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EXISTENCE AND MULTIPLICITY OF POSITIVE RADIAL SOLUTIONS TO NONLOCAL BOUNDARY-VALUE PROBLEMS IN EXTERIOR DOMAINS

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Abstract. In this article, we consider nonlocal p-Laplacian boundary-value problems with integral boundary conditions and a non-negative real-valued boundary condition as a parameter. The main purpose is to study the existence, nonexistence and multiplicity of positive solutions as the boundary parameter varies. Moreover, we prove a sub-super solution theorem, using fixed point index theorems.

1. Introduction

Elliptic partial differential equations (PDEs) are fundamental for modeling natural phenomena. In mathematical modeling, elliptic PDEs are used together with boundary conditions specifying the solution on the boundary of the domain. Dirichlet, Neumann and Robin conditions are well-known classical boundary conditions. It frequently happens that the process or phenomena cannot be described precisely in classical boundary conditions. Thus, mathematical models of various physical, chemical, biological or environmental processes often require nonclassical conditions such as nonlocal conditions which express situations when the information on the boundary cannot be measured directly, or when it depends on the data inside the domain. The study on nonlocal boundary-value problems was initiated by Picone [22], and later continued by Bicadze and Samarskiĭ [2], Il'in and Moiseev [9] and Gupta [8]. Since then, the existence of solutions for nonlocal boundary-value problems has received a great deal of attention in the literature. We refer the reader to [1, 4, 5, 6, 11, 12, 13, 14, 18, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] and the references therein for more recent results.

We consider the quasilinear elliptic problems in an exterior domain

$$\operatorname{div}(|\nabla z|^{p-2}\nabla z) + K(|x|)f_1(|x|, z) = 0 \quad \text{in } \Omega \tag{1.1}$$

with one of the following nonlocal boundary conditions

$$z(x) = \mu \text{ on } \partial\Omega, \quad \lim_{|x|\to\infty} z(x) = \int_\Omega l(|y|)z(y)dy, \tag{1.2}$$

$$z(x) = \int_\Omega l(|y|)z(y)dy \ \text{ on } \partial\Omega, \quad \lim_{|x|\to\infty} z(x) = \mu, \tag{1.3}$$

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where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, $r_0 > 0$, $N > p > 1$, μ is a non-negative real parameter, $K(\not\equiv 0) \in C((r_0,\infty),\mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$, $l \in L^1((r_0,\infty),\mathbb{R}_+)$, and $f_1 \in C([r_0,\infty)\times\mathbb{R}_+,\mathbb{R}_+)$ which satisfies $f_1(r,s) > 0$ for all $(r,s) \in [r_0,\infty)\times(0,\infty)$.

In this article, we use the assumption

(A1) $K \not\equiv 0$ and there exists $\alpha_1 > p-1$ such that $\int_{r_0}^\infty r^{\alpha_1} K(r)dr < \infty$.

Note that if $K_1(x) = K(|x|) \in L^1(\Omega)$, then

$$\int_{r_0}^\infty r^{N-1}K(r)dr < \infty,$$

and (A1) holds for some $\alpha_1 \in (p-1, N-1)$ by the fact $N > p$.

Applying consecutive changes of variables, problems (1.1)-(1.2)-(1.3) are equivalently transformed into the singular p-Laplacian problem

$$\begin{gathered}(\varphi_p(u'(t)))' + h(t)f(t,u(t)) = 0, \quad t \in (0,1),\\ u(0) = \alpha[u], \quad u(1) = \mu,\end{gathered} \tag{1.4}$$

where $p > 1$, $\varphi_p : \mathbb{R} \to \mathbb{R}$ is a homeomorphism defined by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, $\alpha[u] := \int_0^1 k(s)u(s)ds$, $k \in L^1((0,1),\mathbb{R}_+)$, $h(\not\equiv 0) \in C((0,1),\mathbb{R}_+)$ and $f \in C([0,1]\times\mathbb{R}_+,\mathbb{R}_+)$ satisfies that $f(t,s) > 0$ for all $(t,s) \in [0,1]\times(0,\infty)$. Since K satisfies (A1), the weight function h satisfies

$$\int_0^1 s^a(1-s)^b h(s)ds < \infty \tag{1.5}$$

for some $a, b \in [0, p-1)$, and h may be singular at 0 and/or 1. It is well known that h satisfies (1.5), then $h \in \mathcal{A}$, where

$$\mathcal{A} = \Big\{h : \int_0^{1/2} \varphi_p^{-1}\Big(\int_s^{1/2} h(\tau)d\tau\Big)ds + \int_{1/2}^1 \varphi_p^{-1}\Big(\int_{1/2}^s h(\tau)d\tau\Big)ds < \infty\Big\}$$

(for more details, see Section 4).

Throughout this article, we assume that $h \in \mathcal{A}\setminus\{0\} \cap C((0,1),\mathbb{R}_+)$ unless otherwise stated.

Recently, Zhang and Feng [31] studied the one-dimensional singular p-Laplacian problem

$$\begin{gathered}\lambda(\varphi_p(u'(t)))' + w(t)f(t,u(t)) = 0, \quad t \in (0,1),\\ au(0) - bu'(0) = \int_0^1 g(t)u(t)dt, \quad u'(1) = 0,\end{gathered}$$

where λ is a positive parameter, $a, b > 0$, $w, g \in L^1((0,1),\mathbb{R}_+)$. Using fixed point index theory on cones of Banach spaces, they obtained several results about the existence, multiplicity, and nonexistence of positive solutions under various assumptions on the nonlinearity which satisfies L^1-Carathéodory condition. Note that $L^1(0,1) \subsetneq \mathcal{A}$. When $k \equiv 0$, i.e., $\alpha[u] \equiv 0$, problem (1.4) has been studied in [15, 16] for the case that $h \in \mathcal{A}$ and the nonlinearity $f(t,s) = s^q$ with $q > p-1$.

Motivated by [15, 16, 31], using sub-supersolution method and fixed point index theory, we study the existence, nonexistence and multiplicity of positive solutions to (1.4) with an integral boundary condition and a boundary value μ as a parameter, where the nonlinear term may not satisfy the L^1-Carathéodory condition and it does not need the superlinear condition at ∞ (see (A5) below). It seems not obvious that sub-supersolution method can be applicable to p-Laplacian problem (1.4) with an integral boundary condition. Thus we prove a sub-supersolution theorem (see Theorem 2.11). For sub-supersolution methods concerning semilinear problems with nonlocal boundary conditions, we refer to [19, 20, 21].

This article is organized as follows. In Section 2, well-known theorems such as generalized Picone identity and a fixed point index theorem are given, and a solution operator related to problem (1.4) is introduced. In addition, a sub-supersolution theorem to the problem (1.4) is proved. In Section 3, main results in this paper are given. Finally, in Section 4, applications for p-Laplacian problems (1.1)-(1.2)-(1.3) defined in an exterior domain in $\mathbb{R}^N$ are provided.

2. Preliminaries

First, we introduce the generalized Picone identity and some results from the theory of the fixed-point index for completely continuous maps.

Theorem 2.1 (generalized Picone identity [10, 17]). *Let us define*

$$l_p[y] = (\varphi_p(y'))' + b_1(t)\varphi_p(y),$$
$$L_p[z] = (\varphi_p(z'))' + b_2(t)\varphi_p(z),$$

where b_1, b_2 are continuous functions on a nonemtpy interval I. Let y and z be functions such that $y, z, \varphi_p(y')$, $\varphi_p(z')$ are differentiable on I and $z(t) \neq 0$ for $t \in I$. Then the generalized Picone identity can be written as

$$\begin{aligned} \frac{d}{dt}\Big\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\Big\} &= (b_1 - b_2)|y|^p - \Big[|y'|^p + (p-1)|\frac{yz'}{z}|^p \\ &\quad - py'\varphi_p\big(\frac{yz'}{z}\big)\Big] - yl_p[y] + \frac{|y|^p}{\varphi_p(z)}L_p[z]. \end{aligned} \tag{2.1}$$

Remark 2.2. By Young's inequality,

$$y'\varphi_p\big(\frac{yz'}{z}\big) \leq |y'\varphi_p\big(\frac{yz'}{z}\big)| \leq \frac{|y'|^p}{p} + \big(1 - \frac{1}{p}\big)|\frac{yz'}{z}|^p.$$

Thus,

$$|y'|^p + (p-1)|\frac{yz'}{z}|^p - py'\varphi_p\big(\frac{yz'}{z}\big) \geq 0,$$

and the equality holds if and only if $\operatorname{sgn} y' = \operatorname{sgn}(yz'/z)$ and $|y'|^p = |yz'/z|^p$.

