THE $\infty(x)$-EQUATION IN GRUSHIN-TYPE SPACES

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ABSTRACT. We employ Grushin jets which are adapted to the geometry of
Grushin-type spaces to obtain the existence-uniqueness of viscosity solutions
to the $\infty(x)$-Laplace equation in Grushin-type spaces. Due to the differences
between Euclidean jets and Grushin jets, the Euclidean method of proof is not
valid in this environment.

1. INTRODUCTION

Recently, the $\mathfrak{p}(x)$-Laplace equation and its limit equation, the $\infty(x)$-Laplace
equation, have been the focus of much attention as a tool for exploring applications
such as image restoration [7] and electrorheological fluid flow [10]. Linqvist and
Luukari [9] recently proved existence-uniqueness of viscosity solutions to the $\infty(x)$-
Laplace equation in (Euclidean) $\mathbb{R}^n$. However, this proof is not valid in general
Carnot-Carathéodory spaces, such as Grushin-type spaces, because it relies on two
important Euclidean properties, namely that the so-called viscosity penalty function
is the square of the intrinsic distance and that the two first-order jet elements
derived from the penalty function are equal. (These two phenomena are discussed
more below.) The main result of this paper is that the lack of these phenomena in
Grushin-type spaces can be overcome to produce existence-uniqueness of viscosity
solutions in this environment. In particular, we prove the following theorem.

Theorem 1.1. *Let Ω be a bounded domain in the Grushin-type space G_n and let
$f : \partial\Omega \to \mathbb{R}$ be a (Grushin) Lipschitz function. Then the Dirichlet problem*

$$-\Delta_{\infty(x)}u = 0 \quad in \ \Omega$$
$$u = f \quad on \ \partial\Omega$$

has a unique viscosity solution u.

In Section 2, we review the geometry of Grushin-type spaces and definitions of
various viscosity solutions. Section 3 collects all the Grushin tools we will be using
in our proof of existence-uniqueness, found in Section 4. Section 5 details some
further properties of the viscosity solutions.

2. Grushin-type spaces

2.1. **The Environment.** We begin by constructing the Grushin-type spaces. We consider $\mathbb{R}^n$ with coordinates $(x_1, x_2, \ldots, x_n)$ and the vector fields

$$X_i = \rho_i(x_1, x_2, \ldots, x_{i-1})\frac{\partial}{\partial x_i}$$

for $i = 2, 3, \ldots, n$ where $\rho_i(x_1, x_2, \ldots, x_{i-1})$ is a (possibly constant) polynomial. We decree that $\rho_1 \equiv 1$ so that

$$X_1 = \frac{\partial}{\partial x_1}.$$

A quick calculation shows that when $i < j$, the Lie bracket is given by

$$X_{ij} \equiv [X_i, X_j] = \rho_i(x_1, x_2, \ldots, x_{i-1})\frac{\partial \rho_j(x_1, x_2, \ldots, x_{j-1})}{\partial x_i}\frac{\partial}{\partial x_j}. \qquad (2.1)$$

Because the ρ_i's are polynomials, at each point there is a finite number of iterations of the Lie bracket so that $\frac{\partial}{\partial x_i}$ has a non-zero coefficient. This is easily seen for X_1 and X_2, and the result is obtained inductively for X_i. (It is noted that the number of iterations necessary is a function of the point.) Thus, Hörmander's condition is satisfied by these vector fields. Endowing $\mathbb{R}^n$ with an inner product (singular where the polynomials vanish) so that the $X_i's$ are orthonormal produces a manifold that we shall call g_n. This is the tangent space to a generalized Grushin-type space G_n. Points in G_n will also be denoted by $x = (x_1, x_2, \ldots, x_n)$ with a fixed point denoted $x_0 = (x_1^0, x_2^0, \ldots, x_n^0)$.

Even though G_n is not a group, it is a metric space with the natural metric being the Carnot-Carathéodory distance, which is defined for the points x and y as follows:

$$d_C(x, y) = \inf_\Gamma \int_0^1 \|\gamma'(t)\| dt$$

where the set Γ is the set of all curves γ such that $\gamma(0) = x, \gamma(1) = y$ and $\gamma'(t)$ is in span$\{\{X_i(\gamma(t))\}_{i=1}^n\}$. By Chow's theorem (see, for example, [1]) any two points can be connected by such a curve, which means $d_C(x, y)$ is an honest metric. Using this metric, we can define a Carnot-Carathéodory ball of radius r centered at a point x_0 by

$$B_r = B(x_0, r) = \{p \in G_n : d_C(x, x_0) < r\}$$

and similarly, we shall denote a bounded domain in G_n by Ω. The Carnot-Carathéodory metric behaves differently when the polynomials $\rho_i(x_1, x_2, \ldots, x_{i-1})$ vanish. Fixing a point x_0, consider the n-tuple $r_{x_0} = (r_{x_0}^1, r_{x_0}^2, \ldots, r_{x_0}^n)$ where $r_{x_0}^i$ is the minimal length of the Lie bracket iteration required to produce

$$[X_{j_1}, [X_{j_2}, [\cdots [X_{j_{r_{x_0}^i}}, X_i] \cdots](x_0) \neq 0.$$

Note that even though the minimal length is unique, the iteration used to obtain that minimum is not unique. Note also that

$$\rho_i(x_1^0, x_2^0, x_3^0, \ldots, x_{i-1}^0) \neq 0 \leftrightarrow r_{x_0}^i = 0.$$

Using [1, Theorem 7.34] we obtain the local estimate at x_0,

$$d_C(x_0, x) \sim \sum_{i=1}^n |x_i - x_i^0|^{\frac{1}{1+r_{x_0}^i}}. \qquad (2.2)$$

Given a smooth function f on G_n, we define the horizontal gradient of f as

$$\nabla_0 f(x) = (X_1 f(x), X_2 f(x), \ldots, X_n f(x))$$

and the symmetrized second order (horizontal) derivative matrix by

$$((D^2 f(x))^\star)_{ij} = \frac{1}{2}(X_i X_j f(x) + X_j X_i f(x))$$

for $i, j = 1, 2, \ldots n$. We can then define function spaces C^k and the Sobolev spaces $W^{1,\mathrm{p}}$, etc with respect to these vector fields in the usual way.

