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PERRON'S METHOD FOR p-HARMONIOUS FUNCTIONS

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Abstract. We show that Perron's method produces continuous p-harmonious functions for $1 < p < 2$. Such functions approximate p-harmonic functions and satisfy a functional equation involving a convex combination of the mean and median, generalizing the classical mean-value property of harmonic functions. Simple sufficient conditions for the existence of barriers are given. The $p = 1$ situation, in which solutions to the Dirichlet problem may not be unique, is also considered. Finally, the relationship between 1-harmonious functions and functions satisfying a local median value property is discussed.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $h > 0$, $q \in (0,1)$, and $g \in C(\partial\Omega)$ be given. Consider the Dirichlet problem

$$\begin{aligned} u &= M_q^h u \quad \text{in } \Omega\,, \\ u &= g \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1.1}$$

where the statistical operator $M_q^h \colon C(\Omega) \to C(\Omega)$ is defined by

$$\left(M_q^h \varphi\right)(x) := (1-q)\,\mathrm{median}_{\partial B_x^h}\{\varphi\} + q \fint_{\partial B_x^h} \varphi(y)\,dy\,, \quad \text{for } x \in \Omega, \tag{1.2}$$

and the open balls $B_x^h \subset \Omega$ are defined by

$$B_x^h := B(x, r^h(x)), \quad \text{with } r^h(x) := \begin{cases} \sqrt{2h} & \text{if } \operatorname{dist}(x,\partial\Omega) \geq \sqrt{2h}\,, \\ \operatorname{dist}(x,\partial\Omega) & \text{otherwise}\,. \end{cases} \tag{1.3}$$

We seek a solution $u \in C(\overline{\Omega})$. Because of their connection to p-harmonic functions, functions satisfying $u = M_q^h$ are called p-harmonious. In [10] we showed that $C(\overline{\Omega})$ solutions to (1.1) are unique, and that a solution exists when there exist both a subsolution $v \in C(\overline{\Omega})$ with $v = g$ on $\partial\Omega$ and a supersolution $w \in C(\overline{\Omega})$ with $w = g$ on $\partial\Omega$. When Ω is strictly convex, such a sub/supersolution pair can be found for any $g \in C(\partial\Omega)$ by solving Dirichlet problems for a Monge-Ampère equation. The main difficulty in applying this result when Ω is not strictly convex is finding a subsolution and supersolution. In this article, we show that a $C(\overline{\Omega})$ solution can be produced using Perron's method under less restrictive hypotheses.

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Note that if $q = 1$ in (1.1), a harmonic function $u \in C(\overline{\Omega})$ with boundary values g is a solution. Thus, the equation $u = M_q^h u$ is a generalization of the well-known mean-value property of harmonic functions. However, note that if $u = M_q^h u$, $u(x)$ need only equal its nonlinear average over a single sphere centered at x, rather than all such spheres contained in the domain.

We call solutions of (1.1) *p-harmonious* because, as $h \downarrow 0$, the solutions u_h converge locally uniformly to the unique p-harmonic function u in Ω such that $u = g$ on $\partial\Omega$, with $p \in (1, 2)$ determined by q and N; the convergence proofs in [10] and [28] rely on averaging over spheres with the particular function $r^h(\cdot)$ defined above. The term *p-harmonious* first appeared in [24], where the authors employed a convex combination of the mean and the midrange of φ over a ball B (defined by $(1/2)(\sup_B \varphi + \inf_B \varphi)$)) instead of our convex combination of the median and mean as in (1.2), thereby resulting in a different statistical functional equation. Solutions of these equations approximate p-harmonic functions when $p > 2$ (see also [10]). These equations are closely related to tug-of-war games, and have also been studied in [21].

The game-theoretic approach to the p-Laplacian ($1 \le p \le \infty$), pioneered by Kohn and Serfaty [15], Peres and Sheffield [26], and Peres, Schramm, Sheffield and Wilson [25], has led to much recent activity and new insights about nonlinear elliptic and parabolic equations, with potentially far-reaching consequences. See, for example, [1, 2, 3, 20, 23, 8, 19, 17, 18, 6, 4, 5, 27].

Recall that u is p-harmonic in Ω if and only if it is a solution of the p-Laplace equation [11, 16],

$$-\Delta_p u = 0 \quad \text{in } \Omega\,; \tag{1.4}$$

for $p \in (1, \infty)$, the p-Laplacian $-\Delta_p$ is the operator defined formally by

$$-\Delta_p \varphi := -\operatorname{div}\left(|D\varphi|^{p-2} D\varphi\right)\,. \tag{1.5}$$

Since the p-Laplacian is singular when $p < 2$ and degenerate when $p > 2$, solutions of (1.4) must be defined in a weak sense; one can choose either the viscosity or the variational definition, as Juutinen et al [12, 13] showed that these definitions are equivalent.

