Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 133, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

POSITIVE SOLUTIONS FOR SYSTEMS OF COMPETITIVE FRACTIONAL DIFFERENTIAL EQUATIONS

MAJDA CHAIEB, ABDELWAHEB DHIFLI, MALEK ZRIBI

Abstract. Using potential theory arguments, we study the existence and boundary behavior of positive solutions in the space of weighted continuous functions, for the fractional differential system

$$D^\alpha u(x) + p(x)u^{a_1}(x)v^{b_1}(x) = 0 \quad \text{in } (0,1), \qquad \lim_{x\to 0^+} x^{1-\alpha}u(x) = \lambda > 0,$$
$$D^\beta v(x) + q(x)v^{a_2}(x)u^{b_2}(x) = 0 \quad \text{in } (0,1), \qquad \lim_{x\to 0^+} x^{1-\beta}v(x) = \mu > 0,$$

where $\alpha, \beta \in (0,1)$, $a_i > 1$, $b_i \geq 0$ for $i \in \{1,2\}$ and p, q are positive continuous functions on $(0,1)$ satisfying a suitable condition relying on fractional potential properties.

1. Introduction

Fractional differential equations involving Riemann-Liouville differential operators, D^α of fractional order $0 < \alpha < 1$, are gaining much importance and are emerging as an interesting field of research. In fact, fractional calculus has numerous applications in various disciplines of mathematical modeling of physical, biological phenomena and engineering such as control of dynamical systems, porous media, electrochemistry, viscoelasticity, electromagnetic, etc. Also it provides an excellent tool to describe the hereditary properties of various materials and processes. Concerning the development of theory methods and applications of fractional calculus, we refer to [8, 10, 11, 13, 14, 16, 21, 22, 23, 25, 26].

Therefore, this theory has been developed very quickly and the interest in the existence of solutions of fractional differential equations has recently attracted a considerable attention of researchers (see for instance [4, 7, 13, 15, 22, 27, 30, 31, 32] and the references therein).

The study of coupled systems with fractional differential equations is also important as such systems occur in various problems of applied nature (see [1, 6, 9, 12, 18, 28, 29] and references therein).

For a measurable function v, the Riemann-Liouville fractional integral $I^\alpha v$ and derivative $D^\alpha v$ of order $\alpha > 0$ are respectively defined by

$$I^\alpha v(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}v(t)dt$$

2010 *Mathematics Subject Classification.* 26A33, 31B25, 34A12, 34B18.
Key words and phrases. Fractional differential system; positive solution; potential theory; Schauder fixed point theorem.

Submitted February 27, 2016 Published June 7, 2016.

and

$$\begin{aligned} D^\alpha v(x) &= \frac{1}{\Gamma(n-\alpha)}\big(\frac{d}{dx}\big)^n \int_0^x (x-t)^{n-\alpha-1} v(t)dt \\ &= \big(\frac{d}{dx}\big)^n I^{n-\alpha} v(x), \end{aligned}$$

provided that the integrals exist. Here $[\alpha]$ means the integer part of the number α, and Γ is the Euler Gamma function.

Moreover if v is an integrable function in $(0,1)$, we have

$$I^\alpha I^\beta v(x) = I^{\alpha+\beta} v(x) \quad \text{for } x \in (0,1],\ \alpha+\beta \geq 1; \tag{1.1}$$

$$D^\alpha I^\alpha v(x) = v(x), \quad \text{for a.e. } x \in (0,1),\ \alpha > 0. \tag{1.2}$$

See [13, 22] for more information on fractional derivatives and integrals.

The solvability of nonlinear fractional differential equation of order $0 < \alpha < 1$ of the form

$$D^\alpha u = \varphi(\cdot, u), \tag{1.3}$$

in $(0,\infty)$ or in an interval $(0,h)$ with $h > 0$ and φ is a real function, has attracted many researchers. Several existence and nonexistence results have appeared [7, 17, 30, 31, 32].

When φ is a nonnegative continuous function, many authors proved existence and uniqueness results for (1.3) with suitable initial value condition $\lim_{x\to 0^+} x^{1-\alpha}u(x) = \lambda$, $\lambda \in \mathbb{R}$, (see for example [7, 20, 31, 32]). In [20], the authors considered (1.3) in $(0,1)$, with the nonlinearity $\varphi(x,u) = p(x)u^\sigma$ where p is a positive measurable function on $(0,1)$ and $\sigma < 1$. More precisely, they studied the initial-value problem

$$\begin{gathered} D^\alpha u = p(x)u^\sigma, \quad \text{in } (0,1),\ \sigma < 1 \\ \lim_{x\to 0^+} x^{1-\alpha}u(x) = 0. \end{gathered} \tag{1.4}$$

Without the continuity condition on φ imposed in [7, 31, 32], the authors in [20] proved the existence and uniqueness, and properties of the boundary behaviour of a positive solution for problem (1.4) in the weighted space of continuous functions $C_{1-\alpha}([0,1])$.

In this article, we use the following notation: For $r > 0$, we use $C_r([0,1])$ to denote the set of functions f such that $t \mapsto t^r f(t)$ is continuous in $[0,1]$. We endow the set $C_r([0,1])$ with the norm $\|f\|_r = \sup_{t\in[0,1]} t^r|f(t)|$. We denote by $C_0((0,1])$ the class of all continuous functions in $(0,1]$ vanishing continuously at 0^+.

Also, we refer to $B^+((0,1))$ the collection of all nonnegative measurable functions in $(0,1)$ and $L^1((0,1))$ the collection of all integrable functions in $(0,1)$.

For $\alpha \in (0,1)$, we put ω_α the function defined in $(0,1]$ by $\omega_\alpha(x) = x^{\alpha-1}$ and we enter the functional class

$$\mathcal{H}_\alpha = \{f \in B^+((0,1)) : x \to x^{1-\alpha}(I^\alpha f)(x) \in C_0((0,1])\}.$$

As typical example of functions in $\mathcal{H}_\alpha$, we have

Example 1.1. Let $\lambda < 1$ and f be an integrable function in $(0,1)$ such that $0 \leq f \leq ct^{-\lambda}$, $c > 0$ then $f \in \mathcal{H}_\alpha$, for each $\alpha \in (0,1)$.