We may also define the ∞-Laplace operator

$$\Delta_\infty u = \langle ((D^2 u(x))^\star)\nabla_0 u, \nabla_0 u \rangle.$$

This operator is the "limit" operator of the $\mathbf{p}$-Laplace operator (for $2 < p < \infty$), which is given by

$$\Delta_{\mathbf{p}} u = \|\nabla_0 u\|^{\mathbf{p}-2}\Delta u + (\mathbf{p} - 2)\|\nabla_0 u\|^{\mathbf{p}-4}\Delta_\infty u$$
$$= \operatorname{div}_G (\|\nabla_0 u\|^{\mathbf{p}-2}\nabla_0 u)$$

where the divergence is taken with respect to the Grushin vector fields.

Following [9], we generalize these operators by replacing the constant $\mathbf{p}$ with an appropriate function $\mathbf{p}(x) \in C^1 \cap W^{1,\infty}$ and scalar $k > 1$ to obtain the $\mathbf{p}(x)$-Laplace operator

$$\Delta_{\mathbf{p}(x)} u = \|\nabla_0 u\|^{k\mathbf{p}(x)-2}\Delta u + (k\mathbf{p}(x) - 2)\|\nabla_0 u\|^{k\mathbf{p}(x)-4}\Delta_\infty u$$
$$+ \|\nabla_0 u\|^{k\mathbf{p}(x)-2}\langle \nabla_0 u, \nabla_0 k\mathbf{p}(x) \rangle \ln \|\nabla_0 u\|$$
$$= \operatorname{div}_G (\|\nabla_0 u\|^{k\mathbf{p}(x)-2}\nabla_0 u).$$

The corresponding equation $\Delta_{\mathbf{p}(x)} u = 0$ is the Euler-Lagrange equation associated to the energy functional

$$\Big(\int_\Omega \frac{\|\nabla_0 u\|^{k\mathbf{p}(x)}}{k\mathbf{p}(x)} \, dx\Big)^{1/k}.$$

Allowing $k \to \infty$, one has the tool for analysis of the extremal problem

$$\min_u \max_x \|\nabla_0 u\|^{\mathbf{p}(x)}.$$

Letting $k \to \infty$, we have $\Delta_{\mathbf{p}(x)} u \to \Delta_{\infty(x)} u$ where

$$\Delta_{\infty(x)} u = \Delta_\infty u + \|\nabla_0 u\|^2 \langle \nabla_0 u, \nabla_0 \ln \mathbf{p}(x) \rangle \ln \|\nabla_0 u\|.$$

For more details concerning the geometry of Grushin-type spaces, the interested reader is directed to [4, 6] and the references therein.

2.2. **Viscosity Solutions.** Because we will be considering viscosity solutions, we will recall the main definitions and properties. We begin with the Grushin jets $J^{2,+}$ and $J^{2,-}$. (See [3, 4] for a more complete analysis of such jets.)

Definition 2.1. Let u be an upper semi-continuous function. Consider the set

$$K_{\mathfrak{x}}^{2,+} u(x) = \Big\{\varphi \in C^2 \text{ in a neighborhood of } x, \varphi(x) = u(x),$$

$$\varphi(y) \geq u(y), \ y \neq x \text{ in a neighborhood of } x\Big\}.$$

Each function $\varphi \in K_{\mathfrak{x}}^{2,+} u(x)$ determines a vector-matrix pair (η, X) via the relations

$$\eta = \big(X_1 \varphi(x), X_2 \varphi(x), \ldots, X_n \varphi(x)\big)$$

$$X_{ij} = \frac{1}{2}\big(X_i(X_j(\varphi))(x) + X_j(X_i(\varphi))(x)\big).$$

We then define the *second order superjet of u at x* by

$$J^{2,+}u(x) = \{(\eta, X) : \varphi \in K^{2,+}u(x)\},$$

the *second order subjet of u at x* by

$$J^{2,-}u(x) = -J^{2,+}(-u)(x)$$

and the set-theoretic closure

$$\overline{J}^{2,+}u(x) = \Big\{(\eta, X) : \exists\{x_n, \eta_n, X_n\}_{n\in\mathbb{N}} \text{ with } (\eta_n, X_n) \in J^{2,+}u(x_n)$$

$$\text{and } (x_n, u(x_n), \eta_n, X_n) \to (x, u(x), \eta, X)\Big\}.$$

There is an analogous definition for $\overline{J}^{2,-}v(x)$.

We then use these Grushin jets to define viscosity $\infty(x)$-harmonic functions as follows:

Definition 2.2. A lower semi-continuous function v is *viscosity $\infty(x)$-superharmonic* in a bounded domain Ω if $v \not\equiv \infty$ in each component of Ω and for all $x_0 \in \Omega$, whenever $(\xi, \mathcal{Y}) \in \overline{J}^{2,-}v(x_0)$, we have

$$-\Big(\langle \mathcal{Y}\xi, \xi\rangle + \|\xi\|^2\langle\xi, \nabla_0 \ln \mathbf{p}(x)\rangle \ln\|\xi\|\Big) \geq 0.$$

An upper semi-continuous function u is *viscosity $\infty(x)$-subharmonic* in a bounded domain Ω if $u \not\equiv -\infty$ in each component of Ω and for all $x_0 \in \Omega$, whenever $(\eta, \mathcal{X}) \in \overline{J}^{2,+}u(x_0)$, we have

$$-\Big(\langle \mathcal{X}\eta, \eta\rangle + \|\eta\|^2\langle\eta, \nabla_0 \ln \mathbf{p}(x)\rangle \ln\|\eta\|\Big) \leq 0.$$

A function is *viscosity $\infty(x)$-harmonic* if it is both viscosity $\infty(x)$-subharmonic and viscosity $\infty(x)$-superharmonic.

Similarly, we have the following definition concerning $\Delta_{\mathbf{p}(x)}u$.