For a smooth function u with nonvanishing gradient, we define the 1-Laplacian Δ_1 by $\Delta_1 u = |Du| \operatorname{div}(Du/|Du|)$. Note that this definition of $\Delta_1 u$ differs by a factor of $|Du|$ from what is obtained in (1.5) with $p = 1$. With this definition, a calculation shows that

$$\Delta_p u = |Du|^{p-2}\left(\,(p-1)\Delta u + (2-p)\Delta_1 u\,\right) \tag{1.6}$$

for any smooth u with nonvanishing gradient. A normalized 1-homogeneous p-Laplacian defined by $\Delta_p^N u = |Du|^{2-p}\Delta_p u$ is thus a linear combination of Δu and $\Delta_1 u$. This combination is convex when $1 \le p \le 2$. Note that if Du is never zero, $\Delta_p u = 0$ if and only if $\Delta_p^N u = 0$. Furthermore, this equivalence holds in the weak sense as well as the viscosity sense by the results of [13]. The connection between p-harmonic functions and the statistical functional equations defining p-harmonious functions arises from the relationship between $\Delta u(x)$ and $⨍_{\partial B_h^x} u$ and that between $\Delta_1 u(x)$ and $\operatorname{median}_{\partial B_x^h}\{u\}$, both of which follow from Taylor's theorem. More precisely, as shown in [9], a function is p-harmonic in two dimensions if and only if it satisfies an asymptotic mean/median value property; combining the calculations in [9] with those in [28] shows that this result holds in all dimensions $N \ge 2$.

This asymptotic description of p-harmonic functions was motivated by other asymptotic statistical characterizations of p-harmonic functions [22], which rely on a decomposition of the p-Laplacian that is different than (1.6). For a smooth function with nonzero gradient, we define the ∞-Laplacian Δ_∞ by

$$\Delta_\infty u = |Du|^{-2}\Sigma_{i,j} u_i u_j u_{ij}.$$

Then the equation

$$\Delta_p u = |Du|^{p-2}(\Delta u + (p-2)\Delta_\infty u) \tag{1.7}$$

holds for any smooth u with nonvanishing gradient. Again dividing by $|Du|^{p-2}$ results in a normalized p-Laplacian, equal to the one given above. Exploiting the connection between $\Delta_\infty u(x)$ and the midrange of u over a ball centered at x, Manfredi, Parviainen and Rossi obtained that u is p-harmonic if and only if it is equal, in an asymptotic sense, to a linear combination of its mean and midrange. This combination is convex when $p > 2$. Thus, the median and midrange play similar roles in the $1 < p < 2$ and $p > 2$ realms respectively. We remark that the normalized p-Laplacian can also be decomposed as $\Delta_p^N u = \Delta_1 u + (p-1)\Delta_\infty u$. This identity was used to obtain characterizations of p-harmonic functions for $1 \le p \le \infty$ in terms of lower dimensional averages in [14].

If q is taken to be zero in (1.1), the situation is different. In [29], it was shown that functions satisfying a local median value property are 1-harmonic in the viscosity sense (this is using the definition of 1-harmonic that has the gradient factor in front: $\Delta_1 u = |Du|\operatorname{div}(Du/|Du|)$). It was also demonstrated that solutions to the Dirichlet problem for the local median value property may not exist, and when they do exist, may fail to be unique. Elementary geometric reasoning shows that the 1-parameter family of functions u_α in [29] that satisfy the local median value property are also 1-harmonious, at least for h sufficiently small. These functions all agree on the boundary of their domain, so 1-harmonious functions are not uniquely determined by their boundary values. This implies that a comparison principle for the classes S_g and S^g as used below in Section 3 cannot hold when $q = 0$ (since it would immediately imply uniqueness). However, in this case, the arguments used in Section 3 do lead to the existence of weak solutions that bound any $C(\overline{\Omega})$ solutions (Theorem 4.2).

The rest of the paper is organized as follows. In Section 2, we recall the definition and some fundamental properties of the median of a function and establish properties of additional mean/median operators needed for the proofs of the main results, Theorems 3.1 and 4.2. In the following section, the definitions of sub- and supersolutions are expanded from those used in [10], the upper and lower Perron solutions are defined, and Theorem 3.1, stating that when $q > 0$ and the upper and lower Perron solutions, u^P and u_P respectively, are equal on $\partial\Omega$ they are equal throughout $\overline{\Omega}$ and solve (1.1), is proved. As in the classical Perron method, the equality of u^P and u_P follows from the existence of barriers. However, it is not obvious how to construct barriers for the problem considered here, due to the non-local nature of the operator M_q^h. This issue is also discussed in Section 3. Section 4 concerns the situation when $q = 0$. The relationship between 1-harmonious functions and those satisfying the local median value property is explored in Section 5, where an example of a function satisfying the seemingly stronger requirements of the local median value property but is not 1-harmonious for any $h > 0$ is given.