Remark 1.2. By [7], we remark that for $\alpha \in (0,1)$, the function ω_α is the unique solution in $C_{1-\alpha}([0,1])$ of the Dirichlet fractional problem

$$D^\alpha u = 0, \quad \text{in } (0,1),$$

$$\lim_{x\to 0^+} x^{1-\alpha}u(x) = 1.$$

In this article, we analyze (1.3), when φ is a nonpositive measurable function, of the form $\varphi(x,u) = -p(x)u^\sigma$, where $\sigma > 1$ and p satisfies the assumption:

(H1) p is a nonnegative measurable function in $(0,1)$ such that $p\omega_\alpha^\sigma \in \mathcal{H}_\alpha$, $\alpha \in (0,1)$.

More precisely, we study the semilinear problem

$$\begin{gathered} D^\alpha u + p(x)u^\sigma = 0, \quad \text{in } (0,1) \\ \lim_{x\to 0^+} x^{1-\alpha}u(x) = \lambda, \end{gathered} \tag{1.5}$$

where $\lambda > 0$, $\sigma > 1$ and p satisfies (H1). Our first goal is to prove the following result.

Theorem 1.3. *Under assumption* (H1)*, problem* (1.5) *has a unique solution u in $C_{1-\alpha}([0,1])$. Moreover, for each $x \in (0,1]$, we have*

$$c_0\lambda\omega_\alpha(x) \le u(x) \le \lambda\omega_\alpha(x),$$

where $c_0 = \exp(-\sigma\lambda^{\sigma-1}\|I^\alpha(p\omega_\alpha^\sigma)\|_{1-\alpha})$.

Motivated by recent works dealing with coupled systems with fractional differential equations, our second goal is to study, the semilinear fractional system

$$\begin{gathered} D^\alpha u + p(x)u^{a_1}v^{b_1} = 0, \quad \text{in } (0,1) \\ D^\beta v + q(x)v^{a_2}u^{b_2} = 0, \quad \text{in } (0,1) \\ \lim_{x\to 0^+} x^{1-\alpha}u(x) = \lambda > 0, \\ \lim_{x\to 0^+} x^{1-\beta}v(x) = \mu > 0, \end{gathered} \tag{1.6}$$

where $\alpha,\beta \in (0,1)$, $a_i > 1$, $b_i \ge 0$ for $i \in \{1,2\}$ and p,q satisfy the assumption

(H2) $p,q \in C((0,1])$ such that $p\omega_\alpha^{a_1}\omega_\beta^{b_1} \in \mathcal{H}_\alpha$ and $q\omega_\alpha^{b_2}\omega_\beta^{a_2} \in \mathcal{H}_\beta$.

An iterative argument combined with Theorem 1.3 yields to the second main result.

Theorem 1.4. *Under assumption* (H2)*, system* (1.6) *has a positive continuous solution (u,v) in $C_{1-\alpha}([0,1]) \times C_{1-\beta}([0,1])$. Moreover, there exist $c_1, c_2 \in (0,1)$ such that for each $x \in (0,1]$, we have*

$$\begin{gathered} c_1\lambda\omega_\alpha(x) \le u(x) \le \lambda\omega_\alpha(x), \\ c_2\mu\omega_\beta(x) \le v(x) \le \mu\omega_\beta(x). \end{gathered}$$

The outline of this article is as follows. In section 2, we give some preliminary results related to potential theory associated with D^α. In section 3, we prove Theorem 1.3 by converting problem (1.5) into a suitable integral equation and then using potential theory tools. In the last section, inspired by techniques used in [2] and using Theorem 1.3, we prove Theorem 1.4.

Throughout this paper, the letter c is a generic positive constant which may vary from line to line.

2. Potential theory associated with D^α

In this section We present some well known properties, pertaining with potential theory associated with D^α. For more details see [3, 7, 20, 24].

2.1. **The semi-group $(P_t^\alpha)_{t>0}$.** Let $(P_t)_{t>0}$ be the semi group of translation to the left, defined on $B^+((0,1))$ by

$$P_t f(x) = 1_{[0,x)}(t) f(x-t), \quad x \in (0,1).$$

The infinitesimal generator of $(P_t)_{t>0}$ is the derivative operator $\frac{d}{dx}$.

Let $(\eta_t^\alpha)_{t>0}$ be the convolution semi group of probability measures defined on $(0,\infty)$ and satisfying for every $t, s > 0$,

$$\int_0^\infty \eta_t^\alpha(u) e^{-su} du = e^{-ts^\alpha} \quad \text{and} \quad \int_0^\infty \eta_s^\alpha(t) ds = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}.$$

Subordinating $(P_t)_{t>0}$ by means of $(\eta_t^\alpha)_{t>0}$, we obtain the semi group $(P_t^\alpha)_{t>0}$ defined on $B^+((0,1))$ by

$$P_t^\alpha f(x) = \int_0^\infty P_s f(x) \eta_t^\alpha(s) ds, \quad x \in (0,1).$$

The infinitesimal generator associated with the semi group $(P_t^\alpha)_{t>0}$ is the fractional power

$$\big(\frac{d}{dx}\big)^\alpha = D^\alpha.$$

Indeed, it is known from [24] that for every function ϕ of class C^∞ with compact support in $(0,1)$.

$$\big(\frac{d}{dx}\big)^\alpha \phi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty t^{-\alpha-1}(\phi(x) - P_t\phi(x)) dt,$$

which means that

$$\big(\frac{d}{dx}\big)^\alpha \phi(x) = \frac{-\alpha}{\Gamma(1-\alpha)} \int_0^\infty t^{-\alpha-1}\Big(\int_0^t \frac{d}{ds} P_s\phi(x) ds\Big) dt.$$