Definition 2.3. Let E and F be normed spaces and $U \subset E$. The mapping $T : \bar{U} \to F$ is completely continuous if it is continuous and the closure of $T(D)$ is compact for every bounded subset D in U

Theorem 2.4 ([7]). *Let K be a cone in X of a real Banach space X. Then, for relatively bounded open subset U of K and completely continuous operator $A : \bar{U} \to K$ which has no fixed points on ∂U. Then the following properties hold.*

(i) *Normality: $i(A, U, K) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \bar{U}$.*
(ii) *Additivity: $i(A, U, K) = i(A, U_1, K) + i(A, U_2, K)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\bar{U} \setminus (U_1 \cup U_2)$.*
(iii) *Homotopy invariance: $i(H(t, \cdot), U, K)$ is independent of t $(0 \leq t \leq 1)$ whenever $H : [0, 1] \times \bar{U} \to K$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0, 1] \times \partial U$.*
(iv) *Excision property: $i(A, U, K) = i(A, U_0, K)$ whenever U_0 is an open subset of U such that A has no fixed points in $\bar{U} \setminus U_0$.*

(v) *Solution property: If* $i(A, U, K) \neq 0$*, then* A *has at least one fixed point in* U*.*

Theorem 2.5 ([7]). *Let* X *be a Banach space,* $\mathcal{K}$ *an order cone in* X *and* $\mathcal{O}$ *bounded open in* X*. Let* $0 \in \mathcal{O}$ *and* $A : \bar{\mathcal{O}} \cap \mathcal{K} \to \mathcal{K}$ *be completely continuous. Suppose that* $Ax \neq \nu x$ *for all* $x \in \partial\mathcal{O} \cap \mathcal{K}$ *and all* $\nu \geq 1$*. Then* $i(A, \mathcal{O} \cap \mathcal{K}, \mathcal{K}) = 1$*.*

2.1. **Operator.** In this subsection, we define an operator related to problem (1.4) and prove its complete continuity. Throughout this paper, we assume

(A2) $\int_0^1 (1-s)k(s)ds \in [0, 1)$,

and $f \in C([0, 1] \times \mathbb{R}_+, \mathbb{R}_+)$ satisfies that $f(t, s) > 0$ for all $(t, s) \in [0, 1] \times (0, \infty)$. Then there exists a continuous extension $\bar{f} : [0, 1] \times \mathbb{R} \to \mathbb{R}_+$ of the nonlinearity f such that $\bar{f}(t, s) = f(t, 0)$ for $(t, s) \in [0, 1] \times (-\infty, 0)$ and $\bar{f}(t, s) = f(t, s)$ for $(t, s) \in [0, 1] \times \mathbb{R}_+$. Without loss of generality, we denote $\bar{f}$ by f again and consider $f : [0, 1] \times \mathbb{R} \to \mathbb{R}_+$, since we only consider nonnegative solutions to (1.4).

Let $C[0, 1]$ denote the Banach space of continuous functions u defined on $[0, 1]$ with usual maximum norm $\|u\|_\infty := \max_{t\in[0,1]} |u(t)|$, and let $\mathcal{K} := \{u \in C[0, 1] : u$ is a nonnegative function on $[0, 1]\}$. Then, $\mathcal{K}$ is a cone in $C[0, 1]$.

We consider the problem

$$\begin{gathered}(\varphi_p(v'(t) - \beta[\mu, v] + \mu))' + h(t)F(\mu, v)(t) = 0, \quad t \in (0, 1),\\ v(0) = v(1) = 0,\end{gathered} \tag{2.2}$$

where $F : \mathbb{R}_+ \times C[0, 1] \to \mathcal{K}$ is defined by

$$F(\mu, v)(t) := f(t, v(t) + (1-t)\beta[\mu, v] + t\mu) \tag{2.3}$$

for $(\mu, v) \in \mathbb{R}_+ \times C[0, 1]$ and $t \in [0, 1]$, and $\beta : \mathbb{R}_+ \times C[0, 1] \to \mathbb{R}$ is defined by

$$\beta[\mu, v] := \frac{1}{1 - \int_0^1 (1-s)k(s)ds}\Big(\int_0^1 k(s)(v(s) + \mu s)ds\Big)$$

for $(\mu, v) \in \mathbb{R}_+ \times C[0, 1]$. We notice that $\beta[\mu_n, v_n] \to \beta[\mu_0, v_0]$ as $(\mu_n, v_n) \to (\mu_0, v_0)$ in $\mathbb{R}_+ \times C[0, 1]$ and $\beta[\mu, v] \in \mathbb{R}_+$ for all $(\mu, v) \in \mathbb{R}_+ \times \mathcal{K}$.

By a positive solution to problem (1.4) (or (2.2)), we mean a function $u \in C[0, 1] \cap C^1(0, 1)$ which satisfies (1.4) (or (2.2)) and $u > 0$ in $(0, 1)$. Note that all solutions to problem (1.4) (or (2.2)) are concave functions on $(0, 1)$. Indeed, if v is a solution of (2.2), then $(\varphi_p(v'(t) - \beta[\mu, v] + \mu))' = -h(t)F(\mu, v)(t) \leq 0$ for all $t \in (0, 1)$, and $\varphi_p(v'(t) - \beta[\mu, v] + \mu)$ is monotonically decreasing on $(0, 1)$. Since $\varphi_p : \mathbb{R} \to \mathbb{R}$ is a strictly increasing bijective mapping for any $p > 1$, v' is monotonically decreasing on $(0, 1)$, and thus v is a concave function on $(0, 1)$. In a similar manner, it is shown that all solutions to (1.4) are concave functions on $(0, 1)$.

We consider transformations $L_i : \mathbb{R}_+ \times C[0, 1] \to C[0, 1]$, $i = 1, 2$, which are defined by

$$v := L_1(\mu, u) := u - ((1-t)\alpha[u] + t\mu) \quad \text{for } (\mu, u) \in \mathbb{R}_+ \times C[0, 1] \tag{2.4}$$

and

$$u := L_2(\mu, v) := v + (1-t)\beta[\mu, v] + t\mu \quad \text{for } (\mu, v) \in \mathbb{R}_+ \times C[0, 1]. \tag{2.5}$$

Then, it follows that $\alpha[u] = \beta[\mu, L_1(\mu, u)]$ and $\beta[\mu, v] = \alpha[L_2(\mu, v)]$. Moreover, u is a solution to (1.4) if and only if v is a solution to problem (2.2) under the transformations (2.4) and (2.5), respectively.

Define $T:\mathbb{R}_+\times C[0,1]\to C[0,1]$, for $(\mu,v)\in\mathbb{R}_+\times C[0,1]$, by

$$T(\mu,v)(t):=\begin{cases}(\beta[\mu,v]-\mu)t+\int_0^t\varphi_p^{-1}\Big(\varphi_p(\mu-\beta[\mu,v]) \\ +\int_s^M h(\tau)F(\mu,v)(\tau)d\tau\Big)ds, & 0\le t\le M,\\ -(\beta[\mu,v]-\mu)(1-t)+\int_t^1\varphi_p^{-1}\Big(\varphi_p(-\mu+\beta[\mu,v]) \\ +\int_M^s h(\tau)F(\mu,v)(\tau)d\tau\Big)ds, & M\le t\le 1,\end{cases}$$

where $\varphi_p^{-1}:\mathbb{R}\to\mathbb{R}$ is an inverse function of φ_p which is defined by $\varphi_p^{-1}(s)=|s|^{\frac{2-p}{p-1}}s$ for $s\neq 0$ and $\varphi_p^{-1}(0)=0$, and $M=M(\mu,v)\in[0,1]$ is a constant satisfying

$$\begin{aligned}&(\beta[\mu,v]-\mu)M+\int_0^M\varphi_p^{-1}\Big(\varphi_p(\mu-\beta[\mu,v])+\int_s^M h(\tau)F(\mu,v)(\tau)d\tau\Big)ds\\ &=-(\beta[\mu,v]-\mu)(1-M)+\int_M^1\varphi_p^{-1}\Big(\varphi_p(-\mu+\beta[\mu,v])+\int_M^s h(\tau)F(\mu,v)(\tau)d\tau\Big)ds.\end{aligned}\tag{2.6}$$