Definition 2.3. A lower semi-continuous function v is *viscosity $\mathbf{p}(x)$-superharmonic* in a bounded domain Ω if $v \not\equiv \infty$ in each component of Ω and for all $x_0 \in \Omega$, whenever $(\xi, \mathcal{Y}) \in \overline{J}^{2,-}v(x_0)$, we have

$$-\Big(\|\xi\|^{k\mathbf{p}(x)-2}\operatorname{tr}\mathcal{Y} + (k\mathbf{p}(x)-2)\|\xi\|^{k\mathbf{p}(x)-4}\langle\mathcal{Y}\xi, \xi\rangle$$

$$+ \|\xi\|^{k\mathbf{p}(x)-2}\langle\xi, \nabla_0 k\mathbf{p}(x)\rangle \ln\|\xi\|\Big) \geq 0.$$

An upper semi-continuous function u is *viscosity $\mathbf{p}(x)$-subharmonic* in a bounded domain Ω if $u \not\equiv -\infty$ in each component of Ω and for all $x_0 \in \Omega$, whenever $(\eta, \mathcal{X}) \in \overline{J}^{2,+}u(x_0)$, we have

$$-\Big(\|\eta\|^{k\mathbf{p}(x)-2}\operatorname{tr}\mathcal{X} + (k\mathbf{p}(x)-2)\|\eta\|^{k\mathbf{p}(x)-4}\langle\mathcal{X}\eta, \eta\rangle$$

$$+ \|\eta\|^{k\mathbf{p}(x)-2}\langle\eta, \nabla_0 k\mathbf{p}(x)\rangle \ln\|\eta\|\Big) \leq 0.$$

A function is *viscosity $\mathbf{p}(x)$-harmonic* if it is both viscosity $\mathbf{p}(x)$-subharmonic and viscosity $\mathbf{p}(x)$-superharmonic.

Remark 2.4. In the above definitions, we may replace the right-hand side of each inequality by an arbitrary function. In that case, we use the term viscosity $\infty(x)$-subsolution, etc.

3. KEY GRUSHIN TOOLS AND RESULTS

The Euclidean proof of the main theorem relies on two important facts, neither of which hold in Grushin-type spaces. The first is that the square of the intrinsic distance is a valid viscosity "penalty function". By equation (2.2), the square of the Grushin distance may not be sufficiently smooth. This can be rectified by choosing a higher power of the Grushin distance, so that the penalty function is smooth. This choice, however, is at odds with the reason for choosing the square of the distance function: the derivative of the penalty function must be comparable to the intrinsic distance so that we can exploit the Lipschitz property to obtain estimates that are controllable.

We now present the Grushin tools and results we will need to overcome this issue. The first is the Iterated Maximum Principle.

Lemma 3.1 (Iterated Maximum Principle [4]). *Let u be an upper-semicontinuous function in a domain Ω and v be a lower-semicontinuous function in Ω. Let*

$$\sup_{\overline{\Omega}}(u(x) - v(x)) = u(x_0) - v(x_0) > 0$$

occur in the interior of Ω. Consider a real vector $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with non-negative components and the points x and y with coordinates $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. We define the following functions for $j = 1, 2, 3, \ldots, n$:

$$\varphi_{\alpha_j, \alpha_{j+1}, \ldots, \alpha_n}(x, y) = \sum_{i=j}^{n} \frac{1}{2}\alpha_i(x_i - y_i)^2.$$

Using these functions and upper-semicontinuity on a compact set, we can consider the following well-defined functions for $j = 1, 2, 3, \ldots, n$:

$$
\begin{aligned}
&M_{\alpha_j, \alpha_{j+1}, \ldots, \alpha_n} \\
&= \sup_{\overline{\Omega} \times \overline{\Omega}} (u(x) - v(y) - \varphi_{\alpha_j, \alpha_{j+1}, \ldots, \alpha_n}(x, y) : x_k = y_k \quad \text{for } k = 1, 2, \ldots j - 1) \\
&= u(x_{\alpha_j, \alpha_{j+1}, \ldots, \alpha_n}) - v(y_{\alpha_j, \alpha_{j+1}, \ldots, \alpha_n}) \\
&\quad - \varphi_{\alpha_j, \alpha_{j+1}, \ldots, \alpha_n}(x_{\alpha_j, \alpha_{j+1}, \ldots, \alpha_n}, y_{\alpha_j, \alpha_{j+1}, \ldots, \alpha_n}).
\end{aligned}
$$

We then have

$$\lim_{\alpha_n \to \infty} \lim_{\alpha_{n-1} \to \infty} \cdots \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} \varphi_{\alpha_1, \alpha_2, \ldots, \alpha_n}(x_{\alpha_1, \alpha_2, \ldots, \alpha_n}, y_{\alpha_1, \alpha_2, \ldots, \alpha_n}) = 0,$$

$$\lim_{\alpha_n \to \infty} \lim_{\alpha_{n-1} \to \infty} \cdots \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} M_{\alpha_1, \alpha_2, \ldots, \alpha_n} = u(x_0) - v(x_0).$$

Corollary 3.2. *Under the hypotheses of Lemma 3.1, all iterated limits exist and the full limit exists and equals the common value of the iterated limits. That is,*

$$\lim_{\alpha_1, \alpha_2, \ldots, \alpha_n \to \infty} M_{\alpha_1, \alpha_2, \ldots, \alpha_n} = \sup_{\overline{\Omega}}(u(x) - v(x)) = u(x_0) - v(x_0),$$

$$\lim_{\alpha_1, \alpha_2, \ldots, \alpha_n \to \infty} \varphi_{\alpha_1, \alpha_2, \ldots, \alpha_n}(x_{\alpha_1, \alpha_2, \ldots, \alpha_n}, y_{\alpha_1, \alpha_2, \ldots, \alpha_n}) = 0.$$

Remark 3.3. As a consequence of the Iterated Maximum Principle, its proof, and Corollary 3.2, if we denote the points $x_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ and $y_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ by $x^{\vec{\alpha}}$ and $y^{\vec{\alpha}}$, respectively, then we have

$$\lim_{\alpha_1 \to \infty} x^{\vec{\alpha}} = x_{\alpha_2,\alpha_3,\ldots,\alpha_n} = (x_1^0, x_2^{\vec{\alpha}}, x_3^{\vec{\alpha}}, x_4^{\vec{\alpha}}, \ldots, x_n^{\vec{\alpha}})$$

$$\lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} x^{\vec{\alpha}} = x_{\alpha_3,\alpha_4,\ldots,\alpha_n} = (x_1^0, x_2^0, x_3^{\vec{\alpha}}, x_4^{\vec{\alpha}}, \ldots, x_n^{\vec{\alpha}})$$

$$\vdots$$

$$\lim_{\alpha_n \to \infty} \lim_{\alpha_{n-1} \to \infty} \cdots \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} x^{\vec{\alpha}} = x_0 = (x_1^0, x_2^0, x_3^0, \ldots, x_n^0).$$