Then by Fubini's theorem, we deduce that

$$\begin{aligned}
\big(\frac{d}{dx}\big)^\alpha \phi(x) &= \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \frac{d}{ds} P_s\phi(x) ds \\
&= \frac{-1}{\Gamma(1-\alpha)} \int_0^x s^{-\alpha} \frac{d}{ds} \phi(x-s) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^x s^{-\alpha} \phi'(x-s) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} \phi'(t) dt \\
&= I^{1-\alpha}\phi'(x).
\end{aligned}$$

On the other hand, we know that $D^\alpha\phi(x) = I^{1-\alpha}\phi'(x)$. Hence we obtain that

$$\big(\frac{d}{dx}\big)^\alpha \phi(x) = D^\alpha \phi(x).$$

In what follows, we recall the definition of excessive functions with respect to $(P_t^\alpha)_{t>0}$.

Definition 2.1. A function v in $B^+((0,1))$ is said to be excessive with respect to $(P_t^\alpha)_{t>0}$ if v satisfies

$$P_t^\alpha v(x) \le v(x), \quad t > 0,\ x \in (0,1)$$

and $\lim_{t\downarrow 0} P_t^\alpha v(x) = v(x)$.

We use $\mathcal{S}^\alpha$ to denote the cone of all excessive functions with respect to $(P_t^\alpha)_{t>0}$.

Example 2.2. The function w_α is excessive with respect to $(P_t^\alpha)_{t>0}$. Indeed, for $x \in (0,1)$ we have

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)} P_t^\alpha w_\alpha(x) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty P_s w_\alpha(x) \eta_t^\alpha(s) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \eta_t^\alpha(s) ds \\
&= I^\alpha \eta_t^\alpha(x) \\
&= \int_0^\infty P_s^\alpha \eta_t^\alpha(x) ds \\
&= \int_0^\infty \Big(\int_0^\infty P_z \eta_t^\alpha(x) \eta_s^\alpha(z) dz \Big) ds \\
&= \int_0^\infty (\int_0^x \eta_t^\alpha(x-z) \eta_s^\alpha(z) dz) ds \\
&= \int_0^\infty \eta_{t+s}^\alpha(x) ds \\
&= \int_t^\infty \eta_s^\alpha(x) ds.
\end{aligned}$$

Then it follows that $P_t^\alpha w_\alpha(x) \leq \Gamma(\alpha) \int_0^\infty \eta_s^\alpha(x) ds = w_\alpha(x)$ and $\lim_{t\to 0} P_t^\alpha w_\alpha(x) = w_\alpha(x)$.

2.2. **The potential kernel I^α.** Let $f \in B^+((0,1))$. The potential of f associated with $(P_t^\alpha)_{t>0}$ is given by

$$\int_0^\infty P_t^\alpha f(x) dt = \int_0^1 G_\alpha(x,y) f(y) dy,$$

where $G_\alpha(x,y)$ is the Green function associated with $(P_t^\alpha)_{t>0}$ given on $(0,1)\times(0,1)$ by

$$G_\alpha(x,y) = \frac{1}{\Gamma(\alpha)} 1_{(0,x)}(y)(x-y)^{\alpha-1}.$$

So we deduce that the potential kernel associated with $(P_t^\alpha)_{t>0}$ is none other the operator I^α on $B^+((0,1))$.

It is clear that the Green function G_α is lower semi-continuous on $(0,1)\times(0,1)$, which implies that for $f \in B^+((0,1))$, $I^\alpha f$ is also lower semi-continuous on $(0,1]$. Moreover, since for $y \in (0,1)$, the function $x \mapsto G_\alpha(x,y)$ is in $\mathcal{S}^\alpha$ (see [3]), it is the same for $I^\alpha f$, for $f \in B^+((0,1))$.

Proposition 2.3 ([7])**.** *Let f be a function in $C((0,1]) \cap L^1((0,1))$ such that $D^\alpha f$ belongs to $C((0,1]) \cap L^1((0,1))$. Then there exists a unique constant c such that for $x \in (0,1]$*

$$I^\alpha D^\alpha f(x) = f(x) + c\omega_\alpha(x).$$

Proposition 2.4. *If f and g are in $B^+((0,1))$ such that $g \leq f$ and $I^\alpha f \in C((0,1])$, then $I^\alpha g$ is also in $C((0,1])$.*

Proof. Let $\theta \in B^+((0,1))$ such that $f = g + \theta$. So, we have $I^\alpha f = I^\alpha g + I^\alpha \theta$. Now since $I^\alpha\theta$ and $I^\alpha g$ are lower semi-continuous in $(0,1]$, we deduce that $I^\alpha g \in C((0,1])$. $\square$

The potential kernel I^α satisfies the complete maximum principle. That is, for each function $f \in B^+((0,1))$ and $v \in \mathcal{S}^\alpha$ such that $I^\alpha f \leq v$ in $\{f > 0\}$, we have $I^\alpha f \leq v$ in $(0,1)$; see [3, chap.2, proposition 7.1]. Consequently, we deduce the following result.