Indeed, for each $(\mu,v)\in\mathbb{R}_+\times C[0,1]$, there exists a constant $M:=M(\mu,v)\in[0,1]$ satisfying (2.6). Let $(\mu,v)\in\mathbb{R}_+\times C[0,1]$ be fixed, and let us define $x=x_{\mu,v}:(0,1)\to\mathbb{R}$ by

$$x(t)=\beta[\mu,v]-\mu+x_1(t)+x_2(t),\ t\in(0,1),$$

where

$$\begin{aligned}x_1(t)&=\int_0^t\varphi_p^{-1}\Big(\varphi_p(\mu-\beta[\mu,v])+\int_s^t h(\tau)F(\mu,v)(\tau)d\tau\Big)ds,\\ x_2(t)&=\int_t^1\varphi_p^{-1}\left(\varphi_p(\mu-\beta[\mu,v])-\int_t^s h(\tau)F(\mu,v)(\tau)d\tau\right)ds.\end{aligned}$$

Clearly, $\lim_{t\to0^+}x_1(t)=\lim_{t\to1^-}x_2(t)=0$, $\lim_{t\to1^-}x_1(t)\ge\mu-\beta[\mu,v]$, and $\lim_{t\to0^+}x_2(t)\le\mu-\beta[\mu,v]$. Consequently,

$$\lim_{t\to0^+}x(t)\le0\quad\text{and}\quad\lim_{t\to1^-}x(t)\ge0.$$

For $0\le t_1<t_2$, one has

$$x_1(t_2)-x_1(t_1)\ge\int_{t_1}^{t_2}\varphi_p^{-1}\Big(\varphi_p(\mu-\beta[\mu,v])+\int_s^{t_2}h(\tau)F(\mu,v)(\tau)d\tau\Big)ds\tag{2.7}$$

and

$$x_2(t_2)-x_2(t_1)\ge-\int_{t_1}^{t_2}\varphi_p^{-1}\Big(\varphi_p(\mu-\beta[\mu,v])+\int_s^{t_1}h(\tau)F(\mu,v)(\tau)d\tau\Big)ds.\tag{2.8}$$

Thus, $x(t_2)-x(t_1)\ge0$ for $0\le t_1<t_2$, and there exists a $M=M(\mu,v)\in[0,1]$ such that $x(M)=0$, i.e., M satisfies (2.6).

Remark 2.6. We notice that $M=M(\mu,v)$ may not be unique in $[0,1]$. However, if M^1 and M^2 are the zeroes of $x=x_{\mu,v}$ satisfying $M^1<M^2$, then $h(t)F(\mu,v)(t)=0$ for a.e. $t\in[M^1,M^2]$. Indeed, if $h(t)F(\mu,v)(t)\neq0$ for a.e. $t\in[M^1,M^2]$, then $x(M^1)<x(M^2)$ by (2.7) and (2.8) with $t_1=M^1$ and $t_2=M^2$. This is a contradiction to the fact $x(M^1)=x(M^2)=0$. In this case, one can see that $T(\mu,v)(t)\equiv\|T(\mu,v)\|_\infty$ for all $t\in[M^1,M^2]$, and the operator $T(\mu,v)$ is independent of the choice of zero of $x=x(\mu,v)$.

By the definition of $M = M(\mu, v)$, we can easily show that T is well defined, $T(\mu, v)(0) = T(\mu, v)(1) = 0$, $\|T(\mu, v)\|_\infty = T(\mu, v)(M)$, and $(T(\mu, v))'(M) = 0$. Moreover, since $(T(\mu, v))'$ is deceasing in $(0, 1)$, $T(\mu, v)$ is a concave function and $T(\mu, v) \in \mathcal{K}$ for all $(\mu, v) \in \mathbb{R}_+ \times C[0, 1]$. Then we have the following lemma.

Lemma 2.7. *Assume that* (A2) *holds and* $\mu \in \mathbb{R}_+$. *Then* (2.2) *has a positive solution* v *if and only if* $T(\mu, \cdot)$ *has a fixed point* v *in* $\mathcal{K} \setminus \{0\}$.

Proof. Let v be a positive solution to problem (2.2). Then there exists $M \in (0, 1)$ such that $v'(M) = 0$, since v is a concave function and $v(0) = v(1) = 0$. By straightforward integration,

$$v(t) = \begin{cases} (\beta[\mu, v] - \mu)t + \int_0^t \varphi_p^{-1}\Big(\varphi_p(\mu - \beta[\mu, v]) \\ + \int_s^M h(\tau)F(\mu, v)(\tau)d\tau\Big)ds, & 0 \le t \le M, \\ -(\beta[\mu, v] - \mu)(1 - t) + \int_t^1 \varphi_p^{-1}\Big(\varphi_p(-\mu + \beta[\mu, v]) \\ + \int_M^s h(\tau)F(\mu, v)(\tau)d\tau\Big)ds, & M \le t \le 1, \end{cases}$$

and we conclude that $v = T(\mu, v)$ on $[0, 1]$ and $v \in \mathcal{K} \setminus \{0\}$.

Conversely, let $v = T(\mu, v)$ on $[0, 1]$ and $v \neq 0$. Since $T(\mu, v)(0) = T(\mu, v)(1) = 0$ and $T(\mu, v)$ is a concave function on $(0, 1)$, $v > 0$ on $(0, 1)$. From straightforward differentiation, it follows that v is a positive solution to (2.2). □

Recall that a mapping $T : \mathbb{R}_+ \times C[0, 1] \to \mathcal{K}$ is said to be completely continuous if it is continuous and the closure of $T(\Sigma)$ is compact for every bounded subset Σ in $\mathbb{R}_+ \times C[0, 1]$. To show that $T : \mathbb{R}_+ \times C[0, 1] \to \mathcal{K}$ is completely continuous, we first prove the following lemma.

Lemma 2.8. *Assume that* (A2) *holds. Let* $C > 0$ *be given and let* $\{(\mu_n, v_n)\}$ *be a sequence in* $\mathbb{R}_+ \times C[0, 1]$ *with* $\mu_n + \|v_n\|_\infty \le C$. *If* $M_n = M(\mu_n, v_n) \to 0$ *(or 1) as* $n \to \infty$, *then* $\|T(\mu_n, v_n)\|_\infty \to 0$ *as* $n \to \infty$.

Proof. We only prove the case $M_n \to 0$, since the other case is similar. Since $\mu_n + \|v_n\|_\infty \le C$, there exists $N > 0$ such that $\|F(\mu_n, v_n)\|_\infty + |\beta[\mu_n, v_n]| \le N$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} &\|T(\mu_n, v_n)\|_\infty \\ &= T(\mu_n, v_n)(M_n) \\ &= (\beta[\mu_n, v_n] - \mu_n)M_n + \int_0^{M_n} \varphi_p^{-1}\Big(\varphi_p(\mu_n - \beta[\mu_n, v_n]) \\ &\quad + \int_s^{M_n} h(\tau)F(\mu_n, v_n)(\tau)d\tau\Big)ds \\ &\le (N + C)M_n + \gamma_{\frac{1}{p-1}} \int_0^{M_n} \Big((C + N) + \varphi_p^{-1}(N)\varphi_p^{-1}\Big(\int_s^{M_n} h(\tau)d\tau\Big)\Big)ds, \end{aligned}$$

where $\gamma_q = \max\{1, 2^{q-1}\}$ for $q > 0$. It follows from $h \in \mathcal{A}$ that $\|T(\mu_n, v_n)\|_\infty \to 0$ as $n \to \infty$, and thus the proof is complete. □

Lemma 2.9. *Assume that* (A2) *holds. Then* $T : \mathbb{R}_+ \times C[0, 1] \to \mathcal{K}$ *is completely continuous.*

Proof. Let Σ be bounded in $\mathbb{R}_+ \times C[0,1]$. We first show that $\overline{T(\Sigma)}$ is compact. Since Σ is bounded in $\mathbb{R}_+ \times C[0,1]$, there exists $C > 0$ such that $\mu + \|v\|_\infty \le C$ for all $(\mu, v) \in \Sigma$. By the continuity of f, there exists $N > 0$ such that $|\beta[\mu, v] - \mu| + \|F(\mu, v)\|_\infty \le N$ for all $(\mu, v) \in \Sigma$. Then $T(\Sigma)$ is bounded.

