Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 141, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

PERIODIC SOLUTIONS FOR NONLINEAR DIRAC EQUATION WITH SUPERQUADRATIC NONLINEARITY

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ABSTRACT. This article concerns the periodic solutions for a nonlinear Dirac equation. Under suitable assumptions on the nonlinearity, we show the existence of nontrivial and ground state solutions.

1. Introduction and statement of main results

In this article we sutdy the existence of periodic states to the stationary Dirac equation

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + V(x)u = F_u(x,u) \tag{1.1}$$

for $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, where $\partial_k=\frac{\partial}{\partial x_k}$, $a>0$ is a constant, $\alpha_1,\alpha_2,\alpha_3$ and β are the 4×4 Pauli-Dirac matrices:

$$\beta=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \alpha_k=\begin{pmatrix} 0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\quad k=1,2,3,$$

with

$$\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

Such problem arises in the study of standing wave solutions to the nonlinear Dirac equation which describes the self-interaction in quantum electrodynamics and has been used as effective theories in atomic, nuclear and gravitational physics (see [19]). Its most general form is

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - M(x)\psi + R_\psi(x,\psi). \tag{1.2}$$

where ψ represents the wave function of the state of an electron, c denotes the speed of light, $m>0$, the mass of the electron, $\hbar$ is Planck's constant. Assuming that $R(x,e^{i\theta}\psi)=R(x,\psi)$ for all $\theta\in[0,2\pi]$, a standing wave solutions of is a solution of form $\psi(t,x)=e^{\frac{i\theta t}{\hbar}}u(x)$. It is clear that $\psi(t,x)$ solves (1.2) if and only if $u(x)$ solves (1.1) with $a=mc/\hbar$, $V(x)=M(x)/c\hbar+\theta I_4/\hbar$ and $F(x,u)=R(x,u)/c\hbar$.

There are many papers focused on the existence of standing wave solutions of Dirac equation under various hypotheses on the external field and nonlinearity, see [1, 2, 3, 4, 9, 10, 11, 12, 13, 14, 15, 17, 21, 22, 23, 24, 25, 26, 27, 28, 29] and their

2010 *Mathematics Subject Classification.* 35Q40, 49J35.
Key words and phrases. Nonlinear Dirac equation; periodic solutions; variational method.

Submitted August 6, 2015. Published June 10, 2016.

references. It is worth pointing out that in these papers the solutions u are in $L^2(\mathbb{R}^3, \mathbb{C}^4)$. However, to the best of our knowledge, there is only a little works concerning on periodic solutions to the nonlinear Dirac equation. Here we say that a solution u of problem (1.1) is called periodic if $u(x+k) = u(x)$ for any $k \in \mathbb{Z}^3$. Recently, Ding and Liu [7, 8] first studied the subject for superquadratic, subquadratic and concave-convex nonlinearities case, respectively. The authors obtained the existence of the sequence of periodic solutions with large and small energy by using variational method.

Motivated by the above papers, in the present paper we will continue to consider the existence of periodic solutions of problem (1.1) under conditions different from those previously assumed in [7, 8], and we use a general superquadratic condition instead of the Ambrosetti-Rabinowitz condition. Moreover, the existence of ground state solution is also explored. Before going further, for notation convenience, we denote $\Omega = [0,1] \times [0,1] \times [0,1]$, if u is a solution of problem (1.1), its energy will be denoted by

$$\Phi(u) = \int_\Omega \Big(\frac{1}{2}(-i\sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + V(x)u) \cdot u - F(x,u)\Big) dx,$$

where (and in the following) by $u \cdot v$ we denote the scalar product in $\mathbb{C}^4$ of u and v. To state our results, we need the following assumptions:

- (A1) $V \in C(\mathbb{R}^3, [0,\infty))$, and $V(x)$ is 1-periodic in x_i, $i = 1,2,3$;
- (A2) $F \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, [0,\infty))$ is 1-periodic in x_i, $i = 1,2,3$, and $|F_u(x,u)| \le c(1 + |u|^{p-1})$ for some $c > 0$, $2 < p < 3$;
- (A3) $|F(x,u)| \le \frac{1}{2}\eta|u|^2$ if $|u| < \delta$ for some $0 < \eta < \mu_1$, where $\delta > 0$ and μ_1 will be defined later in (2.1);
- (A4) $\frac{F(x,u)}{|u|^2} \to \infty$ as $|u| \to \infty$ uniformly in x;
- (A5) $F(x,u+z) - F(x,u) - rF_u(x,u)\cdot z + \frac{(r-1)^2}{2} F_u(x,u) \cdot u \ge -W(x)$, $r \in [0,1]$, $W(x) \in L^1(\Omega)$.

On the existence of nontrivial periodic solutions we have the following result.

Theorem 1.1. *Let* (A1)–(A5) *be satisfied. Then* (1.1) *has at least one nontrivial periodic solution.*

Let

$$\mathcal{K} := \{u \in E : \Phi'(u) = 0, u \neq 0\}$$

be the critical points set of Φ and let

$$m := \inf\{\Phi(u), u \in \mathcal{K} \setminus \{0\}\},$$

where E is a set to be defined later. On the existence of ground state solutions we have the following result.