Proposition 2.5. *Let $h \in B^+((0,1))$ and $v \in \mathcal{S}^\alpha$. Let ω be a Borel measurable function in $(0,1)$ such that $I^\alpha(h|\omega|) < \infty$ and $v = \omega + I^\alpha(h\omega)$. Then ω satisfies*

$$0 \leq \omega \leq v.$$

Proof. Put $\omega^+ = \sup(\omega, 0)$ and $\omega^- = \sup(-\omega, 0)$. Since $I^\alpha(h|\omega|) < \infty$, we have

$$I^\alpha(h\omega^+) \leq v + I^\alpha(h\omega^-) \quad \text{in } \{\omega > 0\} = \{\omega^+ > 0\}.$$

Then we deduce by the complete maximum principle that

$$I^\alpha(h\omega^+) \leq v + I^\alpha(h\omega^-) \quad \text{in } (0,1).$$

That is,

$$I^\alpha(h\omega) \leq v = \omega + I^\alpha(h\omega).$$

Hence, we obtain $0 \leq \omega \leq \omega + I^\alpha(h\omega) = v$. $\square$

2.3. **The resolvent $(V_h^\alpha)_h$.** Let $(X_t^\alpha,\ t > 0)$ be the Markov process associated with the semigroup $(P_t^\alpha)_{t>0}$ and E^x is the expectation with respect to $(X_t^\alpha, t > 0)$ starting from x. For $h \in B^+((0,1))$, we define the potential kernel V_h^α by

$$V_h^\alpha f(x) := \int_0^\infty E^x(e^{-\int_0^t h(X_s^\alpha)ds} f(X_t^\alpha))dt, \quad x \in (0,1). \tag{2.1}$$

We note that for $h = 0$, we find again the potential kernel I^α. In the remaining of the paper, we use the notation

$$V^\alpha := V_0^\alpha = I^\alpha.$$

If $h \in B^+((0,1))$ satisfies $V^\alpha h < \infty$, we have the following resolvent equation (see [5, 19])

$$V^\alpha = V_h^\alpha + V_h^\alpha(hV^\alpha) = V_h^\alpha + V^\alpha(hV_h^\alpha). \tag{2.2}$$

In particular, for each function u in $B^+((0,1))$ such that $V^\alpha(hu) < \infty$, we have

$$(I - V_h^\alpha(h.))(I + V^\alpha(h.))u = (I + V^\alpha(h.))(I - V_h^\alpha(h.))u = u. \tag{2.3}$$

Lemma 2.6. *Let $h \in B^+((0,1))$ such that $V^\alpha h < \infty$ and $v \in S^\alpha$. Then for each $x \in (0,1)$ such that $0 < v(x) < \infty$, we have*

$$\exp\Big(-\big(\frac{V^\alpha(hv)}{v}\big)(x)\Big)v(x) \leq v(x) - V_h^\alpha(hv)(x) \leq v(x).$$

In particular, if $\sup_{x\in(0,1)}(\frac{V^\alpha(hv)}{v})(x) < \infty$, then

$$mv(x) \leq v(x) - V_h^\alpha(hv)(x) \leq v(x),$$

where $m = \exp\big(-\sup_{x\in(0,1)}(\frac{V^\alpha(hv)}{v})(x)\big)$.

Proof. Let v be a function in S^α, then by [3], there exists a sequence of functions v_n in $B^+((0,1))$ such that $v = \sup_n V^\alpha v_n$. Let $x \in (0,1)$ satisfying $0 < v(x) < \infty$, then there exists $n_0 \in \mathbb{N}$ such that $0 < V^\alpha v_n(x) < \infty$, for each $n > n_0$. Now fix $n > n_0$ and consider the function θ defined on $[0,\infty)$ by $\theta(t) = V_{th}^\alpha v_n(x)$. Then by

(2.1), the function θ is completely monotone on $[0,\infty)$ and so $\log\theta$ is convex on $[0,\infty)$. Therefore,

$$\theta(0) \le \theta(1)\exp\Big(-\frac{\theta'(0)}{\theta(0)}\Big).$$

Which implies

$$V^\alpha v_n(x) \le V_h^\alpha v_n(x)\exp\Big(\frac{V^\alpha(hV^\alpha v_n)(x)}{V^\alpha v_n(x)}\Big).$$

Hence by (2.2)) we obtain

$$\begin{aligned}\exp\Big(-\frac{V^\alpha(hV^\alpha v_n)(x)}{V^\alpha v_n(x)}\Big)V^\alpha v_n(x) &\le V_h^\alpha v_n(x)\\ &= V^\alpha v_n(x) - V_h^\alpha(hV^\alpha v_n)(x) \le V^\alpha v_n(x).\end{aligned}$$

The result holds by letting $n\to\infty$. □

3. Proof of Theorem 1.3

Let p be a function satisfying (H1). We divide the proof into three steps.

3.1. **Converting to integral equation.**

We shall convert problem (1.5) into a suitable integral equation. This follows by the following Lemma.

Lemma 3.1. *Suppose that p satisfies* (H1) *and let u be a positive function in $C_{1-\alpha}([0,1])$. Then u is a solution of problem* (1.5) *if and only if u satisfies the integral equation*

$$u(x) + V^\alpha(pu^\sigma)(x) = \lambda\omega_\alpha(x), \quad x\in(0,1]. \tag{3.1}$$

Proof. Suppose that u satisfies (3.1). Then $u\le\lambda\omega_\alpha$ and so $V^\alpha(pu^\sigma)\le\lambda^\sigma V^\alpha(p\omega_\alpha^\sigma)$. Using (H1), this implies that $\lim_{x\to0^+}\frac{V^\alpha(pu^\sigma)(x)}{\omega_\alpha(x)}=0$ and

$$\int_0^1 (pu^\sigma)(t)dt \le \int_0^1 (1-t)^{\alpha-1}(pu^\sigma)(t)dt < \infty.$$

Returning to (3.1), we deduce that $\lim_{x\to0^+}x^{1-\alpha}u(x)=\lambda$. On the other hand applying D^α on both sides of (3.1), we conclude by (1.2) and Remark 1.2 that the function u satisfies the fractional equation $D^\alpha u + pu^\sigma = 0$. Hence u is a positive solution of problem (1.5)

Conversely, suppose that u is a positive solution of problem (1.5) in $C_{1-\alpha}([0,1])$. Then there exists a positive constant c such that $u\le c\omega_\alpha$ on $[0,1]$. Using (H1), we obtain that $pu^\sigma\in L^1((0,1))$ and $\lim_{x\to0^+}\frac{V^\alpha(pu^\sigma)(x)}{\omega_\alpha(x)}=0$. So the function $u+V^\alpha(pu^\sigma)$ satisfies

$$\begin{gathered}D^\alpha\big(u+V^\alpha(pu^\sigma)\big)=0, \quad \text{in } (0,1),\\ \lim_{x\to0^+}x^{1-\alpha}\big(u+V^\alpha(pu^\sigma)\big)(x)=\lambda.\end{gathered}$$