We now show the equicontinuity of $T(\Sigma)$ on $[0, 1]$. Let $\epsilon > 0$ be given. By Lemma 2.8, there exists $\bar\delta > 0$ such that if $M = M(\mu, v) \in [0, \bar\delta) \cup (1 - \bar\delta, 1]$, then $\|T(\mu, v)\|_\infty < \epsilon$. For $(\mu, v) \in \Sigma$ with $M \in [0, \bar\delta) \cup (1 - \bar\delta, 1]$, we have $|T(\mu, v)(t) - T(\mu, v)(s)| < 2\epsilon$ for all $t, s \in [0, 1]$. For $(\mu, v) \in \Sigma$ with $M \in [\bar\delta, 1 - \bar\delta]$, we have

$$
\begin{aligned}
|(T(\mu, v))'(t)| &= \Big|\beta[\mu, v] - \mu + \varphi_p^{-1}\Big(\varphi_p(\mu - \beta[\mu, v]) + \int_t^M h(\tau)F(\mu, v)(\tau)d\tau\Big)\Big| \\
&\le N + \varphi_p^{-1}\Big(\varphi_p(N) + N\int_{\bar\delta}^{1-\bar\delta} h(\tau)d\tau\Big) =: C_1
\end{aligned}
$$

for all $t \in [\bar\delta, 1 - \bar\delta]$. Therefore, if $t_1, t_2 \in [\bar\delta, 1 - \bar\delta]$ with $|t_1 - t_2| < \epsilon/C_1$, then $|T(\mu, v)(t_1) - T(\mu, v)(t_2)| < \epsilon$ by the Mean Value Theorem. For $t_3, t_4 \in [0, \bar\delta]$ with $t_3 < t_4$,

$$
\begin{aligned}
&|T(\mu, v)(t_3) - T(\mu, v)(t_4)| \\
&\le N|t_3 - t_4| + \int_{t_3}^{t_4} \varphi_p^{-1}\Big(\varphi_p(N) + N\int_s^{1-\bar\delta} h(\tau)d\tau\Big)ds \\
&\le N(1 + \gamma_{\frac{1}{p-1}})|t_3 - t_4| + \gamma_{\frac{1}{p-1}}\varphi_p^{-1}(N)\int_{t_3}^{t_4} \varphi_p^{-1}\Big(\int_s^{1-\bar\delta} h(\tau)d\tau\Big)ds.
\end{aligned}
$$

Since $h \in \mathcal{A}$, there exists $\delta_1 > 0$ such that if $t, s \in [0, \bar\delta]$ with $|t - s| < \delta_1$, then $|T(\mu, v)(t) - T(\mu, v)(s)| < \epsilon$. Similarly, there exists $\delta_2 > 0$ such that if $t, s \in [1 - \bar\delta, 1]$ with $|t - s| < \delta_2$, then $|T(\mu, v)(t) - T(\mu, v)(s)| < \epsilon$. Let $\delta_3 = \min\{\epsilon/C_1, \delta_1, \delta_2\} > 0$, then for $t, s \in [0, 1]$ with $|t - s| < \delta_3$, we have

$$
|T(\mu, v)(t) - T(\mu, v)(s)| < 3\epsilon
$$

for all $(\mu, v) \in \Sigma$. This shows that $T(\Sigma)$ is equicontinuous on $[0, 1]$, and, by Ascoli-Arzelà theorem, $\overline{T(\Sigma)}$ is compact.

We finally show that T is continuous. Let $\{(\mu_n, v_n)\}$ be a sequence with (μ_n, v_n) converges to (μ_0, v_0) in $\mathbb{R}_+ \times C[0, 1]$. Since $\{(\mu_n, v_n)\}$ is bounded, there exists $C_2 > 0$ such that $|\beta[\mu_n, v_n] - \mu_n| + \|F(\mu_n, v_n)\|_\infty < C_2$ for all $n \in \mathbb{N}$, and by the compactness of T, there exists a subsequence, say again, $\{(\mu_n, v_n)\}$ such that $T(\mu_n, v_n)$ converges to V in $C[0, 1]$ as $n \to \infty$. We may assume that $M_n = M(\mu_n, v_n)$, appeared in the definition of operator T, converges to $M_0 \in [0, 1]$ as $n \to \infty$. First consider the case $M_0 = 0$ (or 1). In this case, by Lemma 2.8, $T(\mu_n, v_n) \to 0$ in $C[0, 1]$ as $n \to \infty$. Since M_n is a zero of x_{μ_n, v_n}, it follows from (2.6) that $h(t)F(\mu_n, v_n)(t) \to 0$ for a.e. in $[0, 1]$ as $M_n \to 0$ (or 1). Since $(\mu_n, v_n) \to (\mu_0, v_0)$ in $C[0, 1]$, $h(t)F(\mu_0, v_0)(t) = 0$ for a.e. in $[0, 1]$. Thus $T(\mu_0, v_0) \equiv 0$, and $T(\mu_n, v_n) \to T(\mu_0, v_0)$ in $C[0, 1]$ as $n \to \infty$. Next, consider $M_0 \in (0, 1)$. We will prove $V \equiv T(\mu_0, v_0)$ on $[0, 1]$. Without loss of generality, we may choose a monotone increasing subsequence $\{M_{n_j}\}$ with $M_{n_j} \to M_0$ as $n_j \to \infty$. Then

$$
\Big|\varphi_p^{-1}\Big(\varphi_p(\mu_{n_j} - \beta[\mu_{n_j}, v_{n_j}]) + \int_s^{M_{n_j}} h(\tau)F(\mu_{n_j}, v_{n_j})(\tau)d\tau\Big)\Big|
$$

$$\le \varphi_p^{-1}\Big(\varphi_p(C_2) + C_2\int_s^{M_0} h(\tau)d\tau\Big)$$

for $0 \le s \le M_{n_j}$. Since $\beta[\mu_n, v_n] \to \beta[\mu_0, v_0]$ as $n \to \infty$, by Lebesgue dominated convergence theorem,

$$\begin{aligned} V(t) &= \lim_{j\to\infty} T(\mu_{n_j}, v_{n_j})(t) \\ &= (\beta[\mu_0, v_0] - \mu_0)t + \int_0^t \varphi_p^{-1}\Big(\varphi_p(\mu_0 - \beta[\mu_0, v_0]) \\ &\quad + \int_s^{M_0} h(\tau)F(\mu_0, v_0)(\tau)d\tau\Big)ds \end{aligned} \tag{2.9}$$

for $0 \le t \le M_0$. Note that for fixed $J > 0$ and for all $a > 0$,

$$\int_a^1 \varphi_p^{-1}\Big(\varphi_p(C_2) + C_2\int_{M_{n_J}}^s h(\tau)d\tau\Big)ds < \infty$$

and

$$\begin{aligned} &\Big|\varphi_p^{-1}\Big(\varphi_p(-\mu_{n_j} + \beta[\mu_{n_j}, v_{n_j}]) + \int_{M_{n_j}}^s h(\tau)F(\mu_{n_j}, v_{n_j})(\tau)d\tau\Big)\Big| \\ &\le \varphi_p^{-1}\Big(\varphi_p(C_2) + C_2\int_{M_{n_J}}^s h(\tau)d\tau\Big) \end{aligned}$$

for $M_{n_j} \le s \le 1$ and $j \ge J$. Again applying Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} V(t) &= \lim_{j\to\infty} T(\mu_{n_j}, v_{n_j})(t) \\ &= -(\beta[\mu_0, v_0] - \mu_0)(1-t) + \int_t^1 \varphi_p^{-1}\Big[\varphi_p(-\mu_0 + \beta[\mu_0, v_0]) \\ &\quad + \int_{M_0}^s h(\tau)F(\mu_0, v_0)(\tau)d\tau\Big]ds \end{aligned} \tag{2.10}$$

for $M_0 \le t \le 1$. From (2.9) and (2.10), we have

$$\begin{aligned} V(M_0) &= (\beta[\mu_0, v_0] - \mu_0)M_0 \\ &\quad + \int_0^{M_0} \varphi_p^{-1}\Big(\varphi_p(\mu_0 - \beta[\mu_0, v_0]) + \int_s^{M_0} h(\tau)F(\mu_0, v_0)(\tau)d\tau\Big)ds \\ &= -(\beta[\mu_0, v_0] - \mu_0)(1 - M_0) \\ &\quad + \int_{M_0}^1 \varphi_p^{-1}\Big(\varphi_p(-\mu_0 + \beta[\mu_0, v_0]) + \int_{M_0}^s h(\tau)F(\mu_0, v_0)(\tau)d\tau\Big)ds. \end{aligned}$$