Theorem 1.2. *Let* (A1)–(A5) *be satisfied, and* $|F_u(x,u)| = o(|u|)$ *as* $|u| \to 0$ *uniformly in* x*. Then* (1.1) *has one ground state solution* u *such that* $\Phi(u) = m$*.*

For the nonlinearity $F(x,u)$, it is not difficult to find that there exist some functions satisfying conditions (A2)–(A5) if we take $0 \le W(x) \in L^1(\Omega)$, for example:

(1) Let $F(x,u) = q(x)|u|^p$, where $p \in (2,3)$, $q(x) > 0$ is 1-periodic with respect to x_i, $i = 1,2,3$.

(2) Let $F(x,u)=q(x)\big(|u|^p+(p-2)|u|^{p-\varepsilon}\sin^2(\frac{|u|^\varepsilon}{\varepsilon})\big)$, where $0<\varepsilon<p-2$, $p\in(2,3)$, $q(x)>0$ is 1-periodic with respect to x_i, $i=1,2,3$.

Here we only check the (1) satisfies condition (A5). Indeed, a straightforward computation deduces that $F(x,u)=q(x)|u|^p$ satisfies the following relation (it have already been proved in [24])

$$F(x,(s+1)u+v)-F(x,u)-F_u(x,u)\cdot\big(s(\frac{s}{2}+1)u+(s+1)v\big)\geq 0,\quad s\geq -1.$$

If we take $r=s+1$ and $v=(1-r)u+z$, then

$$F(x,u+z)-F(x,u)-rF_u(x,u)\cdot z+\frac{(r-1)^2}{2}F_u(x,u)\cdot u\geq 0,\quad r\geq 0,$$

which implies (A5) holds if we take $W(x)=0$ and $r\in[0,1]$. For the Ex2, the proof is similar. Additionally, the Ex2 does not satisfy the Ambrosetti-Rabinowitz type superquadratic condition.

The rest of this article is organized as follows. In Section 2, we establish the variational framework associated with problem (1.1), and we also give some preliminary lemmas, which are useful in the proofs of our main results. In Section 3, we give the detailed proofs of our main results.

2. Variational setting and preliminary results

We first introduce a variational structure for problem (1.1). Let

$$L^p(\Omega):=\{u\in L^p_{\rm loc}(\mathbb{R}^3,\mathbb{C}^4):u(x+\hat e_i)=u(x)\text{ a.e., } i=1,2,3\},$$

where $\hat e_1=(1,0,0)$, $\hat e_2=(0,1,0)$, $\hat e_3=(0,0,1)$. In what follows by $\|\cdot\|_q$ we denote the usual L^q-norm for $q\in[1,\infty]$, and $(\cdot,\cdot)_2$ denote the usual L^2 inner product, c, C_i stand for different positive constants. For convenience, let Dirac operator

$$A_0=-i\sum_{k=1}^{3}\alpha_k\partial_k+a\beta\quad\text{and}\quad A_V=A_0+V.$$

Clearly, A_0 and A_V are selfadjoint operator on $L^2(\Omega)$ with domain

$$\begin{aligned}\mathcal{D}(A_V)&=\mathcal{D}(A_0)=H^1(\Omega)\\&:=\{u\in H^1_{\rm loc}(\mathbb{R}^3,\mathbb{C}^4):u(x+\hat e_i)=u(x)\text{ a.e., } i=1,2,3\}.\end{aligned}$$

It is clear that A_0^2 has only eigenvalues of finite multiplicity arranged by

$$a^2<\nu_1<\nu_2\leq\nu_3\leq\dots\to\infty.$$

By the spectral theory of self adjoint operators, A_0 has only eigenvalues $\pm\ell_j=\pm\sqrt{\nu_j}$, $j\in\mathbb{N}$. Moreover, since $H^1(\Omega)$ embeds compactly in $L^2:=L^2(\Omega)$, and the multiplication operator V is bounded in L^2, hence compact relative to A_0, the spectrum of the self-adjoint operator A_V consists of eigenvalues of finite multiplicity. We arrange the eigenvalues as

$$\dots\mu_{-j}\leq\dots\leq\mu_{-1}<\mu_0=0<\mu_1\leq\dots\leq\mu_j\dots\tag{2.1}$$

with $\mu_{\pm j}\to\pm\infty$ as $j\to\infty$, and corresponding eigenfunctions $\{e_{\pm j}\}_{j\in\mathbb{N}}$ form an orthogonal basis in L^2. Observe that we have an orthogonal decomposition

$$L^2=L^-\oplus L^0\oplus L^+\quad\text{and}\quad u=u^-+u^0+u^+,$$

such that A_V is negative definite on L^- and positive definite on L^+ and $L^0 = \ker(A_V)$. Set $E := \mathcal{D}(|A_V|^{1/2})$ be the domain of the selfadjoint operator $|A_V|^{1/2}$ which is a Hilbert space equipped with the inner product