2. Preliminaries

Throughout this article $\Omega \subset \mathbb{R}^N$ is a bounded domain. We begin by reviewing some properties of medians. Recall the following definition [10, 31].

Definition 2.1. If u is a real-valued integrable function on the measurable set $E \subset \mathbb{R}^k$ and $0 < |E| < \infty$, then m is a *median* of u over E if and only if

$$|\{x \in E : u(x) < m\}| \le \frac{1}{2}|E| \quad \text{and} \quad |\{x \in E : u(x) > m\}| \le \frac{1}{2}|E|\,.$$

The set $\operatorname{median}_E\{u\}$ of all medians of u over E is a non-empty compact interval,

$$\operatorname{median}_E\{\alpha u + \beta\} = \alpha \operatorname{median}_E\{u\} + \beta\,, \quad \text{for } \alpha, \beta \in \mathbb{R}\,, \tag{2.1}$$

and it is easy to construct examples of semicontinuous functions with multiple medians over a given set. This does not happen when working with continuous functions, since continuous functions have unique medians on compact connected sets [10].

To deal with non-continuous functions, which do not necessarily have unique medians, we introduce two new mean/median operators: $\underline{M}_{q,h}$ and $\overline{M}_{q,h}$. For an integrable function $u : \overline{\Omega} \to \mathbb{R}$, $q \in [0,1]$ and $h > 0$, let

$$\begin{aligned} \underline{M}_{q,h}u(x) &= q \operatorname{mean}_{\partial B_x^h} u + (1-q) \min \operatorname{median}_{\partial B_x^h} u, \\ \overline{M}_{q,h}u(x) &= q \operatorname{mean}_{\partial B_x^h} u + (1-q) \max \operatorname{median}_{\partial B_x^h} u \end{aligned} \tag{2.2}$$

for $x \in \Omega$, and for $x \in \partial\Omega$, let $\underline{M}_{q,h}u(x) = \overline{M}_{q,h}u(x) = u(x)$. These operators satisfy a natural monotonicity property.

Proposition 2.2. *Let u and v be real-valued integrable functions on Ω with $u \le v$. Then for any $q \in [0,1]$, $\underline{M}_{q,h}u(x) \le \underline{M}_{q,h}v(x)$ and $\overline{M}_{q,h}u(x) \le \overline{M}_{q,h}v(x)$ for all $x \in \Omega$.*

Proof. We prove the claim for $\underline{M}_{q,h}$; the one for $\overline{M}_{q,h}$ is similar. Since $u \le v$, $\operatorname{mean}_{\partial B_x^h} u \le \operatorname{mean}_{\partial B_x^h} v$ for any x and h, so the claim holds if $\min \operatorname{median}_{\partial B_x^h} u \le \min \operatorname{median}_{\partial B_x^h} v$. If this is not true, then there exist $x \in \Omega$ and a number m that is a median of v that is too small to be a median of u on ∂B_x^h. Because m is a median of v,

$$|\{y \in \partial B_x^h : v(y) \le m\}| \ge (1/2)|\partial B_x^h|.$$

Since $u \le v$, $\{y \in \partial B_x^h : v(y) \le m\} \subset \{y \in \partial B_x^h : u(y) \le m\}$, so

$$|\{y \in \partial B_x^h : u(y) \le m\}| \ge (1/2)|\partial B_x^h|.$$

On the other hand, since m is too small to be a median of u in ∂B_x^h, we must have that

$$|\{y \in \partial B_x^h : u(y) \le m\}| < (1/2)|\partial B_x^h|.$$

This is a clear contradiction and the claim holds. $\square$

The next result shows that these operators preserve semicontinuity.

Proposition 2.3. *If $u \in LSC(\bar{\Omega}) \cap L^\infty(\Omega)$, then $\underline{M}_{q,h}u \in LSC(\bar{\Omega})$, and if $v \in USC(\bar{\Omega}) \cap L^\infty(\Omega)$, then $\overline{M}_{q,h}v \in USC(\bar{\Omega})$.*

Proof. Let $x \in \Omega$, and suppose $\{x_n\} \subset \Omega$ is such that $x_n \to x$. Define the map $\rho_n : \partial B_x^h \to \partial B_{x_n}^h$ by $\rho_n(y) = x_n + r^h(x_n)/r^h(x)(y-x)$. Define $u_n(y) = u(\rho_n(y))$ for $y \in \partial B_x^h$. Since $\rho_n(y) \to y$ as $n \to \infty$ and u is lower semicontinuous at y, we have that $u(y) \le \liminf_{n\to\infty} u_n(y)$ for all $y \in \partial B_x^h$. Integrating this last inequality and using Fatou's Lemma (note that because u is bounded, we may assume that u (and therefore u_n) is non-negative), we obtain

$$\int_{\partial B_x^h} u(y)\,dy \le \int_{\partial B_x^h} \liminf_{n\to\infty} u_n(y)\,dy \le \liminf_{n\to\infty} \int_{\partial B_x^h} u_n(y)\,dy. \tag{2.3}$$

This implies that the function $z \to \operatorname{mean}_{\partial B_z^h} u$ is lower semicontinuous at x, and hence in Ω.