Similarly,

$$\lim_{\alpha_1 \to \infty} y^{\vec{\alpha}} = y_{\alpha_2,\alpha_3,\ldots,\alpha_n} = (x_1^0, y_2^{\vec{\alpha}}, y_3^{\vec{\alpha}}, y_4^{\vec{\alpha}}, \ldots, y_n^{\vec{\alpha}})$$

$$\lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} y^{\vec{\alpha}} = y_{\alpha_3,\alpha_4,\ldots,\alpha_n} = (x_1^0, x_2^0, y_3^{\vec{\alpha}}, y_4^{\vec{\alpha}}, \ldots, y_n^{\vec{\alpha}})$$

$$\cdots$$

$$\lim_{\alpha_n \to \infty} \lim_{\alpha_{n-1} \to \infty} \cdots \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} y^{\vec{\alpha}} = x_0 = (x_1^0, x_2^0, x_3^0, \ldots, x_n^0).$$

The importance of the Iterated Maximum Principle is that it will allow us to isolate the vector fields by direction. In the Euclidean case, it is sufficient to use the Lipschitz property and then take the full limit, which is independent of direction. This approach is incompatible with the fact that the Grushin distance estimates vary at each point. Thus, we must use the Lipschitz property with the directions independently. This will allow us to overcome the first technical challenge. To do this, we need the following lemma and corollary.

Lemma 3.4 ([4, Lemma 3.3]). *Assume the hypotheses of the Iterated Maximum Principle (Lemma 3.1) and the notation of Remark 3.3. Suppose that at least one of u or v is (Grushin) Lipschitz. Define the point $(x \diamond_i y)$ by*

$$(x \diamond_i y) = (x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

That is, $(x \diamond_i y)$ coincides with y in the i-th coordinate and coincides with x elsewhere. Then there is a finite positive constant K so that

$$\alpha_i (x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \leq K d_C((x^{\vec{\alpha}} \diamond_i y^{\vec{\alpha}}), x^{\vec{\alpha}}).$$

The following corollary follows immediately from Equation (2.2).

Corollary 3.5. *When $\rho_i(x_1^\alpha, x_2^\alpha, \ldots, x_{i-1}^\alpha) \neq 0$, locally, we have*

$$\alpha_i (x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \leq K d_C((x^{\vec{\alpha}} \diamond_i y^{\vec{\alpha}}), x^{\vec{\alpha}}) = C|x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}}|.$$

The second important fact used in the Euclidean proof is that the first-order jet elements of a viscosity $\infty(x)$-superharmonic and viscosity $\infty(x)$-subharmonic are identical. Because the polynomials are non-constant, this is not the case in the Grushin environment. Using [5, Theorem 3.4] (or the main lemma in [3]), we have for an upper semicontinuous function u and a lower semicontinuous function v,

$$(\Upsilon_{x^{\vec{\alpha}}}, \mathcal{X}^{\vec{\alpha}}) \in \overline{J}^{2,+} u(x^{\vec{\alpha}}), \quad (\Upsilon_{y^{\vec{\alpha}}}, \mathcal{Y}^{\vec{\alpha}}) \in \overline{J}^{2,-} v(y^{\vec{\alpha}})$$

where $(x^{\vec{\alpha}}, y^{\vec{\alpha}})$ are the points associated with $M_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ (from Remark 3.3) and

$$(\Upsilon_{x^{\vec{\alpha}}})_i = \rho_i(x_1^{\vec{\alpha}}, x_2^{\vec{\alpha}}, \ldots, x_{i-1}^{\vec{\alpha}})\alpha_i(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}}),$$
$$(\Upsilon_{y^{\vec{\alpha}}})_i = \rho_i(y_1^{\vec{\alpha}}, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}}))\alpha_i(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}}). \tag{3.1}$$

We see that these vectors are, in general, not equal. We will overcome this challenge by producing more complicated estimates that are still controllable.

4. Existence-uniqueness of $\infty(x)$-harmonic functions

Let Ω be a bounded domain in G_n and $f : \partial\Omega \to \mathbb{R}$ be a (Grushin) Lipschitz function. We will first establish the existence of $\infty(x)$-harmonic functions using Jensen's auxiliary equations [8]:

$$\min\{\|\nabla_0 u\|^2 - \varepsilon, -\Delta_{\infty(x)}u\} = 0 \quad \text{and} \quad \max\{\varepsilon - \|\nabla_0 u\|^2, -\Delta_{\infty(x)}u\} = 0$$

for a real parameter $\varepsilon > 0$. The procedure for existence of viscosity solutions to these equations (and viscosity $\infty(x)$-harmonic functions) is identical to [2, Section 4] and [9, Section 2], up to the obvious modifications. For completeness, we state the steps as one theorem and omit the proofs.

Theorem 4.1 ([9, 2]). *We have the following results:*

(1) *Let $\varepsilon \in \mathbb{R}$. If u_k is a continuous potential-theoretic weak sub-(super-)solution with $u \in W^{1,kp(x)}(\Omega)$ to:*

$$-\Delta_{kp(x)}u_k = \varepsilon^{kp(x)-1} \quad \text{in } \Omega$$
$$u = f \quad \text{on } \partial\Omega$$

then it is a viscosity sub-(super-)solution.

(2) *Letting $k \to \infty$, we have $u_k \to u_\infty$ uniformly (possibly up to a subsequence) in Ω with $u_\infty \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$.*

(3) *The function u_∞ is a viscosity solution to*

$$\min\{\|\nabla_0 u_\infty\|^2 - \varepsilon, -\Delta_{\infty(x)}u_\infty\} = 0 \quad \text{when } \varepsilon > 0$$
$$\max\{\varepsilon - \|\nabla_0 u_\infty\|^2, -\Delta_{\infty(x)}u_\infty\} = 0 \quad \text{when } \varepsilon < 0$$
$$-\Delta_{\infty(x)}u_\infty = 0 \quad \text{when } \varepsilon = 0.$$

In light of [2, Lemma 5.6] and [9, Lemma 2.2], the Main Theorem follows from showing the uniqueness of viscosity solutions to the Jensen auxiliary equations. We will establish this result.