$$(u,v) = (|A_V|^{1/2}u, |A_V|^{1/2}v)_2 + (u^0, v^0)_2, \quad \forall u, v \in E$$

and norm $\|u\| = (u,u)^{1/2}$. Let $E^{\pm} := \overline{\text{span}\{e_{\pm k}\}}_{k\in\mathbb{N}^+}$, $E^0 = \ker(A_V)$. Then E^-, E^0 and E^+ are orthogonal with respect to the products $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$. Hence

$$E = E^- \oplus E^0 \oplus E^+$$

is an orthogonal decomposition of E. Note that if $0 \notin \sigma(A_V)$ then $E^0 = \{0\}$, where $\sigma(A_V)$ denote the spectrum of A_V.

To prove our main results, we need the following embedding theorem (see [5]).

Lemma 2.1. *$E = H^{1/2}(\Omega)$ with equivalent norms, hence E embeds compactly into $L^p(\Omega)$ for all $p \in [1,3)$ and continuously into $L^p(\Omega)$ for all $p \in [1,3]$. In particular there is a $c_p > 0$ such that $\|u\|_p \le c_p\|u\|$ for $u \in E$.*

Next, on E we define the functional

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Psi(u), \quad \text{for } u = u^- + u^0 + u^+ \tag{2.2}$$

where $\Psi(u) = \int_\Omega F(x,u)dx$. Clearly, Φ is strongly indefinite, and our hypotheses imply that $\Phi \in C^1(E,\mathbb{R})$, and a standard argument shows that critical points of Φ are solutions of problem (1.1) (see [5, 20]).

To find critical points of Φ, we shall use the following abstract theorem. Let E be a Hilbert space with norm $\|\cdot\|$ and have an orthogonal decomposition $E = N \oplus N^\perp, N \subset E$ being a closed and separable subspace. There exists a norm $|v|_\omega \le \|v\|$ for all $v \in N$ and induces a topology equivalent to the weak topology of N on a bounded subset of N. For $u = v + w \in E = N \oplus N^\perp$ with $v \in N$, $w \in N^\perp$, we define $|u|_\omega^2 = |v|_\omega^2 + \|w\|^2$. Particularly, if $u_n = v_n + w_n$ is $|\cdot|_\omega$-bounded and $u_n \xrightarrow{|\cdot|_\omega} u$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E [18].

Let $E = E^- \oplus E^0 \oplus E^+$, $e \in E^+$ with $\|e\| = 1$. Let $N := E^- \oplus E^0 \oplus \mathbb{R}e$ and $E_1^+ := N^\perp = (E^- \oplus E^0 \oplus \mathbb{R}e)^\perp$. For $R > 0$, let

$$Q := \{u := u^- + u^0 + se : s \in \mathbb{R}^+, u^- + u^0 \in E^- \oplus E^0, \|u\| < R\}.$$

For $0 < s_0 < R$, we define

$$D := \{u := se + w^+ : s \ge 0, w^+ \in E_1^+, \|se + w^+\| = s_0\}.$$

For $\Phi \in C^1(E,\mathbb{R})$, define

$$\begin{aligned}\Gamma := \big\{h : h : [0,1] \times \bar{Q} \to E \text{ is } |\cdot|_\omega \text{ continuous}, h(0,u) = u \text{ and } \Phi(h(s,u)) \le \Phi(u),\\ \text{for all } u \in \bar{Q}, \text{ for any } (s_0,u_0) \in [0,1] \times \bar{Q} \text{ there is a } |\cdot|_\omega \text{ neighborhood}\\ U(s_0,u_0) \text{ such that } \{u - h(x,u) : (x,u) \in U(s_0,u_0) \cap ([0,1] \times \bar{Q})\} \subset E_{\text{fin}}\big\}\end{aligned}$$

where E_{fin} denotes various finite-dimensional subspaces of E; $\Gamma \ne 0$ since $id \in \Gamma$.

Now we state a critical point theorem which will be used later (see [18]).

Theorem 2.2. *The family of C^1-functionals Φ_λ have the form*

$$\Phi_\lambda(u) := \lambda K(u) - J(u), \quad \forall \lambda \in [1, \lambda_0],$$

where $\lambda_0 > 1$. Assume that

(a) $K(u) \geq 0$ *for all* $u \in E$, $\Phi_1 = \Phi$;
(b) $|J(u)| + K(u) \to \infty$ *as* $\|u\| \to \infty$;
(c) Φ_λ *is* $|\cdot|_\omega$*-upper semicontinuous,* Φ'_λ *is weakly sequentially continuous on* E*,* Φ_λ *maps bounded sets to bounded sets;*
(d) $\sup_{\partial Q} \Phi_\lambda < \inf_D \Phi_\lambda$ *for all* $\lambda \in [1, \lambda_0]$.