Using again Remark 1.2, we deduce that u satisfies (3.1). This completes the proof. □

3.2. **Existence result.** We aim to show an existence result for the integral equation (3.1). We define the function θ by $\theta(x) := \sigma\lambda^{\sigma-1}p(x)\omega_\alpha^{\sigma-1}(x)$, for $x \in (0,1)$. Using (H1), we deduce that $\frac{V^\alpha(\theta\omega_\alpha)}{\omega_\alpha}$ is a positive function in $C_0((0,1])$. This implies in particular that

$$V^\alpha(\theta)(x) \leq x^{1-\alpha}V^\alpha(\theta\omega_\alpha)(x) < \infty$$

Consider the closed convex set

$$\Gamma = \{u \in B^+((0,1)) : c_0\lambda\omega_\alpha \leq u \leq \lambda\omega_\alpha\},$$

where c_0 is the constant given in Theorem 1.3. Let T be the operator defined on Γ by

$$Tu = \lambda(\omega_\alpha - V_\theta^\alpha(\theta\omega_\alpha)) + V_\theta^\alpha(\theta u - pu^\sigma).$$

We claim that Γ is invariant under the operator T. Indeed, for $u \in \Gamma$, we have $u \leq \lambda\omega_\alpha$ and consequently $Tu \leq \lambda\omega_\alpha - V_\theta^\alpha(pu^\sigma) \leq \lambda\omega_\alpha$. Now since for each $x \in (0,1]$, the function $t \to \theta(x)t - p(x)t^\sigma$ is nondecreasing on $[0, \lambda\omega_\alpha(x)]$, we deduce that $\theta u - pu^\sigma \geq 0$. This implies that $Tu \geq \lambda\omega_\alpha - V_\theta^\alpha(\lambda\theta\omega_\alpha)$. Now, since $\omega_\alpha \in \mathcal{S}^\alpha$ we obtain by Lemma 2.6 that

$$c_0\omega_\alpha(x) \leq \omega_\alpha(x) - V_\theta^\alpha(\theta\omega_\alpha)(x) \leq \omega_\alpha(x), \quad x \in (0,1].$$

Hence $Tu \geq c_0\lambda\omega_\alpha$. This shows that $T\Gamma \subset \Gamma$.

Next, we prove that the operator T has a fixed point in Γ. Let u and v be functions in Γ such that $u \geq v$. Then we have $V_\theta^\alpha(\theta u - pu^\sigma) \geq V_\theta^\alpha(\theta v - pv^\sigma)$, which implies that $Tu \geq Tv$. Thus T is nondecreasing on Γ.

Now, consider the sequence (u_n) defined by

$$u_0 = c_0\lambda\omega_\alpha \quad \text{and} \quad u_{n+1} = Tu_n \quad \text{for } , n \in \mathbb{N}.$$

Then, using that Γ is invariant under T and the monotonicity of T, we deduce that

$$c_0\lambda\omega_\alpha \leq u_0 \leq u_1 \leq \dots \leq u_n \leq \lambda\omega_\alpha.$$

Hence the sequence (u_n) converges to a measurable function u in Γ. By the monotone convergence theorem, we deduce that u satisfies the equation

$$u = \lambda\omega_\alpha - V_\theta^\alpha(\lambda\theta\omega_\alpha) + V_\theta^\alpha(\theta u - pu^\sigma);$$

that is,

$$(I - V_\theta^\alpha(\theta.))u + V_\theta^\alpha(pu^\sigma) = (I - V_\theta^\alpha(\theta.))(\lambda\omega_\alpha). \tag{3.2}$$

Applying the operator $(I + V^\alpha(\theta.))$ on both sides of (3.2), we deduce by (2.2) and (2.3) that u satisfies (3.1).

Since $u \leq \lambda\omega_\alpha$, we deduce by (H1) and Proposition 2.4, that $V^\alpha(pu^\sigma) \in C_{1-\alpha}([0,1])$. Hence according to (3.1), the function $u \in C_{1-\alpha}([0,1])$. Finally, by Lemma 3.1, we conclude that u is a positive continuous solution of (1.5).

3.3. **Uniqueness result.** Let $u, v \in C_{1-\alpha}([0,1])$ be two positive solutions of (1.5). Put

$$f(x) := \begin{cases} p(x)\frac{u^\sigma(x) - v^\sigma(x)}{u(x) - v(x)} & \text{if } u(x) \neq v(x) \\ 0 & \text{if } u(x) = v(x). \end{cases}$$

It is clear that $f \in B^+((0,1))$. Using Lemma 3.1 we have that $u, v \leq \lambda\omega_\alpha$ and the function $h = u - v$ satisfies

$$h + V^\alpha(fh) = 0.$$

Since $V^\alpha(f|h|) \leq 2\lambda^\sigma V^\alpha(p\omega_\alpha^\sigma) < \infty$, it follows by Proposition 2.5 that $u = v$. This completes the proof.