This implies M_0 is a zero of x_{μ_0,v_0}, and thus $V \equiv T(\mu_0, v_0)$. So far we have shown that if a sequence $\{(\mu_n, v_n)\}$ converges to (μ_0, v_0) in $\mathbb{R}_+ \times C[0,1]$, then there exists a subsequence, say $\{(\mu_{n_j}, v_{n_j})\}$ such that

$$T(\mu_{n_j}, v_{n_j}) \to T(\mu_0, v_0) \text{ in } C[0,1].$$

By a standard argument, we can show that the original sequence also satisfies

$$T(\mu_n, v_n) \to T(\mu_0, v_0) \text{ in } C[0,1],$$

and this completes the proof. ∎

2.2. **Sub-supersolution Theorem.** We shall prove a sub-supersolution result for the singular problem (1.4). First, we give the definition for sub-supersolution.

Definition 2.10. We say that ψ is a *subsolution* to problem (1.4) if $\psi \in C^1(0,1)$ with $\varphi_p(\psi')$ absolutely continuous and

$$\begin{gathered}(\varphi_p(\psi'(t)))' + h(t)f(t,\psi(t)) \geq 0, \quad t \in (0,1)\\ \psi(0) \leq \alpha[\psi], \quad \psi(1) \leq \mu.\end{gathered}$$

We also say that ζ is a *supersolution* to problem (1.4) if $\zeta \in C^1(0,1)$ with $\varphi_p(\zeta')$ absolutely continuous and it satisfies the reverse of the above inequalities.

Now, a sub-supersolution theorem for the singular problem (1.4) is given as follows.

Theorem 2.11. *Assume that* (A2) *holds, and that there exist ψ and ζ,respectively, a subsolution and an upper solution of* (1.4) *such that $0 \leq \psi(t) \leq \zeta(t)$ for all $t \in [0,1]$. Then problem* (1.4) *has at least one solution u such that*

$$\psi(t) \leq u(t) \leq \zeta(t) \quad \textit{for all } t \in [0,1].$$

Proof. Define $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}_+$ by

$$\gamma(t,u) := \begin{cases} \zeta(t), & u \geq \zeta(t),\\ u, & \psi(t) \leq u \leq \zeta(t),\\ \psi(t), & u \leq \psi(t),\end{cases}$$

and consider the modified problem

$$\begin{gathered}(\varphi_p(u'(t)))' + h(t)f(t,\gamma(t,u(t))) = 0, \quad t \in (0,1),\\ u(0) = \alpha_\gamma[u], \quad u(1) = \mu,\end{gathered} \tag{2.11}$$

where $\alpha_\gamma[u] := \int_0^1 k(s)\gamma(s,u(s))ds$ and $\mu \in \mathbb{R}_+$ is a fixed constant. For given $v \in C[0,1]$, define $g_v : \mathbb{R} \to \mathbb{R}$ by

$$g_v(x) := \int_0^1 k(s)\gamma(s, v(s) + (1-s)x + s\mu)ds \quad \text{for } x \in \mathbb{R}.$$

By (A2), g_v is a contraction mapping on $\mathbb{R}$, since $|\gamma(s,x) - \gamma(s,y)| \leq |x-y|$ for any $x, y \in \mathbb{R}$ and $s \in [0,1]$. Then, there exists a unique solution $\beta_\gamma[v]$ of the equation $x = g_v(x)$, and $\beta_\gamma[v]$ satisfies

$$\beta_\gamma[v] = \int_0^1 k(s)\gamma(s, v(s) + (1-s)\beta_\gamma[v] + s\mu)ds.$$

Under the transformation

$$v(t) := u(t) - ((1-t)\alpha_\gamma[u] + t\mu), \tag{2.12}$$

$\alpha_\gamma[u] = \beta_\gamma[v]$ and (2.11) can be rewritten as follows

$$\begin{gathered}(\varphi_p(v'(t) - \beta_\gamma[v] + \mu))' + h(t)F_\gamma(v)(t) = 0, \quad t \in (0,1),\\ v(0) = v(1) = 0,\end{gathered} \tag{2.13}$$

where $F_\gamma(v)(t) := f(t, \gamma(t, v(t) + (1-t)\beta_\gamma[v] + t\mu))$ for $v \in C[0,1]$ and $t \in [0,1]$. Consequently, u is a solution of (2.11) if and only if v is a solution of (2.13) under the transformation (2.12).

Now, define $T_\gamma : \mathcal{K} \to \mathcal{K}$, for $v \in \mathcal{K}$, by

$$T_\gamma(v)(t) := \begin{cases} (\beta_\gamma[v]-\mu)t + \int_0^t \varphi_p^{-1}\Big((\varphi_p(\mu-\beta_\gamma[v]) \\ + \int_s^{M_\gamma} h(\tau)F_\gamma(v)(\tau)d\tau\Big)ds, & 0 \le t \le M_\gamma, \\ -(\beta_\gamma[v]-\mu)(1-t) + \int_t^1 \varphi_p^{-1}\Big((\varphi_p(-\mu+\beta_\gamma[v]) \\ + \int_{M_\gamma}^s h(\tau)F_\gamma(v)(\tau)d\tau\Big)ds, & M_\gamma \le t \le 1, \end{cases}$$

where $M_\gamma = M_\gamma(v)$ is the constant satisfying

$$\begin{aligned} &(\beta_\gamma[v]-\mu)M_\gamma + \int_0^{M_\gamma} \varphi_p^{-1}\Big((\varphi_p(\mu-\beta_\gamma[v]) + \int_s^{M_\gamma} h(\tau)F_\gamma(v)(\tau)d\tau\Big)ds \\ &= -(\beta_\gamma[v]-\mu)(1-M_\gamma) + \int_{M_\gamma}^1 \varphi_p^{-1}\Big((\varphi_p(-\mu+\beta_\gamma[v]) \\ &\quad + \int_{M_\gamma}^s h(\tau)F_\gamma(v)(\mu,v)(\tau)d\tau\Big)ds. \end{aligned}$$

Then v is a fixed point of T_γ in $\mathcal{K}$ if and only if v is a nonnegative solution of (2.13). It follows that T_γ is completely continuous on $\mathcal{K}$ by the same argument as in the proof of Lemma 2.9 and $T_\gamma(\mathcal{K})$ is bounded in $C[0,1]$. Then T_γ has a fixed point v, and consequently (2.11) has a nonnegative solution u. Now if we prove that $\psi(t) \le u(t) \le \zeta(t)$ for $t \in [0,1]$, then, by the definition of γ, (1.4) has a solution u such that $\psi(t) \le u(t) \le \zeta(t)$ for all $t \in [0,1]$ and the proof is complete. To show $u(t) \le \zeta(t)$, set $X(t) := u(t) - \zeta(t)$. Then, since

$$\begin{gathered} X(0) = u(0) - \zeta(0) \le \int_0^1 k(s)[\gamma(s,u(s)) - \zeta(s)]ds \le 0, \\ X(1) = u(1) - \zeta(1) \le 0, \end{gathered}$$

we assume on the contrary that there is $t_0 \in (0,1)$ such that $X(t_0) = u(t_0) - \zeta(t_0) > 0$. Then there exists $\sigma \in (0,1)$ such that

$$X(\sigma) = \max_{t\in[0,1]} X(t) > 0.$$

Then $X'(\sigma) = 0$ and there is $a \in (\sigma, 1)$ such that $X'(t) < 0$ and $X(t) > 0$ for $t \in (\sigma, a]$, which means that

$$u'(\sigma) = \zeta'(\sigma), \quad u'(t) < \zeta'(t), \quad u(t) > \zeta(t) \quad \text{for } t \in (\sigma, a]. \tag{2.14}$$

Then

$$\begin{aligned} -(\varphi_p(u'(t)))' &= h(t)f(t,\gamma(t,u(t))) \\ &= h(t)f(t,\zeta(t)) \\ &\le -(\varphi_p(\zeta'(t)))' \quad \text{for } t \in (\sigma, a]. \end{aligned}$$