Theorem 4.2. *Let $v = u_\infty$ be the viscosity solution from Theorem 4.1 to*

$$\min\{\|\nabla_0 u\|^2 - \varepsilon, -\Delta_{\infty(x)}u\} = 0 \tag{4.1}$$

in a bounded domain Ω. If u is an upper semi-continuous viscosity subsolution to Equation (4.1) in Ω so that $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. Following [9, Lemma 3.1] and [2, Theorem 5.3], we may assume without loss of generality that v is a strict viscosity supersolution. Suppose

$$\sup_\Omega(u - v) > 0$$

and let

$$\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(x, y) = \sum_{i=1}^n \frac{1}{2}\alpha_i(x_i - y_i)^2.$$

be the function in the Iterated Maximum Principle (Theorem 3.1). By [5, Theorem 3.2], we have vectors $\Upsilon_{x^{\vec{\alpha}}}, \Upsilon_{y^{\vec{\alpha}}}$ and symmetric matrices $\mathcal{X}^{\vec{\alpha}}, \mathcal{Y}^{\vec{\alpha}}$ so that

$$(\Upsilon_{x^{\vec{\alpha}}}, \mathcal{X}^{\vec{\alpha}}) \in \overline{J}^{2,+} u(x^{\vec{\alpha}}) \quad \text{and} \quad (\Upsilon_{y^{\vec{\alpha}}}, \mathcal{Y}^{\vec{\alpha}}) \in \overline{J}^{2,-} v(y^{\vec{\alpha}}).$$

The vectors are explicitly given by (3.1).

Since u is a viscosity subsolution and v a strict viscosity supersolution, we have, for some $\mu > 0$,

$$0 \geq \min\{\|\Upsilon_{x^{\vec{\alpha}}}\|^2 - \varepsilon, -\langle \mathcal{X}^{\vec{\alpha}} \Upsilon_{x^{\vec{\alpha}}}, \Upsilon_{x^{\vec{\alpha}}} \rangle - \|\Upsilon_{x^{\vec{\alpha}}}\|^2 \langle \Upsilon_{x^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{x^{\vec{\alpha}}}\|\}$$

$$0 < \mu \leq \min\{\|\Upsilon_{y^{\vec{\alpha}}}\|^2 - \varepsilon, -\langle \mathcal{Y}^{\vec{\alpha}} \Upsilon_{y^{\vec{\alpha}}}, \Upsilon_{y^{\vec{\alpha}}} \rangle - \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \langle \Upsilon_{y^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(y^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{y^{\vec{\alpha}}}\|\}.$$

Subtracting these equations, we obtain

$$\begin{aligned}
0 < \mu \leq \max \Big\{ &\|\Upsilon_{y^{\vec{\alpha}}}\|^2 - \|\Upsilon_{x^{\vec{\alpha}}}\|^2, \langle \mathcal{X}^{\vec{\alpha}} \Upsilon_{x^{\vec{\alpha}}}, \Upsilon_{x^{\vec{\alpha}}} \rangle - \langle \mathcal{Y}^{\vec{\alpha}} \Upsilon_{y^{\vec{\alpha}}}, \Upsilon_{y^{\vec{\alpha}}} \rangle \\
&+ \|\Upsilon_{x^{\vec{\alpha}}}\|^2 \langle \Upsilon_{x^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{x^{\vec{\alpha}}}\| \\
&- \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \langle \Upsilon_{y^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(y^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{y^{\vec{\alpha}}}\| \Big\}.
\end{aligned} \tag{4.2}$$

By [3, Equation 5.6], we have

$$\lim_{\alpha_n \to \infty} \lim_{\alpha_{n-1} \to \infty} \cdots \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} \|\Upsilon_{y_{\vec{\alpha}}}\|^2 - \|\Upsilon_{x_{\vec{\alpha}}}\|^2 = 0. \tag{4.3}$$

Also by [4, Section 3], we have

$$\lim_{\alpha_n \to \infty} \lim_{\alpha_{n-1} \to \infty} \cdots \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} \langle \mathcal{X}^{\vec{\alpha}} \Upsilon_{x^{\vec{\alpha}}}, \Upsilon_{x^{\vec{\alpha}}} \rangle - \langle \mathcal{Y}^{\vec{\alpha}} \Upsilon_{y^{\vec{\alpha}}}, \Upsilon_{y^{\vec{\alpha}}} \rangle = 0.$$

We therefore turn our attention to

$$\|\Upsilon_{x^{\vec{\alpha}}}\|^2 \langle \Upsilon_{x^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{x^{\vec{\alpha}}}\| - \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \langle \Upsilon_{y^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(y^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{y^{\vec{\alpha}}}\|.$$

We begin by expanding this term:

$$\begin{aligned}
&\|\Upsilon_{x^{\vec{\alpha}}}\|^2 \langle \Upsilon_{x^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{x^{\vec{\alpha}}}\| - \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \langle \Upsilon_{y^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(y^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{y^{\vec{\alpha}}}\| \\
&= \|\Upsilon_{x^{\vec{\alpha}}}\|^2 \langle \Upsilon_{x^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle \Big(\ln \|\Upsilon_{x^{\vec{\alpha}}}\| - \ln \|\Upsilon_{y^{\vec{\alpha}}}\| \Big) \\
&\quad + \Big(\|\Upsilon_{x^{\vec{\alpha}}}\|^2 - \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \Big) \langle \Upsilon_{x^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle \ln \|\Upsilon_{y^{\vec{\alpha}}}\| \\
&\quad + \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \Big(\langle \Upsilon_{x^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle - \langle \Upsilon_{y^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle \Big) \ln \|\Upsilon_{y^{\vec{\alpha}}}\| \\
&\quad + \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \Big(\langle \Upsilon_{y^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) \rangle - \langle \Upsilon_{y^{\vec{\alpha}}}, \nabla_0 \ln \mathsf{p}(y^{\vec{\alpha}}) \rangle \Big) \ln \|\Upsilon_{y^{\vec{\alpha}}}\|
\end{aligned}$$

Using the fact that $1 < \mathsf{p}(x) \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$ and the fact that $\ln a^2 = 2 \ln a$, we have the absolute value of these terms is controlled by a finite constant C times

$$\begin{aligned}
\mathcal{T} \equiv &\|\Upsilon_{x^{\vec{\alpha}}}\|^3 \Big| \ln \frac{\|\Upsilon_{x^{\vec{\alpha}}}\|^2}{\|\Upsilon_{y^{\vec{\alpha}}}\|^2} \Big| + \|\Upsilon_{x^{\vec{\alpha}}}\| \big| \|\Upsilon_{x^{\vec{\alpha}}}\|^2 - \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \big| \Big| \ln \|\Upsilon_{y^{\vec{\alpha}}}\| \Big| \\
&+ \|\Upsilon_{y^{\vec{\alpha}}}\|^2 \|\Upsilon_{x^{\vec{\alpha}}} - \Upsilon_{y^{\vec{\alpha}}}\| \Big| \ln \|\Upsilon_{y^{\vec{\alpha}}}\| \Big| \\
&+ \|\Upsilon_{y^{\vec{\alpha}}}\|^3 \|\nabla_0 \ln \mathsf{p}(x^{\vec{\alpha}}) - \nabla_0 \ln \mathsf{p}(y^{\vec{\alpha}})\| \Big| \ln \|\Upsilon_{y^{\vec{\alpha}}}\| \Big|.
\end{aligned} \tag{4.4}$$

We will need to consider several cases. These cases rely on the fact that v is a strict supersolution, so that $\|\Upsilon_{y^{\vec{\alpha}}}\|^2 > \varepsilon > 0$.