4. Proof of Theorem 1.4

Suppose that the functions p and q satisfy (H2). We put

$$\tilde{p} = a_1\lambda^{a_1-1}\mu^{b_1}p\omega_\alpha^{a_1-1}\omega_\beta^{b_1}, \quad \tilde{q} = a_2\mu^{a_2-1}\lambda^{b_2}q\omega_\alpha^{b_2}\omega_\beta^{a_2-1}.$$

Using hypothesis (H2), the functions $V^\alpha(\tilde{p}\omega_\alpha)$ and $V^\beta(\tilde{q}\omega_\beta)$ are in $C_{1-\alpha}([0,1])$ and $C_{1-\beta}([0,1])$, respectively. Then the constants

$$c_1 = \exp(-\|V^\alpha(\widetilde{p}\omega_\alpha)\|_{1-\alpha}) \quad \text{and} \quad c_2 = \exp(-\|V^\beta(\tilde{q}\omega_\beta)\|_{1-\beta})$$

are positive. We consider the closed convex set Λ defined by

$$\begin{aligned}\Lambda = \big\{&(u,v) \in (C([0,1]))^2 : c_1\lambda \le u \le \lambda, \;\; c_2\mu \le v \le \mu, \\ &\lim_{x\to 0^+} u(x) = \lambda, \; \lim_{x\to 0^+} v(x) = \mu\big\},\end{aligned}$$

endowed with the norm $\|(u,v)\| = \|u\|_\infty + \|v\|_\infty$.

Let $(u,v) \in \Lambda$, then the functions $p\omega_\beta^{b_1}v^{b_1} \in \mathcal{H}_\alpha$ and $q\omega_\alpha^{b_2}u^{b_2} \in \mathcal{H}_\beta$. So by Theorem 1.3, the following two problems

$$\begin{gathered} D^\alpha y + (p\omega_\beta^{b_1}v^{b_1})(x)y^{a_1} = 0, \quad \text{in } (0,1) \\ \lim_{x\to 0^+} x^{1-\alpha}y(x) = \lambda \end{gathered}$$

and

$$\begin{gathered} D^\beta z + (q\omega_\alpha^{b_2}u^{b_2})(x)z^{a_2} = 0, \quad \text{in } (0,1) \\ \lim_{x\to 0^+} x^{1-\beta}z(x) = \mu. \end{gathered}$$

have respectively a unique positive solution $y \in C_{1-\alpha}([0,1])$ and $z \in C_{1-\beta}([0,1])$ satisfying for $x \in (0,1]$ the following inequalities

$$c_1\lambda\omega_\alpha(x) \le y(x) \le \lambda\omega_\alpha(x) \quad \text{and} \quad c_2\mu\omega_\beta(x) \le z(x) \le \mu\omega_\beta(x).$$

Let T be the operator defined on Λ by

$$T(u,v) := \Big(\frac{y}{\omega_\alpha}, \frac{z}{\omega_\beta}\Big).$$

Then T is well defined and obviously $T\Lambda \subset \Lambda$.

We aim to show that T has a fixed point in Λ. Let us prove that $T\Lambda$ is relatively compact in $((C([0,1]))^2, \|\cdot\|)$. First, we show that $T\Lambda$ is equicontinuous on $[0,1]$. Let $(u,v) \in \Lambda$ and let $(y,z) \in C_{1-\alpha}([0,1]) \times C_{1-\beta}([0,1])$ such that $T(u,v) = (\frac{y}{\omega_\alpha}, \frac{z}{\omega_\beta})$. Using Lemma 3.1, we have

$$\begin{aligned} y &= \lambda\omega_\alpha - V^\alpha(p\omega_\beta^{b_1}v^{b_1}y^{a_1}), \\ z &= \mu\omega_\beta - V^\beta(q\omega_\alpha^{b_2}u^{b_2}z^{a_2}). \end{aligned}$$

Let $m > 0$ and $x_1, x_2 \in (0,1]$ be such that $m < x_1 < x_2 \le 1$. Then

$$\begin{aligned} &|x_1^{1-\alpha}y(x_1) - x_2^{1-\alpha}y(x_2)| \\ &= |x_1^{1-\alpha}V^\alpha(p\omega_\beta^{b_1}v^{b_1}y^{a_1})(x_1) - x_2^{1-\alpha}V^\alpha(p\omega_\beta^{b_1}v^{b_1}y^{a_1})(x_2)| \\ &\le \frac{\mu^{b_1}\lambda^{a_1}}{\Gamma(\alpha)}\int_0^{x_1} |x_1^{1-\alpha}(x_1-t)^{\alpha-1} - x_2^{1-\alpha}(x_2-t)^{\alpha-1}|(p\omega_\beta^{b_1}\omega_\alpha^{a_1})(t)dt \\ &\quad + x_2^{1-\alpha}\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}(p\omega_\beta^{b_1}\omega_\alpha^{a_1})(t)dt \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\mu^{b_1}\lambda^{a_1}}{\Gamma(\alpha)}\Big(\int_0^{x_1}|(1-\frac{t}{x_1})^{\alpha-1}-(1-\frac{t}{x_2})^{\alpha-1}|(p\omega_\beta^{b_1}\omega_\alpha^{a_1})(t)dt\\
&\quad+\int_{x_1}^{x_2}(1-\frac{t}{x_2})^{\alpha-1}(p\omega_\beta^{b_1}\omega_\alpha^{a_1})(t)dt\Big).
\end{aligned}
$$

Using the fact that for $t>0$, the function $x\mapsto(1-\frac{t}{x})^{\alpha-1}$ is non-increasing in (t,∞), we obtain