Integrating this from σ to $t \in (\sigma, a]$, we obtain

$$\varphi_p(u'(t)) \ge \varphi_p(\zeta'(t)), \quad \text{for } t \in (\sigma, a].$$

Since φ_p is monotone increasing, $u'(t) \ge \zeta'(t)$ for $t \in (\sigma, a]$, and it contradicts (2.14). Thus $u(t) \le \zeta(t)$ for $t \in [0,1]$. In a similar manner, $\psi(t) \le u(t)$ for $t \in [0,1]$ can be proved, and thus the proof is complete. $\square$

3. Main Results

First, we give a list of assumptions which will be used in this section:

(A2') $\int_0^1 (1-s)k(s)ds \in [0, 1/2)$;

(A3) there is a compact interval $I := [\theta_1, \theta_2] \subset (0,1)$ such that

$$m_h := \min_{t\in I} h(t) > 0;$$

(A4) $f_0 := \lim_{s\to 0^+} f(t,s)/s^{p-1} = 0$ uniformly in $t \in [0,1]$;

(A5) there exists $\bar{B} > 0$ such that

$$f(t,s) \ge A_\infty s^{p-1} \quad \text{for } s \ge \bar{B} \text{ uniformly in } t \in I = [\theta_1, \theta_2],$$

where

$$A_\infty := \frac{1}{m_h}\Big(\frac{\pi_p}{\theta_2-\theta_1}\Big)^p > 0, \quad \pi_p := \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}.$$

Remark 3.1. (1) Assumption (A4) implies that $f(\cdot, 0) \equiv 0$. (2) Note that if $k(t) = t^\alpha$ for $t \in [0,1]$, then (A2') holds for any $\alpha > 0$.

Lemma 3.2. *Assume that* (A2), (A3), (A5) *hold. Then* $\|u\|_\infty < \bar{B}/\theta$ *for any solutions* u *to problem* (1.4) *with* $\mu \in \mathbb{R}_+$. *Here* $\bar{B}$ *is the constant in* (A5) *and* $\theta := \min\{\theta_1, 1-\theta_2\} > 0$.

Proof. Assume on the contrary that there exists $\mu_0 \in \mathbb{R}_+$ such that (1.4) with μ_0 instead of μ has a positive solution u_0 satisfying $\|u_0\|_\infty \ge \bar{B}/\theta$. It follows from the concavity of u_0 that

$$u_0(t) \ge \theta\|u_0\|_\infty \ge \bar{B}, \quad t \in I = (\theta_1, \theta_2),$$

which implies that, by (A5),

$$(\varphi_p(u_0'(t)))' + A_\infty m_h \varphi_p(u_0(t)) \le 0, \quad t \in I = (\theta_1, \theta_2).$$

It is easy to check that $w(t) = S_q\big(\frac{\pi_p}{\theta_2-\theta_1}(t-\theta_1)\big)$ is a solution of

$$(\varphi_p(w'(t)))' + \left(\frac{\pi_p}{\theta_2-\theta_1}\right)^p \varphi_p(w(t)) = 0, \quad t \in I = (\theta_1, \theta_2),$$
$$w(\theta_1) = w(\theta_2) = 0,$$

where S_q is the q-sine function with $\frac{1}{p} + \frac{1}{q} = 1$ (e.g., see [3, 30]). Applying $y = w$, $z = u_0$, $b_1 = (\frac{\pi_p}{\theta_2-\theta_1})^p$ and $b_2 = A_\infty m_h$ in (2.1) and integrating it from θ_1 to θ_2, we have

$$\int_{\theta_1}^{\theta_2} \Big(\big(\frac{\pi_p}{\theta_2-\theta_1}\big)^p - A_\infty m_h\Big)|w|^p dt > 0.$$

Thus

$$A_\infty < \frac{1}{m_h}\big(\frac{\pi_p}{\theta_2-\theta_1}\big)^p.$$

This contradicts the choice of A_∞. □

Lemma 3.3. *Assume that* (A2), (A3), (A5) *hold. Then there exists a constant* $\bar{\mu} > 0$ *such that there is no positive solution of* (2.2) *for* $\mu > \bar{\mu}$.

Proof. Suppose on the contrary that there there exists a sequence $\{\mu_n\}$ such that $\mu_n \to \infty$ and (1.4) with μ_n instead of μ has a positive solution u_n. Since $u_n(1) = \mu_n$, $\|u_n\|_\infty \to \infty$, and it contradicts Lemma 3.2. □

Lemma 3.4. *Assume that* (A2') *and* (A4) *hold. Then there exists a constant* $\mu_0 > 0$ *such that* (2.2)*, with* μ_0 *instead of* μ*, has a positive solution.*

Proof. Denote

$$L := \frac{1}{2(1-\int_0^1(1-s)k(s)ds)} > 0, \quad C_0 := 2 + \frac{\int_0^1(1+s)k(s)ds}{1-\int_0^1(1-s)k(s)ds} > 0,$$
$$Q := \max\Big\{ \int_0^{1/2} \varphi_p^{-1}\Big(\int_s^{1/2} h(\tau)d\tau\Big)ds,\ \int_{1/2}^1 \varphi_p^{-1}\Big(\int_{1/2}^s h(\tau)d\tau\Big)ds\Big\} > 0.$$

Since $0 < L < 1$ by (A2'), we can choose $\varepsilon_1 > 0$ such that

$$0 < \varepsilon_1 < \Big(\frac{1-L}{QC_0}\Big)^{p-1}. \tag{3.1}$$

Now consider a function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(t) := \max\{H^1(t), H^2(t)\} \text{ for } t \in \mathbb{R},$$

where

$$H^1(t) := \int_0^{1/2} \varphi_p^{-1}\Big((2L)^{p-1} + tC_0^{p-1}\int_s^{1/2} h(\tau)d\tau\Big)ds,$$
$$H^2(t) := \int_{1/2}^1 \varphi_p^{-1}\Big((2L)^{p-1} + tC_0^{p-1}\int_{1/2}^s h(\tau)d\tau\Big)ds.$$

Since H is continuous and increasing on $\mathbb{R}$ and $H(0) = L < 1$, we may choose $\varepsilon_2 > 0$ such that $H(\varepsilon_2) < 1$. Set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\} > 0$. From (A4), there exists $r > 0$ such that

$$f(t,s) \le \varepsilon s^{p-1} \quad \text{for } 0 \le s \le r \text{ and } t \in [0,1]. \tag{3.2}$$

Define $\mathcal{K}_1 := \{x \in \mathcal{K} : \|x\|_\infty < \frac{r}{C_0}\}$. For $0 < \mu_0 < \frac{r}{C_0}$, consider $T_{\mu_0} : \overline{\mathcal{K}_1} \to \mathcal{K}$ defined by $T_{\mu_0}v = T(\mu_0, v)$, then $T_{\mu_0} : \overline{\mathcal{K}_1} \to \mathcal{K}$ is completely continuous. If $v \in \partial\mathcal{K}_1$, then $\|v\|_\infty = \frac{r}{C_0}$ and by using $0 < \mu_0 < \frac{r}{C_0} = \|v\|_\infty$, we have

$$\begin{aligned} |v(t) + (1-t)\beta[\mu_0, v] + t\mu_0| &\le \|v\|_\infty + |\beta[\mu_0, v]| + \mu_0 \\ &< \Big(2 + \frac{\int_0^1(1+s)k(s)ds}{1-\int_0^1(1-s)k(s)ds}\Big)\|v\|_\infty \\ &= C_0\|v\|_\infty = r \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} |\beta[\mu_0, v] - \mu_0| &\le \frac{\int_0^1 k(s)|v(s)|ds + \mu_0(1-\int_0^1 k(s)ds)}{1-\int_0^1(1-s)k(s)ds} \\ &\le \frac{1}{1-\int_0^1(1-s)k(s)ds}\|v\|_\infty = 2L\|v\|_\infty. \end{aligned} \tag{3.4}$$