Case 1: $\big|\ln\|\Upsilon_{y^{\vec{\alpha}}}\|\big| > |\ln\varepsilon|$ and

$$\ln\frac{\|\Upsilon_{x^{\vec{\alpha}}}\|^2}{\|\Upsilon_{y^{\vec{\alpha}}}\|^2} > 0.$$

Using these hypotheses, we may express $\mathcal{T}$ in terms of coordinates. Namely,

$$\mathcal{T} \equiv \sum_{i=1}^{n}\Big(\alpha_i^2\rho_i^2(x_1^{\vec{\alpha}},x_2^{\vec{\alpha}},\dots,x_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big)^{3/2}$$

$$\times\ln\Big(1+\frac{\sum_{i=2}^{n}\alpha_i^2\Big(\rho_i^2(x_1^{\vec{\alpha}},x_2^{\vec{\alpha}},\dots,x_{i-1}^{\vec{\alpha}})-\rho_i^2(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})\Big)(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2}{\varepsilon}\Big)$$

$$+\Big(\sum_{i=1}^{n}\alpha_i^2\rho_i^2(x_1^{\vec{\alpha}},x_2^{\vec{\alpha}},\dots,x_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big)^{1/2}$$

$$\times\Big|\sum_{i=2}^{n}\alpha_i^2\Big(\rho_i^2(x_1^{\vec{\alpha}},x_2^{\vec{\alpha}},\dots,x_{i-1}^{\vec{\alpha}})-\rho_i^2(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})\Big)(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big|$$

$$\times\frac{1}{2}\Big|\ln\sum_{i=1}^{n}\alpha_i^2\rho_i^2(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big|$$

$$+\sum_{i=1}^{n}\Big(\alpha_i^2\rho_i^2(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big)$$

$$\times\sum_{i=2}^{n}\alpha_i^2\Big(\rho_i(x_1^{\vec{\alpha}},x_2^{\vec{\alpha}},\dots,x_{i-1}^{\vec{\alpha}})-\rho_i(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})\Big)^2(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2$$

$$\times\frac{1}{2}\Big|\ln\sum_{i=1}^{n}\alpha_i^2\rho_i^2(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big|$$

$$+\Big(\sum_{i=1}^{n}\Big(\alpha_i^2\rho_i^2(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big)\Big)^{3/2}$$

$$\times\Big(\sum_{i=1}^{n}\Big(\rho_i(x_1^{\vec{\alpha}},x_2^{\vec{\alpha}},\dots,x_{i-1}^{\vec{\alpha}})\frac{\partial\mathtt{p}(x)}{\partial x_i}\big|_{(x_1^{\vec{\alpha}},x_2^{\vec{\alpha}},\dots,x_i^{\vec{\alpha}})}$$

$$-\rho_i(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})\frac{\partial\mathtt{p}(y)}{\partial y_i}\big|_{(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_i^{\vec{\alpha}})}\Big)^2\Big)^{1/2}$$

$$\times\frac{1}{2}\Big|\ln\sum_{i=1}^{n}\alpha_i^2\rho_i^2(y_1^{\vec{\alpha}},y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big|.$$

Note that some of the sums start at $i=2$ since $\rho_1 \equiv 1$.

Using Corollary 3.5, Remark 3.3, and the fact that $\rho_1 \equiv 1$ and $\mathtt{p}(x) \in C^1\cap W^{1,\infty}$, we have for some finite constant K,

$$0 \le \mathcal{T}_1 = \lim_{\alpha_1\to\infty}\mathcal{T}$$

$$\le \Big(K+\sum_{i=2}^{n}\Big(\alpha_i^2\rho_i^2(x_1^0,x_2^{\vec{\alpha}},\dots,x_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\Big)^{3/2}\Big)$$

$$\times\ln\Big(1+\frac{\sum_{i=3}^{n}\alpha_i^2\Big(\rho_i^2(x_1^0,x_2^{\vec{\alpha}},\dots,x_{i-1}^{\vec{\alpha}})-\rho_i^2(x_1^0,y_2^{\vec{\alpha}},\dots,y_{i-1}^{\vec{\alpha}})\Big)(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2}{\varepsilon}\Big)$$

$$+ \Big(K + \sum_{i=2}^{n} \alpha_i^2 \rho_i^2(x_1^0, x_2^{\vec{\alpha}}, \ldots, x_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big)^{1/2}$$

$$\times \Big| \sum_{i=3}^{n} \alpha_i^2 \Big(\rho_i^2(x_1^0, x_2^{\vec{\alpha}}, \ldots, x_{i-1}^{\vec{\alpha}}) - \rho_i^2(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}}) \Big)(x_i^{\vec{\alpha}} - y_{i-1}^{\vec{\alpha}})^2 \Big|$$

$$\times \frac{1}{2} \Big| \ln \big[K + \sum_{i=2}^{n} \alpha_i^2 \rho_i^2(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \big] \Big|$$

$$+ \Big(K + \sum_{i=2}^{n} \Big(\alpha_i^2 \rho_i^2(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big) \Big)$$

$$\times \sum_{i=3}^{n} \alpha_i^2 \Big(\rho_i(x_1^0, x_2^{\vec{\alpha}}, \ldots, x_{i-1}^{\vec{\alpha}}) - \rho_i(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}}) \Big)^2 (x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2$$

$$\times \frac{1}{2} \Big| \ln \big[K + \sum_{i=2}^{n} \alpha_i^2 \rho_i^2(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \big] \Big|$$

$$+ \Big(K + \sum_{i=2}^{n} \Big(\alpha_i^2 \rho_i^2(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big) \Big)^{3/2}$$

$$\times \Big(\sum_{i=1}^{n} \Big(\rho_i(x_1^0, x_2^{\vec{\alpha}}, \ldots, x_{i-1}^{\vec{\alpha}}) \frac{\partial \mathsf{p}(x)}{\partial x_i} |_{(x_1^0, x_2^{\vec{\alpha}}, \ldots, x_i^{\vec{\alpha}})}$$

$$- \rho_i(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}}) \frac{\partial \mathsf{p}(y)}{\partial y_i} |_{(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_i^{\vec{\alpha}})} \Big)^2 \Big)^{1/2}$$

$$\times \frac{1}{2} \Big| \ln \big[K + \sum_{i=2}^{n} \alpha_i^2 \rho_i^2(x_1^0, y_2^{\vec{\alpha}}, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \big] \Big|.$$