Then for all $\lambda \in [1, \lambda_0]$*, there exists a sequence* $\{u_n\}$ *such that*

$$\sup_n \|u_n\| < \infty, \quad \Phi'_\lambda(u_n) \to 0, \quad \Phi_\lambda(u_n) \to c_\lambda,$$

where

$$c_\lambda := \inf_{h\in\Gamma} \sup_{u\in\bar{Q}} \Phi_\lambda(h(1,u)) \in [\inf_D \Phi_\lambda, \sup_{\bar{Q}} \Phi_\lambda].$$

To apply Theorem 2.2, we shall prove a few lemmas. We select λ_0 such that $1 < \lambda_0 < \min[2, \frac{\mu_1}{\eta}]$. For $1 \leq \lambda \leq \lambda_0$, we consider

$$\Phi_\lambda(u) := \frac{\lambda}{2}\|u^+\|^2 - \Big(\frac{1}{2}\|u^-\|^2 + \int_\Omega F(x,u)dx\Big) := \lambda K(u) - J(u). \tag{2.3}$$

It is easy to see that Φ_λ satisfies condition (a) in Theorem 2.2. To see (c), if $u_n \xrightarrow{|\cdot|_\omega} u$, and $\Phi_\lambda(u_n) \geq c$, then $u_n^+ \to u^+$ and $u_n^- \rightharpoonup u^-$ in E, $u_n(x) \to u(x)$ a.e. on Ω, going to a subsequence if necessary. Using Fatou's lemma, we know $\Phi_\lambda(u) \geq c$, which means that Φ_λ is $|\cdot|_\omega$-upper semicontinuous. By Lemma 2.1 and (A2), Φ'_λ is weakly sequentially continuous on E, and Φ_λ maps bounded sets to bounded sets.

Lemma 2.3. *Assume that* (A1)–(A5) *are satisfied, then*

$$J(u) + K(u) \to \infty \quad \textit{as } \|u\| \to \infty.$$

Proof. Suppose to the contrary that there exists $\{u_n\}$ with $\|u_n\| \to \infty$ such that $J(u_n) + K(u_n) \leq C$ for some $C > 0$. Let $w_n = \frac{u_n}{\|u_n\|} = w_n^- + w_n^0 + w_n^+$, then $\|w_n\| = 1$ and

$$\begin{aligned} \frac{C}{\|u_n\|^2} &\geq \frac{K(u_n) + J(u_n)}{\|u_n\|^2} \\ &= \frac{1}{2}(\|w_n^+\|^2 + \|w_n^-\|^2) + \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}dx \\ &= \frac{1}{2}(\|w_n\|^2 - \|w_n^0\|^2) + \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}dx. \end{aligned} \tag{2.4}$$

Going to a subsequence if necessary, we may assume $w_n \rightharpoonup w$, $w_n^- \rightharpoonup w^-$, $w_n^+ \rightharpoonup w^+$, $w_n^0 \to w^0$ and $w_n(x) \to w(x)$ a.e. on Ω. If $w^0 = 0$, by (A2) and (2.4) we have

$$\frac{1}{2}\|w_n\|^2 + \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}dx \leq \frac{1}{2}\|w_n^0\|^2 + \frac{C}{\|u_n\|^2},$$

which implies $\|w_n\| \to 0$, this contradicts with $\|w_n\| = 1$. If $w^0 \neq 0$, then $w \neq 0$. Therefore, $|u_n| = |w_n|\|u_n\| \to \infty$. By (A2), (A4) and Fatou's lemma we have

$$\int_\Omega \frac{F(x,u_n)}{|u_n|^2}|w_n|dx \to \infty.$$

Hence by (2.4) again, we obtain $0 \geq +\infty$, a contradiction. The proof is complete. □

Note that Lemma 2.3 implies condition (b). To continue the discussion, we still need to verify condition (d).

Lemma 2.4. *Assume that* (A1)–(A5) *are satisfied. Then there are two positive constants $\kappa, \rho > 0$ such that*

$$\Phi_\lambda(u) \geq \kappa, \quad u \in E^+, \quad \|u\| = \rho, \quad \lambda \in [1, \lambda_0].$$

Proof. By (2.1) and the definition of E^+, it is easy to see that

$$\|u\|^2 = (Au, u)_2 \geq \mu_1 \|u\|_2^2, \quad \forall u \in E^+, \tag{2.5}$$

For any $u \in E^+$, by (A2), (A3), (2.5) and Lemma 2.3, we have

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{\lambda}{2}\|u\|^2 - \int_\Omega F(x,u)dx \\
&\geq \frac{1}{2}\|u\|^2 - \int_{\{|u|<\delta\}} F(x,u)dx - \int_{\{|u|\geq\delta\}} F(x,u)dx \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\eta \int_{\{|u|<\delta\}} |u|^2 dx - c\int_{\{|u|\geq\delta\}} |u|^p dx \\
&\geq \frac{1}{2}\|u\|^2 - \frac{\eta}{\mu_1}\frac{1}{2}\|u\|^2 - C'\|u\|^p \\
&= \frac{1}{2}\|u\|^2\Big(1 - \frac{\eta}{\mu_1} - 2C'\|u\|^{p-2}\Big), \quad 0 \leq \eta < \mu_1.
\end{aligned}$$