We now turn to the minimal median. We begin by establishing that for any $\alpha \in \mathbb{R}$,

$$|\partial B_x^h \cap \{u > \alpha\}| \le \liminf_{n\to\infty} |\partial B_x^h \cap \{u_n > \alpha\}|. \tag{2.4}$$

Let $\alpha \in \mathbb{R}$, and let $f_\alpha = \chi_{\partial B_x^h \cap \{u>\alpha\}}$ and $f_\alpha^n = \chi_{\partial B_x^h \cap \{u_n > \alpha\}}$. We have that $f_\alpha(y) \le \liminf_{n\to\infty} f_\alpha^n(y)$ for all $y \in \partial B_x^h$, so (2.4) follows again by Fatou.

Now let $\alpha = \min \operatorname{median}_{\partial B_x^h} u$. Then for any $\varepsilon > 0$, $\alpha - \varepsilon$ is too small to be a median of u on ∂B_x^h, so $|\partial B_x^h \cap \{u > \alpha - \varepsilon\}| > (1/2)|\partial B_x^h|$ and by (2.4),

$$\liminf_{n\to\infty} |\partial B_x^h \cap \{u_n > \alpha - \varepsilon\}| > (1/2)|\partial B_x^h|.$$

So for all but finitely many n, $|\partial B_x^h \cap \{u_n > \alpha - \varepsilon\}| > (1/2)|\partial B_x^h|$. For each such n, $\alpha - \varepsilon$ is too small to be a median, so for these n, $\min \operatorname{median}_{\partial B_x^h} u_n \ge \alpha - \varepsilon$. Since this holds for all but finitely many n, we obtain

$$\liminf_{n\to\infty} \min \operatorname{median}_{\partial B_x^h} u_n \ge \alpha - \varepsilon.$$

Since ε is arbitrary, we conclude

$$\liminf_{n\to\infty} \min \operatorname{median}_{\partial B_x^h} u_n = \liminf_{n\to\infty} \min \operatorname{median}_{\partial B_{x_n}^h} u \ge \alpha.$$

Recalling the definition of α, we see that the minimal median operator is lower semicontinuous at x, and therefore in Ω.

To show $\underline{M}_{q,h} u \in LSC(\overline{\Omega})$, we show that for any $\alpha \in \mathbb{R}$, $\{z \in \overline{\Omega} : \underline{M}_{q,h} u(z) > \alpha\}$ is open relative to $\overline{\Omega}$. We already have that $\underline{M}_{q,h} u$ is LSC in Ω, so if $x \in \Omega$ is such that $\underline{M}_{q,h} u(x) > \alpha$, the same is true in a neighborhood of x in Ω. Now suppose $x \in \partial\Omega$ is such that $u(x) = \underline{M}_{q,h} u(x) > \alpha$. Because $u \in LSC(\overline{\Omega})$, $u > \alpha$ in some neighborhood $\mathcal{N}$ of x in $\overline{\Omega}$. Then for all $y \in \Omega$ sufficiently close to x, $\partial B_y^h \subset \mathcal{N}$, so $\underline{M}_{q,h} u(y) > \alpha$. If $y \in \mathcal{N} \cap \partial\Omega$, $u(y) = \underline{M}_{q,h} u(y) > \alpha$, so we see that x is an interior point of $\{z \in \overline{\Omega} : \underline{M}_{q,h} u(z) > \alpha\}$, and the proof is complete. $\square$

We also need the following result concerning the continuity of the minimal and maximal mean and median operators with respect to monotone sequences of functions.