Now, if $\rho_2(x_1^0) = 0$, the corresponding term vanishes. If $\rho_2(x_1^0) \neq 0$, we again apply Corollary 3.5 and Remark 3.3 to obtain the existence of a finite constant K so that

$$0 \leq \mathcal{T}_2 = \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} \mathcal{T}$$

$$\leq \Big(K + \sum_{i=3}^{n} \Big(\alpha_i^2 \rho_i^2(x_1^0, x_2^0, \ldots, x_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big)^{3/2} \Big)$$

$$\times \ln \Big(1 + \frac{\sum_{i=4}^{n} \alpha_i^2 \Big(\rho_i^2(x_1^0, x_2^0, \ldots, x_{i-1}^{\vec{\alpha}}) - \rho_i^2(x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}}) \Big)(x_i^{\vec{\alpha}} - y_{i-1}^{\vec{\alpha}})^2}{\varepsilon} \Big)$$

$$+ \Big(K + \sum_{i=3}^{n} \alpha_i^2 \rho_i^2(x_1^0, x_2^0, \ldots, x_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big)^{1/2}$$

$$\times \Big| \sum_{i=4}^{n} \alpha_i^2 \Big(\rho_i^2(x_1^0, x_2^0, \ldots, x_{i-1}^{\vec{\alpha}}) - \rho_i^2(x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}}) \Big)(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big|$$

$$\times \frac{1}{2} \Big| \ln \big[K + \sum_{i=3}^{n} \alpha_i^2 \rho_i^2(x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \big] \Big|$$

$$+ \Big(K + \sum_{i=3}^{n} \Big(\alpha_i^2 \rho_i^2(x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big) \Big)$$

$$\times \sum_{i=4}^{n} \alpha_i^2 \Big(\rho_i(x_1^0, x_2^0, \ldots, x_{i-1}^{\vec{\alpha}}) - \rho_i(x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}}) \Big)^2 (x_i^{\vec{\alpha}} - y_{i-1}^{\vec{\alpha}})^2$$

$$\times \frac{1}{2} \Big| \ln \Big[K + \sum_{i=3}^{n} \alpha_i^2 \rho_i^2 (x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big] \Big|$$

$$+ \Big(K + \sum_{i=3}^{n} \Big(\alpha_i^2 \rho_i^2 (x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big) \Big)^{3/2}$$

$$\times \Big(\sum_{i=1}^{n} \Big(\rho_i(x_1^0, x_2^0, \ldots, x_{i-1}^{\vec{\alpha}}) \frac{\partial \mathbf{p}(x)}{\partial x_i} \big|_{(x_1^0, x_2^0, \ldots, x_i^{\vec{\alpha}})}$$

$$- \rho_i(x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}}) \frac{\partial \mathbf{p}(y)}{\partial y_i} \big|_{(x_1^0, x_2^0, \ldots, y_i^{\vec{\alpha}})} \Big)^2 \Big)^{1/2}$$

$$\times \frac{1}{2} \Big| \ln \Big[K + \sum_{i=3}^{n} \alpha_i^2 \rho_i^2 (x_1^0, x_2^0, \ldots, y_{i-1}^{\vec{\alpha}})(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \Big] \Big|.$$

We iterate this process until we arrive at

$$0 \le \mathcal{T}_{n-1} = \lim_{\alpha_{n-1} \to \infty} \lim_{\alpha_{n-2} \to \infty} \cdots \lim_{\alpha_1 \to \infty} \mathcal{T}$$

$$\le \Big(K + \Big(\alpha_n^2 \rho_n^2 (x_1^0, x_2^0, \ldots, x_{n-1}^0)(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 \Big)^{3/2} \Big) \times \ln \Big(1 + \frac{0}{\varepsilon} \Big)$$

$$+ \Big(K + \alpha_n^2 \rho_n^2 (x_1^0, x_2^0, \ldots, x_{n-1}^0)(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 \Big)^{1/2}$$

$$\times \Big| \alpha_n^2 \Big(0 \Big)(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 \Big|$$

$$\times \frac{1}{2} \Big| \ln \Big[K + \alpha_n^2 \rho_n^2 (x_1^0, x_2^0, \ldots, x_{n-1}^0)(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 \Big] \Big|$$

$$+ \Big(K + \Big(\alpha_n^2 \rho_n^2 (x_1^0, x_2^0, \ldots, x_{n-1}^0)(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 \Big) \Big)$$

$$\times \alpha_n^2 \Big(0 \Big)^2 (x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2$$

$$\times \frac{1}{2} \Big| \ln \Big[K + \alpha_n^2 \rho_n^2 (x_1^0, x_2^0, \ldots, x_{n-1}^0)(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 \Big] \Big|$$

$$+ \Big(K + \Big(\alpha_n^2 \rho_i^2 (x_1^0, x_2^0, \ldots, x_{n-1}^0)(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 \Big) \Big)^{3/2}$$

$$\times \Big(\sum_{i=1}^{n} \Big(\rho_i(x_1^0, x_2^0, \ldots, x_{n-1}^0) \frac{\partial \mathbf{p}(x)}{\partial x_i} \big|_{(x_1^0, x_2^0, \ldots, x_i^0)}$$

$$- \rho_i(x_1^0, x_2^0, \ldots, x_{n-1}^0) \frac{\partial \mathbf{p}(y)}{\partial y_i} \big|_{(x_1^0, x_2^0, \ldots, x_n^0)} \Big)^2 \Big)^{1/2}$$

$$\times \frac{1}{2} \Big| \ln \Big[K + \alpha_n^2 \rho_n^2 (x_1^0, x_2^0, \ldots, x_{n-1}^0)(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 \Big] \Big|$$

$$= 0 + 0 + 0 + 0.$$

We then conclude that

$$\lim_{\alpha_n \to \infty} \lim_{\alpha_{n-1} \to \infty} \cdots \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} \mathcal{T} = 0.$$

Case 2: $\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big| \le |\ln\varepsilon|$ and

$$\ln\frac{\|\Upsilon_{x^{\bar{\alpha}}}\|^2}{\|\Upsilon_{y^{\bar{\alpha}}}\|^2} > 0.$$