We only give the proof for the case that $M = M(\mu_0, v) \le \frac{1}{2}$, since the case $M = M(\mu_0, v) > 1/2$ can be proved in a similar manner. For $v \in \partial\mathcal{K}_1$ and $\mu_0 \in (0, \frac{r}{C_0})$, we have two cases: either (i) $\beta[\mu_0, v] - \mu_0 \ge 0$ or *(ii)* $\beta[\mu_0, v] - \mu_0 < 0$. First, we assume that $\beta[\mu_0, v] - \mu_0 \ge 0$. Recall that

$$F(\mu_0, v)(t) = f(t, v(t) + (1-t)\beta[\mu_0, v] + t\mu_0) \text{ for } t \in [0,1].$$

Then, by (3.1), (3.2), (3.3) and (3.4), we have

$$\begin{aligned}\|T_{\mu_0}v\|_\infty &= T(\mu_0,v)(M)\\ &\le (\beta[\mu_0,v]-\mu_0)M + \int_0^M \varphi_p^{-1}\Big(\int_s^M h(\tau)F(\mu_0,v)(\tau)d\tau\Big)ds\\ &\le \frac{1}{2}(\beta[\mu_0,v]-\mu_0) + \varepsilon^{\frac{1}{p-1}}C_0\|v\|_\infty \int_0^{1/2}\varphi_p^{-1}\Big(\int_s^{1/2} h(\tau)d\tau\Big)ds\\ &\le L\|v\|_\infty + \varepsilon^{\frac{1}{p-1}}C_0Q\|v\|_\infty\\ &= (L+\varepsilon^{\frac{1}{p-1}}C_0Q)\|v\|_\infty < \|v\|_\infty.\end{aligned}$$

Next, we assume that $\beta[\mu_0,v]-\mu_0<0$. Using the fact that $H^1(\varepsilon)\le H(\varepsilon)<1$, by the similar argument above, we have

$$\begin{aligned}\|T_{\mu_0}v\|_\infty &= T(\mu_0,v)(M)\\ &\le \int_0^M \varphi_p^{-1}\Big(\varphi_p(\mu_0-\beta[\mu_0,v]) + \int_s^M h(\tau)F(\mu_0,v)(\tau)d\tau\Big)ds\\ &\le \int_0^{1/2}\varphi_p^{-1}\Big((2L)^{p-1}+\varepsilon C_0^{p-1}\int_s^{1/2} h(\tau)d\tau\Big)ds\|v\|_\infty\\ &= H^1(\varepsilon)\|v\|_\infty < \|v\|_\infty.\end{aligned}$$

Thus $\|T_{\mu_0}v\|_\infty < \|v\|_\infty$ for $v\in\partial\mathcal{K}_1$ and $\mu_0\in(0,\frac{r}{C_0})$. In view of Theorem 2.5,

$$i(T_{\mu_0},\mathcal{K}_1,\mathcal{K})=1.$$

Since 0 is not a solution of (2.2) with μ_0 instead of μ, this problem has a positive solution in $\mathcal{K}_1$, and the proof is complete. □

Now we give the first main result.

Theorem 3.5. *Assume that* (A2'), (A3)–(A5) *hold. Then there exists a constant* $\mu^*>0$ *such that problem* (2.2) *has at least one positive solution for* $\mu\in(0,\mu^*]$ *and no positive solution for* $\mu>\mu^*$.

Proof. Let $\Lambda=\{\mu : \text{(2.2) has a positive solution}\}$ and $\mu^*=\sup\Lambda$. Then, by Lemma 3.3 and Lemma 3.4, $\Lambda\ne\varnothing$ and $0<\mu^*<\infty$. We show that (2.2), with μ^* instead of μ, has a positive solution. Indeed, there is a sequence $\{\mu_n\}$ in Λ such that $\mu_n\to\mu^*$, and let v_n be a positive solution of (2.2) with μ_n instead of μ. By Lemma 3.2, $\|v_n\|_\infty<C$ for some $C>0$. By complete continuity of T, $\{v_n\}$ has a convergent subsequence converging to, say v^* and v_* is a solution of (2.2) with μ^* instead of μ. Let $u_*:=L_2(\mu^*,v_*)$. Then u_* is a positive solution of (2.2) with μ^* instead of μ, and thus $\mu^*\in\Lambda$. For $0<\mu<\mu^*$, $\psi\equiv 0$ is a trivial subsolution of (2.2) by (A4), and the positive solution u^* of (2.2), with μ^* instead of μ, is a supersolution of (2.2). Then by Theorem 2.11, (2.2) has a positive solution u such that $0\le u\le u^*$ for $\mu\in(0,\mu^*)$, and thus the proof is complete. □

For a multiplicity result on positive solutions to problem (1.4), we need an additional assumption

(A6) there exists $\delta\in(0,1)$ such that $k(t)>0$ for $t\in(1-\delta,1)$ and for each $t\in[0,1]$, $f(t,u)$ is monotone increasing in $(0,\bar{B}/\theta]$ with respect to u, where $\bar{B}$ is the constant in (A5) and $\theta=\min\{\theta_1,1-\theta_2\}>0$.

Then we have the following main result in this paper.

Theorem 3.6. *Assume that* (A2'), (A3)–(A6) *hold. Then there exists a constant* $\mu^* > 0$ *such that problem* (1.4) *has at least two positive solutions for* $\mu \in (0, \mu^*)$, *one positive solution for* $\mu \in \{0, \mu^*\}$, *and no positive solution for* $\mu > \mu^*$.

Proof. Define $T_1 : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ by

$$T_1(\mu, u)(t) := T(\mu, L_1(\mu, u))(t) + (1 - t)\alpha[u] + t\mu$$

for $(\mu, u) \in \mathbb{R}_+ \times \mathcal{K}$ and $t \in [0, 1]$. Here, $L_1 : \mathbb{R}_+ \times \mathcal{K} \to C[0, 1]$ is the continuous mapping defined in (2.4), and it maps bounded sets in $\mathbb{R}_+ \times C[0, 1]$ into bounded sets in $C[0, 1]$. Then T_1 is completely continuous, $T_1(\mu, u)(0) = \alpha[u]$ and $T_1(\mu, u)(1) = \mu$. Moreover, if $T_1(\mu, u) = u$ in $\mathcal{K}$, then u is a nonnegative solution of (1.4).

Let $\mu_0 \in [0, \mu^*)$ be fixed, where μ^* is the constant defined in Theorem 3.5. Let u_* be a positive solution of problem (2.2) with μ^* instead of μ, and $\epsilon > 0$ be given. Set $\Gamma = \{u \in C[0, 1] : -\epsilon < u(t) < u_*(t),\ t \in [0, 1]\}$. Then Γ is an open set containing 0. We show that $T_1(\mu_0, u) \neq \nu u$ for all $u \in \mathcal{K} \cap \partial\Gamma$ and $\nu \geq 1$. Assume on the contrary that there exists $\nu \geq 1$ and $u \in \mathcal{K} \cap \partial\Gamma$ such that $T_1(\mu_0, u) = \nu u$. Then, u satisfies

$$\begin{gathered}(\varphi_p(u'(t)))' + \nu^{1-p} h(t) f(t, u(t)) = 0, \quad t \in (0, 1),\\ u(0) = \nu^{-1}\alpha[u], \quad u(1) = \nu^{-1}\mu_0.\end{gathered}$$

From the facts that $u \in \mathcal{K} \cap \partial\Gamma$ and $0 \leq u(1) < \mu^* = u_*(1)$, it follows that $0 \leq u(t) \leq u_*(t)$ for all $t \in [0, 1]$ and $0 < u(0) < \alpha[u_*] = u_*(0)$ by the fact that $k(t) > 0$ for t near 1. Then there exists $t_0 \in (0, 1)$ such that $u(t_0) = u_*(t_0)$ and $u'(t_0) = u_*'(t_0)$. By Lemma 3.2 and (A6), $f(t, u(t)) \leq f(t, u_*(t))$ for $t \in [0, 1]$. Thus, for $t \in [0, 1]$, we have

$$(\varphi_p(u_*'(t)))' - (\varphi_p(u'(t)))' = -h(t)(f(t, u_*(t)) - \nu^{1-p} f(t, u(t))) \leq 0.$$

For $t \in (t_0, 1)$, integrating it from t_0 to t,

$$\varphi_p(u_*'(t)) \leq \varphi_p(u'(t)),$$

and $u_*'(t) \leq u'(t)$ for $t \in (t_0, 1)$. Integrating this again from t_0 to 1, $u_*(1) \leq u(1)$, which contradicts the fact that $u(1) < u_*(1)$. Thus, by Theorem 2.5,

$$i(T_1(\mu_0, \cdot), \mathcal{K} \cap \Gamma, \mathcal{K}) = 1. \tag{3.5}$$

Thus (2.2) has a positive solution in Γ for all $\mu_0 \in (0, \mu^*)$.

