at $\widetilde{\zeta}=\pi(\zeta)$.

We obtain the formula for the solution of (3.1)–(3.2) on a k-symmetric region G.

Theorem 4.1. *Let G be a k-symmetric region bounded by a finite number of σ-rectifiable Jordan curves. Let f be a k-symmetric, continuous function on G and h be a k-symmetric, continuous function on ∂G. There is a unique function u on $\overline{G}$, such that $\Delta u=f$ on G and $u=h$ on ∂G. For all ζ in G,*

$$u(\zeta)=\frac{1}{2\pi}\iint_G f(z)g_G^{(k)}(z,\widetilde{\zeta})\,dx\,dy+\frac{1}{2\pi}\int_{\partial G}h(z)\frac{\partial g_G^{(k)}(z,\widetilde{\zeta})}{\partial n_\sigma}d\sigma. \tag{4.1}$$

Proof. By definition,

$$\frac{1}{2}\big[g_G(z,\zeta)+g_G(z,k(\zeta))\big]=g_G^{(k)}(z,\widetilde{\zeta})$$

is the k-invariant Green function for the region G, with singularities $-\frac{1}{2}\ln|z-\zeta|$ and $-\frac{1}{2}\ln|z-k(\zeta)|$ at ζ and $k(\zeta)$, respectively. We combine the solution of the Dirichlet problem for harmonic functions given by

$$u(\zeta)=\frac{1}{2\pi}\int_{\partial G}h(z)\frac{\partial g_G^{(k)}(z,\widetilde{\zeta})}{\partial n_\sigma}d\sigma$$

for $\zeta\in G$, see [4], with the solution of the Poisson's equation which is zero on the boundary given by Theorem 3.4. ☐

Next we derive the solution of (2.1)–(2.2) on the region D.

Proposition 4.2. *Let F be the continuous real-valued function on D, defined by the relation $f=F\circ\pi$ and let H be the continuous real-valued function on ∂D,defined by the relation $h=H\circ\pi$. The solution of (2.1)–(2.2) is the function U defined on $\overline{D}$, by the relation $u=U\circ\pi$, where π is the canonical projection of O_2 on X and u is the solution (4.1) of (3.1)–(3.2) on the k-symmetric region G of O_2.*

Proof. The k-symmetry of the function f on G, yields

$$\Delta U(\widetilde{\zeta})=\Delta u(\zeta)=f(\zeta)=f(k(\zeta))=F(\widetilde{\zeta}),$$

for all $\widetilde{\zeta}\in D$, where $\widetilde{\zeta}=\pi(\zeta)$. Also, the k-symmetry of the function h on ∂G, yields

$$U(\widetilde{\zeta})=u(\zeta)=h(\zeta)=h(k(\zeta))=H(\widetilde{\zeta}),$$

for all $\widetilde{\zeta}\in\partial D$.Due to the uniqueness, the function U defined on $\overline{D}$ by

$$U(\widetilde{\zeta})=u(\zeta),$$

for all $\widetilde{\zeta}$ in $\overline{D}$, where $\widetilde{\zeta}=\pi(\zeta)$, is the solution of (2.1)–(2.2). ☐

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Monica Roşiu
Department of Mathematics, University of Craiova, Street A.I. Cuza No 13, Craiova 200585, Romania
E-mail address: monica_rosiu@yahoo.com