We then have

$$\begin{aligned}
0 \le \mathcal{T} \le {} & \|\Upsilon_{x^{\bar{\alpha}}}\|^3\big|\ln\frac{\|\Upsilon_{x^{\bar{\alpha}}}\|^2}{\|\Upsilon_{y^{\bar{\alpha}}}\|^2}\big| + \|\Upsilon_{x^{\bar{\alpha}}}\|\big|\|\Upsilon_{x^{\bar{\alpha}}}\|^2 - \|\Upsilon_{y^{\bar{\alpha}}}\|^2\big||\ln\varepsilon| \\
& + \|\Upsilon_{y^{\bar{\alpha}}}\|^2\|\Upsilon_{x^{\bar{\alpha}}} - \Upsilon_{y^{\bar{\alpha}}}\||\ln\varepsilon| \\
& + \|\Upsilon_{y^{\bar{\alpha}}}\|^3\|\nabla_0\ln\mathsf{p}(x^{\vec{\alpha}}) - \nabla_0\ln\mathsf{p}(y^{\vec{\alpha}})\||\ln\varepsilon|.
\end{aligned}$$

We then proceed as in Case 1.

Case 3: $\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big| > |\ln\varepsilon|$ and

$$\ln\frac{\|\Upsilon_{x^{\bar{\alpha}}}\|^2}{\|\Upsilon_{y^{\bar{\alpha}}}\|^2} < 0.$$

From these hypotheses, we have

$$\begin{aligned}
\mathcal{T} = {} & \|\Upsilon_{x^{\bar{\alpha}}}\|^3\ln\frac{\|\Upsilon_{y^{\bar{\alpha}}}\|^2}{\|\Upsilon_{x^{\bar{\alpha}}}\|^2} + \|\Upsilon_{x^{\bar{\alpha}}}\|\big|\|\Upsilon_{x^{\bar{\alpha}}}\|^2 - \|\Upsilon_{y^{\bar{\alpha}}}\|^2\big|\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big| \\
& + \|\Upsilon_{y^{\bar{\alpha}}}\|^2\|\Upsilon_{x^{\bar{\alpha}}} - \Upsilon_{y^{\bar{\alpha}}}\|\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big| \\
& + \|\Upsilon_{y^{\bar{\alpha}}}\|^3\|\nabla_0\ln\mathsf{p}(x^{\vec{\alpha}}) - \nabla_0\ln\mathsf{p}(y^{\vec{\alpha}})\|\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big|.
\end{aligned}$$

We will use the following lemma.

Lemma 4.3. *Given that* $\|\Upsilon_{y^{\bar{\alpha}}}\|^2 > \varepsilon$, *we have*

$$\|\Upsilon_{x^{\bar{\alpha}}}\|^2 \ge \frac{\varepsilon}{2}.$$

Proof. Suppose not. Then $\|\Upsilon_{x^{\bar{\alpha}}}\|^2 < \varepsilon/2$. We then have

$$0 < \frac{\varepsilon}{2} \le \|\Upsilon_{y^{\bar{\alpha}}}\|^2 - \|\Upsilon_{x^{\bar{\alpha}}}\|^2$$

Taking iterated limits of this inequality contradicts (4.3).   □

Using this lemma, we have

$$\begin{aligned}
0 \le \mathcal{T} \le {} & \|\Upsilon_{x^{\bar{\alpha}}}\|^3\ln\Big(1 + \frac{\|\Upsilon_{y^{\bar{\alpha}}}\|^2 - \|\Upsilon_{x^{\bar{\alpha}}}\|^2}{\frac{\varepsilon}{2}}\Big) \\
& + \|\Upsilon_{x^{\bar{\alpha}}}\|\big|\|\Upsilon_{x^{\bar{\alpha}}}\|^2 - \|\Upsilon_{y^{\bar{\alpha}}}\|^2\big|\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big| \\
& + \|\Upsilon_{y^{\bar{\alpha}}}\|^2\|\Upsilon_{x^{\bar{\alpha}}} - \Upsilon_{y^{\bar{\alpha}}}\|\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big| \\
& + \|\Upsilon_{y^{\bar{\alpha}}}\|^3\|\nabla_0\ln\mathsf{p}(x^{\vec{\alpha}}) - \nabla_0\ln\mathsf{p}(y^{\vec{\alpha}})\|\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big|.
\end{aligned}$$

This case then proceeds as in Case 1.

Case 4: $\big|\ln\|\Upsilon_{y^{\bar{\alpha}}}\|\big| \le |\ln\varepsilon|$ and

$$\ln\frac{\|\Upsilon_{x^{\bar{\alpha}}}\|^2}{\|\Upsilon_{y^{\bar{\alpha}}}\|^2} < 0.$$

This case is similar to Case 3 (cf. Case 2) and we omit it.

Equation (4.2) now produces a contradiction and the theorem then follows.   □

An analogous argument produces the following Corollary.

Corollary 4.4. *Let $v = u_\infty$ be the viscosity solution from Theorem 4.1 to*

$$\max\{\varepsilon - \|\nabla_0 u\|^2, -\Delta_{\infty(x)} u\} = 0 \qquad (4.5)$$

in a bounded domain Ω. If u is an lower semi-continuous viscosity supersolution to Equation (4.5) in Ω so that $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.

5. A Harnack inequality

We include a Harnack inequality for completeness. First, we have the following lemma whose proof is identical to [9, Lemma 4.1] and omitted.

Lemma 5.1. *Let u be a positive viscosity $\infty(x)$-harmonic function and ζ a positive, compactly supported smooth function. Then*

$$\sup_{x \in \Omega} \left| \nabla_0 \zeta(x) \nabla_0 \ln u(x) \right|^{p(x)} \leq \sup_{x \in \Omega} \left| \nabla_0 \zeta(x) + \zeta(x) \ln \left(\frac{\zeta(x)}{u(x)} \right) \nabla_0 \ln p(x) \right|^{p(x)}.$$

As in [9, Section 4], we have the following Harnack inequality as a consequence.

Theorem 5.2. *Let u be a positive viscosity $\infty(x)$-harmonic function. Let B_r be a ball of radius $r > 0$ contained in the bounded domain Ω. Let B_{2r} be the concentric ball of twice the radius also contained in Ω. Then*

$$\sup_{x \in B_r} u(x) \leq C\left(\inf_{x \in B_r} u(x) + r \right)$$

for some constant C depending on $\sup_{x \in B_{2r}} u(x)$.

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