$$
\begin{aligned}
&|x_1^{1-\alpha}y(x_1)-x_2^{1-\alpha}y(x_2)|\\
&\le \frac{\mu^{b_1}\lambda^{a_1}}{\Gamma(\alpha)}\Big[\int_0^{x_1}((1-\frac{t}{x_1})^{\alpha-1}-(1-\frac{t}{x_2})^{\alpha-1})(p\omega_\beta^{b_1}\omega_\alpha^{a_1})(t)dt\\
&\quad+\int_{x_1}^{x_2}(1-\frac{t}{x_2})^{\alpha-1}(p\omega_\beta^{b_1}\omega_\alpha^{a_1})(t)dt\Big]\\
&=\mu^{b_1}\lambda^{a_1}\big(x_1^{1-\alpha}V^\alpha(p\omega_\alpha^{a_1}\omega_\beta^{b_1})(x_1)-x_2^{1-\alpha}V^\alpha(p\omega_\alpha^{a_1}\omega_\beta^{b_1})(x_2)\big)\\
&\quad+\frac{2\mu^{b_1}\lambda^{a_1}}{\Gamma(\alpha)}\int_{x_1}^{x_2}(1-\frac{t}{x_2})^{\alpha-1}(p\omega_\beta^{b_1}\omega_\alpha^{a_1})(t)dt\\
&\le \mu^{b_1}\lambda^{a_1}\big(x_1^{1-\alpha}V^\alpha(p\omega_\alpha^{a_1}\omega_\beta^{b_1})(x_1)-x_2^{1-\alpha}V^\alpha(p\omega_\alpha^{a_1}\omega_\beta^{b_1})(x_2)\big)\\
&\quad+\frac{2\mu^{b_1}\lambda^{a_1}}{\alpha\Gamma(\alpha)}x_2(1-\frac{x_1}{x_2})^\alpha\sup_{[m,1]}(p\omega_\beta^{b_1}\omega_\alpha^{a_1}).
\end{aligned}
$$

Now by (H2), we deduce that $|x_1^{1-\alpha}y(x_1)-x_2^{1-\alpha}y(x_2)|\to 0$ as $|x_1-x_2|\to 0$, uniformly in $(u,v)\in\Lambda$. Similarly we prove that $|x_1^{1-\beta}z(x_1)-x_2^{1-\beta}z(x_2)|\to 0$ as $|x_1-x_2|\to 0$ uniformly in $(u,v)\in\Lambda$.

On the other hand, for $x\in(0,1]$, we have

$$
\begin{gathered}
|x^{1-\alpha}y(x)-\lambda|\le\mu^{b_1}\lambda^{a_1}x^{1-\alpha}V^\alpha(p\omega_\alpha^{a_1}\omega_\beta^{b_1})(x),\\
|x^{1-\beta}z(x)-\mu|\le\lambda^{b_2}\mu^{a_2}x^{1-\beta}V^\beta(q\omega_\alpha^{b_2}\omega_\beta^{a_2})(x).
\end{gathered}
$$

Then using again (H2), we deduce that $|x^{1-\alpha}y(x)-\lambda|\to 0$ and $|x^{1-\beta}z(x)-\mu|\to 0$ as $x\to 0^+$ uniformly in $(u,v)\in\Lambda$. Hence, we conclude that the family $T\Lambda$ is equicontinuous in $[0,1]$.

Since $T\Lambda$ is uniformly bounded, we deduce by Ascoli's theorem that $T\Lambda$ is relatively compact in $((C([0,1]))^2,\|\cdot\|)$.

Next, we prove the continuity of T in Λ. Let (u_k,v_k) be a sequence in Λ that converges to $(u,v)\in\Lambda$ with respect to $\|\cdot\|$. Let (y_k,z_k) and (y,z) in $C_{1-\alpha}([0,1])\times C_{1-\beta}([0,1])$ such that $T(u_k,v_k)=(\frac{y_k}{\omega_\alpha},\frac{z_k}{\omega_\beta})$ and $T(u,v)=(\frac{y}{\omega_\alpha},\frac{z}{\omega_\beta})$. Then

$$
\begin{aligned}
y_k-y&=V^\alpha(p\omega_\beta^{b_1}v^{b_1}y^{a_1})-V^\alpha(p(\omega_\beta^{b_1}v_k{}^{b_1}y_k^{a_1})\\
&=V^\alpha(p\omega_\beta^{b_1}v^{b_1}(y^{a_1}-y_k^{a_1}))+V^\alpha(p\omega_\beta^{b_1}y_k^{a_1}(v^{b_1}-v_k^{b_1})).
\end{aligned}
$$

Now using that

$$
\xi^{a_1}-\nu^{a_1}=a_1(\xi-\nu)\int_0^1(t\xi+(1-t)\nu)^{(a_1-1)}dt,\quad\text{for }\ \nu,\xi\ge 0,
$$

we deduce that

$$
\big(I+V^\alpha(p_k.)\big)(y_k-y)=V^\alpha(p\omega_\beta^{b_1}y_k^{a_1}(v^{b_1}-v_k^{b_1})),\tag{4.1}
$$

where $p_k(x)=a_1p(x)\omega_\beta^{b_1}(x)v^{b_1}(x)\int_0^1(ty(x)+(1-t)y_k(x))^{(a_1-1)}dt$.

Since p satisfies (H2) and the functions $\frac{y_k}{\omega_\alpha}$, $\frac{y}{\omega_\alpha}$, v are in Λ, it follows that $V^\alpha(p_k(y_k-y))<\infty$. So by applying $(I-V^\alpha_{p_k}(p_k.))$ on both sides of equation (4.1), we obtain from equations (2.3) and (2.2) that

$$y_k-y=V^\alpha_{p_k}(p\omega_\beta^{b_1}y_k^{a_1}(v^{b_1}-v_k^{b_1})).$$

On the other hand, for $x\in(0,1]$, we have

$$\begin{aligned} V^\alpha_{p_k}(p\omega_\beta^{b_1}y_k^{a_1}|v^{b_1}-v_k^{b_1}|)(x) &\le V^\alpha(p\omega_\beta^{b_1}y_k^{a_1}|v^{b_1}-v_k^{b_1}|)(x)\\ &\le b_1\lambda^{a_1}\mu^{b_1-1}c_2^{\min(b_1-1,0)}\|v-v_k\|_\infty V^\alpha(p\omega_\beta^{b_1}\omega_\alpha^{a_1})(x). \end{aligned}$$

Hence, by using (H2), we deduce that there exists $c>0$ such that

$$\|\frac{y_k}{\omega_\alpha}-\frac{y}{\omega_\alpha}\|_\infty\le c\|v-v_k\|_\infty.$$

This implies $\|\frac{y_k}{\omega_\alpha}-\frac{y}{\omega_\alpha}\|_\infty\to 0$ as $k\to\infty$. Similarly we prove that $\|\frac{z_k}{\omega_\beta}-\frac{z}{\omega_\beta}\|_\infty\to 0$ as $k\to\infty$. So, we obtain

$$\|T(u_k,v_k)-T(u,v)\|\to 0 \quad \text{as } k\to\infty.$$

Finally, the Schauder fixed point theorem implies the existence of $(u,v)\in\Lambda$ such that $T(u,v)=(u,v)$. It follows that $(y,z)=(\omega_\alpha u,\omega_\beta v)$ is a positive solution in $C_{1-\alpha}([0,1])\times C_{1-\beta}([0,1])$ of system (1.6). This completes the proof.