The conclusion follows if we take $\|u\|$ sufficiently small. □

Lemma 2.5. *Assume that* (A1)–(A5) *are satisfied. Then there exists a constant $R > 0$ such that*

$$\Phi_\lambda(u) \leq 0, \quad u \in \partial Q_R, \quad \lambda \in [1, \lambda_0],$$

where

$$Q_R := \{u := v + se : s \geq 0, v \in E^- \oplus E^0, e \in E^+ \text{ with } \|e\| = 1,\ \|u\| \leq R\}.$$

Proof. By contradiction, we suppose that there exist $R_n \to \infty$, $\lambda_n \in [1, \lambda_0]$ and $u_n = v_n + s_n e = v_n^- + v_n^0 + s_n e \in \partial Q_{R_n}$ such that $\Phi_{\lambda_n}(u_n) > 0$. If $s_n = 0$, by (A2), we obtain

$$\Phi_{\lambda_n}(u_n) = -\frac{1}{2}\|v_n^-\|^2 - \int_\Omega F(x,u_n)dx \leq -\frac{1}{2}\|v_n^-\|^2 \leq 0.$$

Therefore,

$$s_n \neq 0 \quad \text{and} \quad \|u_n\|^2 = \|v_n\|^2 + s_n^2.$$

Let

$$\tilde{u}_n = \frac{u_n}{\|u_n\|} = \tilde{s}_n e + \tilde{v}_n\,.$$

Then

$$\|\tilde{u}_n\|^2 = \|\tilde{v}_n\|^2 + \tilde{s}_n^2 = 1.$$

Thus, passing to a subsequence, we may assume that

$$\begin{gathered}
\tilde{s}_n \to \tilde{s}, \quad \lambda_n \to \lambda, \\
\tilde{u}_n = \frac{u_n}{\|u_n\|} = \tilde{s}_n e + \tilde{v}_n \rightharpoonup \tilde{u} \quad \text{in } E, \\
\tilde{u}_n(x) \to \tilde{u}(x) \quad \text{a.e. on } \Omega.
\end{gathered}$$

It follows from $\Phi_{\lambda_n}(u_n) > 0$ that

$$
\begin{aligned}
0 < \frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2} &= \frac{1}{2}(\lambda_n \tilde{s}_n^2 - \|\tilde{v}_n\|^2) - \int_\Omega \frac{F(x,u_n)}{|u_n|^2}|\tilde{u}_n|^2 dx \\
&= \frac{1}{2}[(\lambda_n + 1)\tilde{s}_n^2 - 1] - \int_\Omega \frac{F(x,u_n)}{|u_n|^2}|\tilde{u}_n|^2 dx.
\end{aligned} \tag{2.6}
$$

From (A2) and (2.6), we know that $(\lambda + 1)\tilde{s}^2 - 1 \geq 0$, that is

$$
\tilde{s}^2 \geq \frac{1}{1+\lambda} \geq \frac{1}{1+\lambda_0} > 0.
$$

Thus $\tilde{u} \neq 0$. It follows from (A4) and Fatou's lemma that

$$
\int_\Omega \frac{F(x,u_n)}{|u_n|^2}|\tilde{u}_n|^2 dx \to \infty \quad \text{as } n \to \infty,
$$

which contradicts to (2.6). The proof is complete. □

Hence, Lemmas 2.4 and 2.5 imply condition (d) of Theorem 2.2. Applying Theorem 2.2, we obtain the following result.

Lemma 2.6. *Assume that* (A1)–(A5) *are satisfied. then for each* $\lambda \in [1, \lambda_0]$, *there exists a sequence* $\{u_n\}$ *such that*

$$
\sup_n \|u_n\| < \infty, \quad \Phi'_\lambda(u_n) \to 0, \quad \Phi_\lambda(u_n) \to c_\lambda,
$$

Lemma 2.7. *Assume that* (A1)–(A5) *are satisfied. then for each* $\lambda \in [1, \lambda_0]$, *there exists a* $u_\lambda \in E$ *such that*

$$
\Phi'_\lambda(u_\lambda) = 0, \quad \Phi_\lambda(u_\lambda) = c_\lambda.
$$

Proof. Let $\{u_n\}$ be the sequence obtained in Lemma 2.6. Since $\{u_n\}$ is bounded, we can assume $u_n \rightharpoonup u_\lambda$ in E and $u_n(x) \to u_\lambda(x)$ a.e. on Ω. By Lemma 2.6 and the fact Φ'_λ is weakly sequentially continuous, we have

$$
\langle \Phi'_\lambda(u_\lambda), \varphi \rangle = \lim_{n\to\infty} \langle \Phi'_\lambda(u_n), \varphi \rangle = 0, \quad \forall \varphi \in E.
$$