On the other hand, by Lemma 3.3, we know that there is $\mu_1 > 0$ such that (2.2) has no positive solution at $\mu = \mu_1$. This implies that $T_1(\mu_1, \cdot)$ has no fixed point in $\mathcal{K}$. Thus, by Solution property, for any open set $\mathcal{U}$ in $C[0, 1]$, we have

$$i(T_1(\mu_1, \cdot), \mathcal{K} \cap \mathcal{U}, \mathcal{K}) = 0. \tag{3.6}$$

Then by Lemma 3.2, we may choose $R > 0$ such that all possible solutions u of (2.2) for all $\mu \in \mathbb{R}_+$ satisfy that $u \in B_R$ and $\overline{\Gamma} \subset B_R$. Here B_R is an open ball with center 0 and radius R. Define a homotopy $g : [0, 1] \times (\overline{B}_R \cap \mathcal{K}) \to \mathcal{K}$ by

$$g(\tau, u) = T_1(\tau\mu_1 + (1 - \tau)\mu_0, u).$$

Then, g is completely continuous on $[0, 1] \times \mathcal{K}$. Furthermore, by Lemma 3.2, $g(\tau, u) \neq u$, for all $(\tau, u) \in [0, 1] \times (\partial B_R \cap \mathcal{K})$. Thus by Homotopy invariance property and (3.6), we have

$$i(T_1(\mu_0, \cdot), B_R \cap \mathcal{K}, \mathcal{K}) = i(T_1(\mu_1, \cdot), B_R \cap \mathcal{K}, \mathcal{K}) = 0. \tag{3.7}$$

From (3.5) and (3.7) with Additivity property, we have

$$i(T_1(\mu_0,\cdot),(B_R\setminus\overline{\Gamma})\cap\mathcal{K},\mathcal{K})=-1.$$

Therefore (2.2), with μ_0 instead of μ, has another positive solution in $(B_R\setminus\overline{\Gamma})\cap\mathcal{K}$ for all $\mu_0\in[0,\mu^*)$, and this completes the proof in view of Theorem 3.5. □

4. Applications

In this section, we illustrate problem (1.1)-(1.2)-(1.3), introduced above. By applying consecutive changes of variables, $r=|x|$, $w(r)=z(x)$ and $t=(\frac{r}{r_0})^{\frac{-N+p}{p-1}}$, $u(t)=w(r)$, problem (1.1)-(1.2) is equivalently transformed into problem (1.4), where f, h and k are given by

$$\begin{gathered}
f(t,u)=f_1\Big(r_0t^{\frac{-(p-1)}{N-p}},u\Big),\\
h(t)=\Big(\frac{p-1}{N-p}\Big)^p r_0^p t^{\frac{-p(N-1)}{N-p}}K\Big(r_0t^{\frac{-(p-1)}{N-p}}\Big),\\
k(t)=\Big(\frac{p-1}{N-p}\Big)r_0^N t^{\frac{-p(N-1)}{N-p}}l\Big(r_0t^{\frac{-(p-1)}{N-p}}\Big).
\end{gathered}$$

If $K=K(r)$ satisfies (A1), then there exists $a\in[0,p-1)$ such that $\int_0^1 s^a h(s)ds<\infty$, which implies that $h\in\mathcal{A}$. In fact, since $a\in[0,p-1)$, $-\frac{a}{p-1}>-1$ and

$$\begin{aligned}
\int_0^{1/2}\varphi_p^{-1}\Big(\int_s^{1/2}h(\tau)d\tau\Big)ds &\le \int_0^{1/2}\varphi_p^{-1}\Big(\int_s^{1/2}(\frac{\tau}{s})^a h(\tau)d\tau\Big)ds\\
&\le \varphi_p^{-1}\Big(\int_0^{1/2}\tau^a h(\tau)d\tau\Big)\int_0^{1/2}s^{-\frac{a}{p-1}}ds<\infty.
\end{aligned}$$

Also, if l satisfies

- (A7) $1-2\int_{r_0}^\infty\big(1-(\frac{r}{r_0})^{\frac{p-N}{p-1}}\big)l(r)r^{N-1}dr>0$, and there exists $\delta_1>0$ such that $l(r)>0$ for $r\in(r_0,r_0+\delta_1)$,

then (A2') holds and $k(t)>0$ for all t near 1.

On the other hand, if $K=K(r)$ and $f_1=f_1(r,s)$ satisfy

- (A8) there is a compact interval $J:=[\rho_1,\rho_2]\subset(r_0,\infty)$ such that
$$m_K:=\min_{r\in J}K(r)>0;$$
- (A4') $\lim_{s\to0^+}\frac{f_1(r,s)}{s^{p-1}}=0$ uniformly in $r\in[r_0,\infty)$;
- (A5') there exists $\bar{B}>0$ such that
$$f_1(r,s)\ge A_1s^{p-1}\quad\text{for } s\ge\bar{B}\text{ uniformly in } r\in J,$$
where
$$A_1:=\frac{1}{m_K}\Big(\frac{\pi_p}{\hat\theta_2-\hat\theta_1}\Big)^p>0,\quad \hat\theta_1:=\Big(\frac{\rho_2}{r_0}\Big)^{\frac{-N+p}{p-1}},\quad \hat\theta_2:=\Big(\frac{\rho_1}{r_0}\Big)^{\frac{-N+p}{p-1}};$$
- (A6') for each $r\in[r_0,\infty)$, $f_1(r,u)$ is monotone increasing in $(0,\bar{B}/\hat\theta)$ with respect to u, where $\hat\theta:=\min\{\hat\theta_1,1-\hat\theta_2\}$.

then (A3) holds and (A4)–(A6) hold. Thus, in view of Theorem 3.6, we have the following corollary.

Corollary 4.1. *Assume that* (A1), (A7), (A8), (A4'), (A5'), (A6') *hold. Then there exists a constant* $\mu^* > 0$ *such that problem* (1.1)-(1.2) *has at least two positive radial solutions for* $\mu \in (0, \mu^*)$*, one positive radial solution for* $\mu \in \{0, \mu^*\}$*, and no positive radial solution for* $\mu > \mu^*$*.*

In a similar manner, by applying consecutive changes of variables, $r = |x|$, $w(r) = z(x)$ and $t = 1 - \big(\frac{r}{r_0}\big)^{\frac{-N+p}{p-1}}$, $u(t) = w(r)$, problem (1.1), (1.3) is equivalently transformed into problem (1.4), where f, h and k are given by

$$f(t,u) = f_1\Big(r_0(1-t)^{\frac{-(p-1)}{N-p}}, u\Big),$$
$$h(t) = \Big(\frac{p-1}{N-p}\Big)^p r_0^p (1-t)^{\frac{-p(N-1)}{N-p}} K\Big(r_0(1-t)^{\frac{-(p-1)}{N-p}}\Big),$$
$$k(t) = \Big(\frac{p-1}{N-p}\Big) r_0^N (1-t)^{\frac{-p(N-1)}{N-p}} l\Big(r_0(1-t)^{\frac{-(p-1)}{N-p}}\Big).$$

Then we have the following corollary.

Corollary 4.2. *Assume that* (A1), (A8), (A4'), (A5'), (A6') *with*

$$\hat\theta_1 = 1 - \Big(\frac{\rho_1}{r_0}\Big)^{\frac{-N+p}{p-1}} \quad \textit{and} \quad \hat\theta_2 = 1 - \Big(\frac{\rho_2}{r_0}\Big)^{\frac{-N+p}{p-1}}.$$

Also assume

(A7') $1 - 2\int_{r_0}^{\infty} (\frac{r}{r_0})^{\frac{p-N}{p-1}} l(r) r^{N-1} dr > 0$*, and there exists* $r_1 > 0$ *such that* $l(r) > 0$ *for* $r \in (r_1, \infty)$*.*

Then there exists a constant $\mu^* > 0$ *such that problem* (1.1)*,* (1.3) *has at least two positive radial solutions for* $\mu \in (0, \mu^*)$*, one positive radial solution for* $\mu \in \{0, \mu^*\}$*, and no positive radial solution for* $\mu > \mu^*$*.*

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