Proposition 2.4. *Suppose that $\{v_n\} \subset LSC(\overline{\Omega})$ is a nondecreasing sequence of bounded functions converging pointwise to the bounded function $v \in LSC(\overline{\Omega})$. Then for all $x \in \Omega$, $\underline{M}_q^h v_n(x) \to \underline{M}_q^h v(x)$. Suppose that $\{w_n\} \subset USC(\overline{\Omega})$ is a nonincreasing sequence of bounded functions converging pointwise to the bounded function $w \in USC(\overline{\Omega})$. Then for all $x \in \Omega$, $\overline{M}_q^h w_n(x) \to \overline{M}_q^h w(x)$.*

Proof. Again, we prove the claim for $\underline{M}_q^h$. By Proposition 2.2, for each $x \in \Omega$, the sequence $\{\underline{M}_q^h v_n(x)\}$ is nondecreasing and bounded above by $\underline{M}_q^h v(x)$. Therefore $\lim_{n\to\infty} \underline{M}_q^h v_n(x)$ exists and is less than or equal to $\underline{M}_q^h v(x)$. Fix $x \in \Omega$ and let $m = \min \operatorname{median}_{\partial B_x^h} v$. The means of v_n over ∂B_x^h converge to the mean of v on ∂B_x^h by the dominated convergence theorem (see the proof of [10, Theorem 3.2]), so the claim holds if $\min \operatorname{median}_{\partial B_x^h} v_n \to \min \operatorname{median}_{\partial B_x^h} v$. Suppose that this is not the case. Then there exists $\varepsilon > 0$ such that $m - \varepsilon > m_* := \lim_{n\to\infty} \min \operatorname{median}_{\partial B_x^h} v_n$. The number $m - \varepsilon$ is too small to be a median of v on ∂B_x^h, so

$$|\partial B_x^h \cap \{v \le m - \varepsilon\}| < (1/2)|\partial B_x^h|.$$

However, $m - \varepsilon$ is not too small to be a median of v_n on ∂B_x^h so

$$|\partial B_x^h \cap \{v_n \le m - \varepsilon\}| \ge (1/2)|\partial B_x^h|.$$

Note that this is true for all n, since $\min \operatorname{median}_{\partial B_x^h} v_n \le m_*$ for all n by the monotonicity of $\{v_n\}$. Therefore,

$$\lim_{n\to\infty} |\partial B_x^h \cap \{v_n \le m - \varepsilon\}| \ge (1/2)|\partial B_x^h|.$$

But, again using the monotonicity of the sequence and continuity from above, we have

$$\begin{aligned}\lim_{n\to\infty} |\partial B_x^h \cap \{v_n \le m - \varepsilon\}| &= |\cap_{n=1}^{\infty} (\partial B_x^h \cap \{v_n \le m - \varepsilon\})| \\ &= |\partial B_x^h \cap \{v \le m - \varepsilon\}| < (1/2)|\partial B_x^h|,\end{aligned}$$

contradicting the previous line. □

3. Perron method and barriers

In this section we assume that $0 < q \le 1$. We begin by defining *subsolutions* and *supersolutions* for the problem (1.1) and the Perron solutions. Let

$$S_g = \{v \in C(\overline{\Omega}) : v \le M_q^h v \text{ and } v \le g \text{ on } \partial\Omega\},$$
$$S^g = \{w \in C(\overline{\Omega}) : w \ge M_q^h w \text{ and } w \ge g \text{ on } \partial\Omega\}.$$

The *lower Perron solution* is then

$$u_P(x) = \sup\{v(x) : v \in S_g\}$$

and the *upper Perron solution* is

$$u^P(x) = \inf\{w(x) : w \in S^g\}.$$

The Perron solutions are well-defined and finite at all $x \in \Omega$. First note that the constant functions $\min_{\partial\Omega} g$ and $\max_{\partial\Omega} g$ belong to S_g and S^g respectively so these sets of functions are nonempty. By a simple modification of the proof of the comparison principle (Theorem 3.1) in [10], we have that $v \le w$ for all $v \in S_g$ and $w \in S^g$. It follows that $u_P \le u^P$ in $\overline{\Omega}$ and that both u_P and u^P are bounded (by $\min_{\partial\Omega} g$ and $\max_{\partial\Omega} g$). By construction we have that $u_P \in LSC(\overline{\Omega})$ and $u^P \in USC(\overline{\Omega})$.

Observe that if (1.1) has a solution $u \in C(\overline{\Omega})$, then $u \in S_g \cap S^g$, and that any function in $S_g \cap S^g$ is a $C(\overline{\Omega})$ solution. Also note that if (1.1) has a solution, then that solution must coincide with u_P and u^P. To see this, suppose $u \in C(\overline{\Omega})$ is a solution of (1.1). Then $u \in S_g$, so that $u \le u_P$. Similarly, $u \in S^g$, so $u \ge u^P$, but $u_P \le u^P$, so we see $u = u_P = u^P$.

The following theorem is the main result of this article.

Theorem 3.1. *If $u_P = u^P$ $(= g)$ on $\partial\Omega$, then $u_P = u^P$ in $\overline{\Omega}$ and the Perron solutions solve* (1.1).