Acknowledgments. The authors thank Professor Maâgli for stimulating discussions and useful suggestions. Also the authors are greatly indebted to the anonymous referees for their valuable suggestions and comments which surely improved the quality of the presentation.

References

[1] B. Ahmad, J. J. Nieto; *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl. 58 (2009), 1838-1843.

[2] R. Alsaedi, H. Mâagli, N. Zeddini; *Positive solutions for some competitive elliptic systems*, Math. Slovaca 64 No. 1 (2014), 61–72.

[3] J. Bliedtner, W. Hansen; *Potential Theory. An Analytic and Probabilistic Approach to Balayage*, Springer Berlin 1986.

[4] L. M. C. M. Campos; *On the solution of some simple fractional differential equations*, Int. J. Math. Sci. 13 (1990), 481–496.

[5] K. L. Chung, Z. Zhao; *From Brownian motion to Schrödinger's equation*, Springer Berlin 1995.

[6] V. Daftardar-Gejji; *Positive solution of a system of non-autonomous fractional differential equations*, J. Math. Anal. Appl. 302 (2005), 56–64.

[7] D. Delbosco, L. Rodino; *Existence and uniqueness for a nonlinear fractional differential equation*, J. Math. Anal. App. 204 (1996), 609–625.

[8] K. Diethelm, A. D. Freed; *On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity*, F. Keil, W. Mackens, H. Voss, J. Werther (Eds.), Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer-Verlag Heidelberg (1999), 217–224.

[9] V. Gafiychuk, B. Datsko, V. Meleshko; *Mathematical modeling of time fractional reaction-diffusion systems*, J. comput. Appl. Math. 220 (2008), 215–225.

[10] R. Hilfer; *Applications of fractional calculus in Physics*, World Scientific 2000.

[11] R. Hilfer; *Experimental evidence for fractional time evolution in glass forming materials*, Chem. Physics 284 (2002), 399–408.

[12] H. Jafari, V. Daftardar-Gejji; *Solving a system of nonlinear fractional differential equations using a domain decomposition*, J. Comput. Appl. Math. 196 (2) (2006), 644–651.

[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and applications of fractional differential equations*, Elsevier, Amsterdam 2006.
[14] R. C. Koeller; *Application of fractional calculus to the theory of viscoelasticity*, J. App. Mech. 51 (5) (1984), 299–307.
[15] A. Kristály, V. Rădulescu, C. Varga; *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems, Encyclopedia of Mathematics and its Applications*, No. 136, Cambridge University Press, Cambridge 384 pp. (2010).
[16] P. Kumar, O. P. Agarwal; *An approximate method for numerical solution of fractional differential equation*, Signal Process. 86 (2006), 2602–2610.
[17] V. Lakshmikantham, A. S. Vatsala; *General uniqueness and monotone iterative technique for fractional differential equations*, Applied Mathematics Letters 21 (2008), 828–834.
[18] M. P. Lazarevi; *Finite time stability analysis of PD^{α} fractional control of robotic time-delay systems*, Mech. Res. Comm. 33 (2006), 269–279.
[19] H. Mâagli; *Perturbation semi-liné aire des résolvantes et des semi-groupes*, Potential Anal. 3 (1994), 61–87.
[20] H. Mâagli, M. Chaieb, A. Dhifli, S. Zermani; *Existence and boundary behavior of positive solutions for a semilinear fractional differential equation*, Mediterr. J. Math. 12 (2015), 1265–1285.
[21] F. Metzler, W. Schick, H.G. Kilian, T. F. Nonnenmacher; *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys. 103 (1995), 7180–7186.
[22] K. S. Miller, B. Ross; *An introduction to the fractional calculus and fractional differential equation*, Wiley, New York 1993.
[23] G. Molica Bisci, D. Repovš; *On doubly nonlocal fractional elliptic equations*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 26 (2015), 161–176.
[24] A. Pazy; *Semigroups of linear operators and applications to partial differential equations*, Springer, New York 1983.
[25] I. Podlubny; *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. App. Anal. 5 (2002) 367-386.
[26] P. Pucci, M. Xiang, B. Zhang; *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations*, Adv. Nonlinear Anal. 5 (2016), 27–55.
[27] V. Rădulescu; *Qualitative analysis of nonlinear elliptic partial differential equations: Monotonicity, Analytic, and Variational Methods, Contemporary Mathematics and its Applications*, vol. 6, Hindawi Publ. Corp., 210 pp. (2008).
[28] X. Su; *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Lett. 22 (2009) 64–69.
[29] G. Wang, Ravi P. Agarwal, A. Cabada; *Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations*, App. Math. Lett. 25 (2012), 1019–1024.
[30] C. Yu, G. Gao; *Existence of fractional differential equations*, J. Math. Anal. Appl. 310 (2005), 26–29.
[31] S. Zhang; *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl. 252 (2000), 804–812.
[32] S. Zhang; *Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives*, Nonlinear. Anal. 71 (2009), 2087-2093.

Majda Chaieb
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia
E-mail address: majda.chaieb@gmail.com

Abdelwaheb Dhifli
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia
E-mail address: dhifli_waheb@yahoo.fr

Malek Zribi
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 2092 Tunis, Tunisia
E-mail address: malek.zribi@insat.rnu.tn