That is $\Phi'_\lambda(u_\lambda) = 0$. By Lemma 2.6 again, we have

$$
\Phi_\lambda(u_n) - \frac{1}{2}\langle \Phi'_\lambda(u_n), u_n \rangle = \int_\Omega \Big(\frac{1}{2}(F_u(x,u_n), u_n) - F(x,u_n) \Big) dx \to c_\lambda.
$$

On the other hand, by Lemma 2.1, it is easy to prove that

$$
\int_\Omega \frac{1}{2} F_u(x,u_n) \cdot u_n dx \to \int_\Omega \frac{1}{2} F_u(x,u_\lambda) \cdot u_\lambda dx, \tag{2.7}
$$

$$
\int_\Omega F(x,u_n) dx \to \int_\Omega F(x,u_\lambda) dx, \tag{2.8}
$$

Therefore, by (2.7), (2.8) and the fact $\Phi'_\lambda(u_\lambda) = 0$, we obtain

$$
\begin{aligned}
\Phi_\lambda(u_\lambda) &= \Phi_\lambda(u_\lambda) - \frac{1}{2}\langle \Phi'_\lambda(u_\lambda), u_\lambda \rangle \\
&= \int_\Omega \Big(\frac{1}{2}(F_u(x,u_\lambda) \cdot u_\lambda - F(x,u_\lambda) \Big) dx = c_\lambda.
\end{aligned}
$$

The proof is complete. □

Applying Lemma 2.7, we obtain the following result.

Lemma 2.8. *Assume that* (A1)–(A5) *are satisfied. Then for each* $\lambda \in [1, \lambda_0]$, *there exists sequences* $u_n \in E$ *and* $\lambda_n \in [1, \lambda_0]$ *with* $\lambda_n \to \lambda$ *such that*

$$\Phi'_{\lambda_n}(u_n) = 0, \quad \Phi_{\lambda_n}(u_n) = c_{\lambda_n}.$$

Lemma 2.9. *Suppose* (A5) *holds. Then*

$$\int_\Omega \Big(F(x,u) - F(x,rw) + r^2 F_u(x,u)\cdot w - \frac{1+r^2}{2} F_u(x,u)\cdot u\Big)dx \le C,$$

where $u \in E, w \in E^+, 0 \le r \le 1$ *and the constant* C *does not depend on* u, w, r.

Proof. The inequality follows from (A5) if we take $u = u$ and $z = rw - u$, and $C = \int_\Omega |W(x)|dx$. □

Lemma 2.10. *Assume that* (A1)–(A5) *are satisfied. Then the sequences* $\{u_n\}$ *given in Lemma 2.8 are bounded.*

Proof. Suppose to the contrary that $\{u_n\}$ is unbounded. Without loss of generality, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|} = v_n^+ + v_n^0 + v_n^-$, then $\|v_n\| = 1$. Going to a subsequence if necessary, we can assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in L^p for $p \in [1,3)$, $v_n(x) \to v(x)$ a.e. on Ω. For v, we have only the following two cases: $v \ne 0$ and $v = 0$.

First, we consider $v \ne 0$. It follows from (A4) and Fatou's Lemma that

$$\int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}dx = \int_\Omega \frac{F(x,u_n)}{|u_n|^2}|v_n|^2 dx \to \infty \quad \text{as } n \to \infty,$$

which, together with Lemmas 2.4 and 2.8 imply

$$0 \le \frac{c_{\lambda_n}}{\|u_n\|^2} = \frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2} = \frac{\lambda_n}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}dx \to -\infty$$

as $n \to \infty$. Which is a contradiction.

Next we assume that $v = 0$. We claim that there exist a constant c independent of u_n and λ_n such that

$$\Phi_{\lambda_n}(ru_n^+) - \Phi_{\lambda_n}(u_n) \le c, \quad \forall r \in [0,1]. \tag{2.9}$$

Since

$$\frac{1}{2}\langle \Phi'_{\lambda_n}(u_n), \varphi\rangle = \frac{1}{2}\lambda_n(u_n^+, \varphi^+) - \frac{1}{2}(u_n^-, \varphi^-) - \frac{1}{2}\int_\Omega F_u(x,u_n)\cdot\varphi dx = 0, \quad \forall \varphi \in E,$$

it follows from the definition of Φ_λ that

$$\begin{aligned}
&\Phi_{\lambda_n}(ru_n^+) - \Phi_{\lambda_n}(u_n)\\
&= \frac{1}{2}\lambda_n(r^2-1)\|u_n^+\|^2 + \frac{1}{2}\|u_n^-\|^2 + \int_\Omega \big[F(x,u_n) - F(x,ru_n^+)\big]\,dx\\
&\quad + \frac{1}{2}\lambda_n(u_n^+,\varphi^+) - \frac{1}{2}(u_n^-,\varphi^-) - \frac{1}{2}\int_\Omega F_u(x,u_n)\cdot\varphi dx.
\end{aligned} \tag{2.10}$$