Proof. By assumption, $u_P = g$ on $\partial\Omega$. By Proposition 2.2 and the definition of u_P, $v \le M_q^h v = \underline{M}_q^h v \le \underline{M}_q^h u_P$ for any $v \in S_g$. Taking the sup over all $v \in S_g$, we get $u_P \le \underline{M}_q^h u_P$. We now define a sequence of $LSC(\overline{\Omega})$ functions by taking

$$u_1 = u_P \quad \text{and} \quad u_{j+1} = \underline{M}_q^h u_j, \quad j \ge 1.$$

Each u_j belongs to $LSC(\overline{\Omega})$ by Proposition 2.3, $u_j = g$ on $\partial\Omega$ by construction, and the sequence $\{u_j\}$ is nondecreasing by Proposition 2.2. Furthermore, since $u_1 \le \max_{\partial\Omega} g$, we get that $u_j \le \max_{\partial\Omega} g$ for all j. Therefore u_j converges pointwise to some bounded function $u \in LSC(\overline{\Omega})$ with $u = g$ on $\partial\Omega$. By Proposition 2.4, we have that $\underline{M}_q^h u_j \to \underline{M}_q^h u$ pointwise, so that

$$u(x) = \lim_{j\to\infty} u_{j+1}(x) = \lim_{j\to\infty} \underline{M}_q^h u_j(x) = \underline{M}_q^h u(x)$$

for all $x \in \Omega$. Thus $u(x) = \underline{M}_q^h u(x)$ for all $x \in \Omega$. Similarly, define a sequence $U_j \in USC(\overline{\Omega})$ by

$$U_1 = u^P \quad \text{and} \quad U_{j+1} = \overline{M}_q^h U_j, \quad j \ge 1.$$

Letting $U(x) = \lim_{j\to\infty} U_j(x)$, we get that $U \in USC(\overline{\Omega})$, $U = \overline{M}_q^h U$ in Ω and $U = g$ on $\partial\Omega$. Note that by construction, $u_j \le U_j$ for all j, so that $u \le U$. We are now in exactly the same situation as in the proof of [10, Theorem 3.2], following equation (3.18) in that paper. Running precisely the same argument as in that proof, we obtain $u = U$ in $\overline{\Omega}$, which implies that they belong to $C(\overline{\Omega})$, and therefore that $u = M_q^h u = M_q^h U = U$ in Ω, so that u solves (1.1). Finally, this implies that $u \in S_g$, and since $u_P \le u$, we obtain that $u = u_P$ so that u_P is in fact a solution of (1.1). We note also that $u_P = u^P$. □

A natural question then is: Under what conditions can it be guaranteed that $u_P = u^P$ on $\partial\Omega$? As in the classical Perron method, the existence of a solution in $C(\overline{\Omega})$ can be reduced to the existence of barriers at each point. Let $x \in \partial\Omega$. A *lower barrier at* x is a function $v \in S_g$ with $v(x) = g(x)$, and an *upper barrier at* x is a function $w \in S^g$ with $w(x) = g(x)$. If at each point of $\partial\Omega$, there exist both an upper and a lower barrier, then Theorem 3.1 can be applied to produce a $C(\overline{\Omega})$ solution to (1.1). Due to the nonlocal nature of M_q^h, it is unclear how to produce barriers in the standard way (by solving PDE problems). Note that while solutions of the functional equation $u = M_q^h u$ are related to p-harmonic functions, p-harmonic functions in general do not solve this functional equation [9]. Furthermore, it is not obvious how to use the solvability of (1.1) in balls contained in Ω, since the change in domain required to do so also changes the operators.

While necessary and sufficient conditions on g and Ω for the solvability of (1.1) are not known at this time, it is possible to give some simple sufficient conditions. Any linear function L satisfies $L = M_q^h L$ for any $h > 0$ and any $q \in [0,1]$, so linear functions are possible barriers. If g is such that at every $x \in \partial\Omega$ we can find upper and lower linear barriers, then $u_P = u^P = g$ on $\partial\Omega$, and (1.1) has a solution.

More generally, if the convex and concave envelopes of g coincide on $\partial\Omega$, (1.1) has a solution. The convex envelope of g, defined for $x \in \overline{\Omega}$, is

$$g_*(x) = \sup\{v(x) \in C(\overline{\Omega}) : v \text{ is convex, } v \le g \text{ on } \partial\Omega\},$$

and the concave envelope g^* is defined similarly. If Ω is not convex, we consider $v \in C(\overline{\Omega})$ to be convex if v is the restriction of a convex function defined on a convex domain to $\overline{\Omega}$. It was shown in [10] that a convex (concave) function u (v) satisfies $u \le M_q^h u$ ($v \ge M_q^h v$) for any $h > 0$ and any $q \in [0, 1]$. Thus $g_* \le u_P$ and $g^* \ge u^P$, so that if $g_* = g^*$ on $\partial\Omega$ the hypotheses of Theorem 3.1 are met. We also see that regardless of whether $g_* = g^*$ on $\partial\Omega$, if $u \in C(\overline{\Omega})$ is a solution of (1.1), $g_* \le u \le g^*$ in $\overline{\Omega}$.