Taking

$$\varphi = (r^2+1)u_n^- - (r^2-1)u_n^+ + (r^2+1)u_n^0 = (r^2+1)u_n - 2r^2u_n^+,$$

with Lemma 2.9 and (2.10) implies

$$\Phi_{\lambda_n}(ru_n^+) - \Phi_{\lambda_n}(u_n) = -\frac{1}{2}\|u_n^-\|^2 + \int_\Omega \Big(F(x,u_n) - F(x,ru_n^+)$$

$$+ r^2 F_u(x,u_n)\cdot u_n^+ - \frac{1+r^2}{2}F_u(x,u_n)\cdot u_n\Big)dx \le C.$$

Hence, (2.9) holds. Let θ be a constant and take

$$r_n := \frac{\theta}{\|u_n\|} \to 0 \quad \text{as } n\to\infty.$$

Therefore, (2.9) implies

$$\Phi_{\lambda_n}(r_n u_n^+) - \Phi_{\lambda_n}(u_n) \le C$$

for all sufficiently large n. From $v_n^+ = \frac{u_n^+}{\|u_n\|}$ and Lemma 2.8 that

$$\Phi_{\lambda_n}(\theta v_n^+) \le C' \tag{2.11}$$

for all sufficiently large n. Note that Lemma 2.4, Lemma 2.8 and (A2) imply

$$\begin{aligned} 0 \le \frac{c_{\lambda_n}}{\|u_n\|^2} &= \frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2} \\ &= \frac{\lambda_n}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}dx \\ &\le \frac{\lambda_0}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2; \end{aligned}$$

thus, $\lambda_0\|v_n^+\| \ge \|v_n^-\|$. If $v_n^+ \to 0$, then from the above inequality, we have $v_n^- \to 0$, and therefore

$$\|v_n^0\|^2 = 1 - \|v_n^+\|^2 - \|v_n^-\|^2 \to 1.$$

Hence, $v_n^0 \to v^0$ because of $\dim E^0 < \infty$. Thus, $v \ne 0$, a contradiction. Therefore, $v_n^+ \nrightarrow 0$ and $\|v_n^+\|^2 \ge \alpha$ for all n and some $\alpha > 0$. By (A2) and (A3), we have

$$\begin{aligned} \int_\Omega F(x,\theta v_n^+)dx &\le \frac{1}{2}\eta\theta^2 \int_{\{|\theta v_n^+|<\delta\}} |v_n^+|^2 dx + \frac{1}{2}c\int_{\{|\theta v_n^+|\ge\delta\}} \theta^p|v_n^+|^p dx \\ &\le \frac{1}{2}\eta\theta^2 \int_{\{|\theta v_n^+|<\delta\}} |v_n^+|^2 dx + C_1'\int_{\{|\theta v_n^+|\ge\delta\}} |v_n^+|^p dx. \end{aligned} \tag{2.12}$$

For all sufficiently large n, from (2.11), (2.12) and the fact $\lambda_n \to \lambda, v_n^+ \to v^+ = 0$ in L^p for $p\in[1,3)$ it follows that

$$\begin{aligned} \Phi_{\lambda_n}(\theta v_n^+) &= \frac{1}{2}\lambda_n\theta^2\|v_n^+\|^2 - \int_\Omega F(x,\theta v_n^+)dx \\ &\ge \frac{1}{2}\lambda_n\theta^2\alpha - \frac{1}{2}\eta\theta^2\int_{\{|\theta v_n^+|<\delta\}} |v_n^+|^2dx - C_1'\int_{\{|\theta v_n^+|\ge\delta\}}|v_n^+|^p dx \\ &\to \frac{1}{2}\lambda\alpha\theta^2, \quad \text{as } n\to\infty. \end{aligned}$$

This implies that $\Phi_{\lambda_n}(\theta v_n^+) \to \infty$ as $\theta\to\infty$, contrary to (2.11). Therefore, $\{u_n\}$ are bounded. The proof is complete. □

3. Proofs of main results

Proof of Theorem 1.1. From Lemma 2.8, there are sequences $1 < \lambda_n \to 1$ and $\{u_n\} \subset E$ such that $\Phi'_{\lambda_n}(u_n) = 0$ and $\Phi_{\lambda_n}(u_n) = c_{\lambda_n}$. By Lemma 2.10, we know

$\{u_n\}$ is bounded in E, thus we can assume $u_n \rightharpoonup u$ in E, $u_n \to u$ in L^p for $p \in [1,3)$, $u_n(x) \to u(x)$ a.e. on Ω. Therefore

$$\langle \Phi'_{\lambda_n}(u_n), \varphi\rangle = \lambda_n(u_n^+, \varphi) - (u_n^-, \varphi) - \int_\Omega F_u(x, u_n)\cdot \varphi dx = 0, \quad \forall \varphi \in E.$$

Hence, in the limit

$$\langle \Phi'(u), \varphi\rangle = (u^+, \varphi) - (u^-, \varphi) - \int_\Omega F_u(x, u_n)\cdot \varphi dx = 0, \quad \forall \varphi \in E.$$