Remark 3.2. We note that if $u_P \ne u^P$, the above method still produces weak solutions to (1.1). Namely by iterating u_P as in the proof of Theorem 3.1, we produce $u \in LSC(\overline{\Omega})$ with $u = \underline{M}_q^h u$ in Ω and $u(x) = \sup_{v \in S_g} v(x)$ for $x \in \partial\Omega$. Similarly, there exists $U \in USC(\overline{\Omega})$ with $U = \overline{M}_q^h U$ in Ω and $U(x) = \inf_{w \in S^g} w(x)$ for $x \in \partial\Omega$.

4. Perron method when $q = 0$

While the results of Section 2 apply when $q = 0$, some of the arguments in Section 3 do not. As indicated near the end of Section 1, $C(\overline{\Omega})$ solutions of (1.1) in the $q = 0$ case, when they exist, need not be unique. This non-uniqueness demonstrates the impossibility of a comparison principle of the type used in Section 3. This comparison principle was used to obtain the boundedness of the Perron solutions as well as the inequality $u_P \le u^P$. Thus, while the Perron solutions can be defined in the $q = 0$ case in precisely the same way as in the previous section (note that S_g and S^g are nonempty since they contain the constant functions $\min_{\partial\Omega} g$ and $\max_{\partial\Omega} g$ respectively), we do not know *a priori* that they are finite. In this section, we show that the Perron solutions are bounded and that the method used in Section 3 produces weak solutions to (1.1) that bound all possible $C(\overline{\Omega})$ solutions. These weak solutions need only be semicontinuous and satisfy functional equations involving the minimal or maximal median operator.

The following maximum and minimum principle immediately implies the finiteness of the Perron solutions.

Proposition 4.1. *If $v \in C(\overline{\Omega})$ satisfies $v \le M_0^h v$ in Ω, then $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$. If $w \in C(\overline{\Omega})$ satisfies $w \ge M_0^h w$ in Ω, then $\min_{\overline{\Omega}} w = \min_{\partial\Omega} w$.*

We remark that the corresponding result also holds when $q > 0$ and can be proved more easily by a standard argument.

Proof. We prove the statement for a subsolution v by contradiction. If the claim is not true, there exists $v \in C(\overline{\Omega})$ that attains a maximum T, where $T > \max_{\partial\Omega} v$, and satisfies $v \le M_0^h v$ in Ω. The set $\Omega_T = \{x \in \overline{\Omega} : v(x) = T\}$ is nonempty, closed and disjoint from $\partial\Omega$. Consider the convex hull of Ω_T, denoted $\operatorname{conv}\Omega_T$. This set is compact, convex, and equal to the convex hull of its extreme points. Note that $\operatorname{conv}\Omega_T$ need not be a subset of $\overline{\Omega}$ if Ω is not convex. Clearly $\Omega_T \subset (\operatorname{conv}\Omega_T) \cap \Omega$.

Let x be an extreme point of $\operatorname{conv}\Omega_T$. Then x cannot be written as a nontrivial convex combination of points in $\operatorname{conv}\Omega_T$ (in other words, $x \ne ty + (1-t)z$ for any $0 < t < 1$ and $y, z \in \operatorname{conv}\Omega_T$). By the definition of convex hull, x must be a convex combination of elements of Ω_T. But, because this convex combination cannot be nontrivial (because Ω_T is contained in $\operatorname{conv}\Omega_T$), x must belong to Ω_T. Because x is an extreme point of $\operatorname{conv}\Omega_T$, there exists a supporting hyperplane H to $\operatorname{conv}\Omega_T$ such that $H \cap \operatorname{conv}\Omega_T = \{x\}$. This implies that the functional inequality $v \le M_0^h v$ cannot hold at x. This is not hard to see: because $\operatorname{conv}\Omega_T \setminus \{x\}$ lies (strictly) on one side of H, the measure of $\partial B_x^h \cap \operatorname{conv}\Omega_T$ is strictly less than half the measure of ∂B_x^h. Recalling that $\Omega_T \subset \operatorname{conv}\Omega_T$, we see that $v < T$ on more than half of ∂B_x^h so that $\operatorname{median}_{\partial B_x^h} v < T = v(x)$. For the facts and definitions from convex geometry used above, we refer to the books [7] and [30]. □

Consider the problem (1.1) when $q = 0$ and define the classes S_g and S^g as in the previous section. As in the $q > 0$ case, any function in $S_g \cap S^g$ is a $C(\overline{\Omega})$ solution to the Dirichlet problem (1.1), and any $C(\overline{\Omega})$ solution must belong to $S_g \cap S^g$. However, as mentioned in Section 1, we don't generally expect $S_g \cap S^g$ to consist of a single element when nonempty. The next result shows that when the upper and lower Perron solutions coincide on the boundary, they solve (1.1) in a weak sense and bound all possible $C(\overline{\Omega})$ solutions.

Theorem 4.2. *If $u_P = u^P$ $(= g)$ on $\partial\Omega$, then there exist $u \in LSC(\overline{\Omega})$ and $U \in USC(\overline{\Omega})$, both equal to g on $\partial\Omega$, such that $u = \underline{M}_0^h u$ in Ω and $U = \overline{M}_0^h U$ in Ω. Furthermore, if $\overline{u}$ is a $C(\overline{\Omega})$ solution of (1.1) with $q = 0$, then $U \le \overline{u} \le u$.*

Proof. The proof of Theorem 3.1 produces the functions u and U with the claimed properties. The part of the argument establishing that $u = U$ does not apply when $q = 0$ however. Suppose that $\overline{u}$ is a $C(\overline{\Omega})$ solution of (1.1). Then $\overline{u} \in S_g \cap S^g$. It follows that $\overline{u} \le u_P$ by the definition of u_P and since u is produced by monotone iteration starting with u_P (see the proof of Theorem 3.1), we have that $\overline{u} \le u$. Similarly, since $\overline{u} \in S^g$, we get that $\overline{u} \ge U$, proving the last claim. □

If (1.1) has multiple $C(\overline{\Omega})$ solutions, the inequality in Theorem 4.2 shows that u and U (and therefore u_P and u^P) do not coincide. An analogous statement to Remark 3.2 also holds when $q = 0$.

It was also shown in [29] that the Dirichlet problem for the local median value property may not have a $C(\overline{\Omega})$ solution. 1-harmonious functions need not satisfy the local median value property (since the defining property for 1-harmonious functions requires that $u(x)$ equal its median over a single sphere centered at x and not all such spheres with sufficiently small radius as is the case for the local median value property, see the next section). However, this suggests that it may be possible for $S_g \cap S^g$ to be empty, so that there is no $C(\overline{\Omega})$ 1-harmonious function in that domain equal to g on the boundary.

5. Local median value property and 1-harmonious functions

Functions satisfying a local median value property were studied in [29]. It was shown there that such functions are (normalized) 1-harmonic in the viscosity sense. A function $u \in C(\Omega)$ satisfies the local median value property if $u(x)$ is equal to its median on $\partial B_\epsilon(x)$ for all $\varepsilon < R(x)$ where $R : \Omega \to \mathbb{R}$ is positive, continuous and satisfies $R(x) \le \operatorname{dist}(x, \partial\Omega)$. This property appears to be stronger than the condition defining 1-harmonious functions since it requires that $u(x)$ equal its median over all sufficiently small spheres centered at x, whereas to satisfy $u = M_0^h u$, $u(x)$ need only equal its median on the single sphere ∂B_x^h. However, for any given

$h > 0$, a function can satisfy the local median value property without satisfying $u = M_0^h u$, since $R(x)$ could be smaller than $r^h(x)$ defined in (1.3). Thus, while these conditions are clearly related, there is no obvious connection between them. Furthermore, as the following example demonstrates, a function u can possess the local median value property without satisfying $u = M_0^h u$ for *any* $h > 0$.

Let $\rho > 0$ be small ($\rho < 1$ is sufficient, but it is easier to see the claim when $\rho \approx 0$). Let $\Omega = B_1((1,0)) \subset \mathbb{R}^2$. Define $u \in C(\overline{\Omega})$ by

$$u(x,y) = \begin{cases} 0 & \text{if } -\rho x \le y \le \rho x, \\ \varepsilon & \text{if } y = \rho x + \varepsilon \text{ for } \varepsilon > 0, \\ -\varepsilon & \text{if } y = -\rho x + \varepsilon \text{ for } \varepsilon < 0. \end{cases}$$

In other words, u is zero in the part of $\overline{\Omega}$ between the lines $y = \rho x$ and $y = -\rho x$. Outside of this sector, $u > 0$ and is constant along lines parallel to the line $y = \rho x$ above $y = \rho x$ and along lines parallel to the line $y = -\rho x$ below $y = -\rho x$. It is not hard to see that u satisfies the local median value property since its level sets are either line segments or the region bounded by the lines $y = \rho x$ and $y = -\rho x$. Given any $h > 0$, there exists $\varepsilon > 0$ such that $B := B^h_{(\varepsilon,0)}$ is tangent to Ω at the origin. Then $u(\varepsilon, 0) = 0$, but because $\rho < 1$, $|\partial B \cap \{u > 0\}| > (1/2)|\partial B|$, so that $\operatorname{median}_{\partial B} u > 0$, and u is not 1-harmonious.

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