Thus $\Phi'(u) = 0$. Note that

$$\Phi_{\lambda_n}(u_n) - \frac{1}{2}\langle \Phi'_{\lambda_n}(u_n), u_n\rangle = \int_\Omega \Big(\frac{1}{2}F_u(x, u_n)\cdot u_n - F(x, u_n)\Big) dx = c_{\lambda_n} \ge c_1. \tag{3.1}$$

Similar to (2.7) and (2.8), we know that

$$\int_\Omega \Big(\frac{1}{2}F_u(x, u_n)\cdot u_n - F(x, u_n)\Big) dx \to \int_\Omega \Big(\frac{1}{2}F_u(x, u)\cdot u - F(x, u)\Big) dx,$$

as $n \to \infty$. It follows from $\Phi'(u) = 0$, (3.1) and Lemma 2.4 that

$$\begin{aligned}
\Phi(u) &= \Phi(u) - \frac{1}{2}\langle \Phi'(u), u\rangle \\
&= \int_\Omega \Big(\frac{1}{2}F_u(x, u)\cdot u - F(x, u)\Big) dx \\
&= \lim_{n\to\infty}\int_\Omega \Big(\frac{1}{2}\big(F_u(x, u_n)\cdot u_n\big) - F(x, u_n)\Big) dx \\
&\ge c_1 \ge \kappa > 0.
\end{aligned}$$

Therefore, $u \ne 0$. □

Proof of Theorem 1.2. Theorem 1.1 shows that $\mathcal{K}$ is not an empty set. Let $m := \inf_{u\in\mathcal{K}} \Phi(u)$. Now suppose that

$$|F_u(x, u)| = o(|u|), \quad \text{as } |u| \to 0.$$

It follows from (A2) that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|F_u(x, u)| = \varepsilon|u| + C_\varepsilon|u|^{p-1}. \tag{3.2}$$

Let $\{u_n\}$ be a sequence in $\mathcal{K}$ such that

$$\Phi(u_n) \to m, \tag{3.3}$$

by Lemma 2.10, the sequence $\{u_n\}$ is bounded in E. Thus, $u_n \rightharpoonup u$ in E, $u_n \to u$ in L^p for $p \in [1,3)$ and $u_n(x) \to u(x)$ a.e. on Ω, after passing to a subsequence. Note that

$$0 = \langle \Phi'(u_n), u_n^+\rangle = \|u_n^+\|^2 - \int_\Omega F_u(x, u_n)\cdot u_n^+ dx, \tag{3.4}$$

this together with (3.2), Hölder inequality and the Sobolev embedding theorem imply

$$\begin{aligned}
\|u_n^+\|^2 &= \int_\Omega F_u(x, u_n)\cdot u_n^+ dx \\
&\le \varepsilon\int_\Omega |u_n||u_n^+| dx + C_\varepsilon\int_\Omega |u_n|^{p-1}|u_n^+| dx \\
&\le \varepsilon\|u_n\|^2 + C_\varepsilon\|u_n\|^p.
\end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\|u_n^-\|^2 \le \varepsilon\|u_n\|^2 + C_\varepsilon\|u_n\|^p. \tag{3.6}$$

It follows from (3.5) and (3.6) that $\|u_n\| \ge c$ for some $c > 0$. A standard argument shows that $u_n \to u$ in E by Lemma 2.1, hence $u \ne 0$. Observe that

$$\langle \Phi'(u_n), \varphi\rangle = (u_n^+, \varphi) - (u_n^-, \varphi) - \int_\Omega F_u(x, u_n)\cdot \varphi dx = 0, \quad \forall \varphi \in E;$$

taking the limit

$$\langle \Phi'(u), \varphi\rangle = (u^+, \varphi) - (u^-, \varphi) - \int_\Omega F_u(x, u)\cdot \varphi dx = 0, \ \forall \varphi \in E.$$

Thus, $\Phi'(u) = 0$ and $u \in \mathcal{K}$. Similar to (2.7) and (2.8), we have

$$\begin{aligned}\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle &= \int_\Omega \Big(\frac{1}{2}F_u(x, u_n)\cdot u_n - F(x, u_n)\Big)dx \\ &\to \int_\Omega \Big(\frac{1}{2}F_u(x, u)\cdot u - F(x, u)\Big)dx \quad \text{as } n \to \infty.\end{aligned}$$

It follows from $\Phi'(u) = 0$ and (3.3) that

$$\begin{aligned}\Phi(u) &= \Phi(u) - \frac{1}{2}\langle \Phi'(u), u\rangle \\ &= \int_\Omega \Big(\frac{1}{2}F_u(x, u)\cdot u - F(x, u)\Big)dx \\ &= \lim_{n\to\infty}\int_\Omega \Big(\frac{1}{2}F_u(x, u_n)\cdot u_n - F(x, u_n)\Big)dx \\ &= \lim_{n\to\infty}\Phi(u_n) = m.\end{aligned}$$

This completes the proof. □

Acknowledgments. This work is partially supported by the National Natural Science Foundation of China (Nos: 11571370, 11201138, 11471137, 11471278, 61472136).

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