Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 144, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NONTRIVIAL CONVEX SOLUTIONS FOR SYSTEMS OF MONGE-AMPÉRE EQUATIONS VIA GLOBAL BIFURCATION

ZEXIN QI

Abstract. We obtain existence results for some systems of Monge-Ampére equations, using bifurcation theorems of Krasnosell'ski-Rabinowitz type.

1. Introduction

We study the system of coupled Monge-Ampére equations,

$$\begin{gathered} \det D^2 u = f(u,v), \quad x \in \Omega, \\ \det D^2 v = g(u,v), \quad x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded, smooth and strictly convex domain in $\mathbb{R}^n$, and $\det D^2 u$ stands for the determinant of Hessian matrix of u. We will restrict system (1.1) to be elliptic and search for nontrivial convex solutions, thus we suppose the functions f and g to be nonnegative.

Monge-Ampére equations have received considerable attention in the previous decades, because of their important applications in geometry and other scientific fields. However, systems coupled by Monge-Ampére equations have only been considered in recent years, see for example [5, 9, 10]. Wang [9] studied the system

$$\begin{gathered} \det D^2 u_1 = f(-u_2), \quad \text{in } B, \\ \det D^2 u_2 = g(-u_1), \quad \text{in } B, \\ u_1 = u_2 = 0, \quad \text{on } \partial B, \end{gathered} \tag{1.2}$$

with $B := \{x \in \mathbb{R}^n : |x| < 1\}$. Under suitable assumptions on f and g, the author found nontrivial radial convex solutions for (1.2), using ODE techniques together with fixed point theorems in a cone. More precisely, he obtained

Theorem 1.1 ([9, Theorem 1.1]). *Suppose $f, g : [0,\infty) \to [0,\infty)$ are continuous.*

(a) *If $f_0 = g_0 = 0$ and $f_\infty = g_\infty = \infty$, then (1.2) has at least one nontrivial radial convex solution.*
(b) *If $f_0 = g_0 = \infty$ and $f_\infty = g_\infty = 0$, then (1.2) has at least one nontrivial radial convex solution.*

2010 *Mathematics Subject Classification.* 35J47, 35J96, 35B32.
Key words and phrases. Monge-Ampére equations; elliptic system; global bifurcation.

Submitted December 21, 2015. Published June 14, 2016.

where

$$f_0 := \lim_{x\to 0^+} \frac{f(x)}{x^n}, \quad f_\infty := \lim_{x\to\infty} \frac{f(x)}{x^n}.$$

The above theorem implies the existence of a radial convex solution for the system

$$\begin{gathered}
\det D^2 u_1 = (-u_2)^\alpha, \quad \text{in } B, \\
\det D^2 u_2 = (-u_1)^\beta, \quad \text{in } B \\
u_1 < 0, \quad u_2 < 0, \quad \text{in } B, \\
u_1 = u_2 = 0, \quad \text{on } \partial B
\end{gathered} \tag{1.3}$$

if one of the following two conditions holds: (1) $\alpha > n, \beta > n$, (2) $\alpha < n, \beta < n$.

Theorem 1.1 was improved later in [10] by a decoupling method. For example, for system (1.3), the authors in [10] proved that it has a radial convex solution if and only if $\alpha > 0, \beta > 0$ and $\alpha\beta \neq n^2$. Moreover, as $\alpha\beta = n^2$, for the eigenvalue problem

$$\begin{gathered}
\det D^2 u_1 = \lambda(-u_2)^\alpha, \quad \text{in } \Omega, \\
\det D^2 u_2 = \mu(-u_1)^\beta, \quad \text{in } \Omega, \\
u_1 < 0, u_2 < 0, \quad \text{in } \Omega, \\
u_1 = u_2 = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.4}$$

with positive parameters λ and μ. They used a nonlinear version of Krein-Rutman theorem developed in [3] to obtain the following result.

Theorem 1.2 ([10, Theorem 1.4]). *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded, smooth and strictly convex domain. If $\alpha > 0, \beta > 0$ and $\alpha\beta = n^2$, then system (1.4) admits a convex solution if and only if $\lambda\mu^{\frac{\alpha}{n}} = C$, where C is a positive constant depending on n, α and Ω.*

The motivation of this article come from a corollary of Theorem 1.2 (see Lemma 2.1 below), as well as the work in [3], where Jacobsen investigated global bifurcation problems for a class of fully nonlinear elliptic equations, including the Monge-Ampére equation. As byproducts, Jacobsen obtaied some interesting existence results. We will show that, under suitable assumptions on the functions f and g, one can generalize part of the work in [3] to get new existence results for problem (1.1), see Theorem 3.8 and 4.6 below.

This article is organized as follows. In Section 2, we give some preliminaries. In Section 3 and 4, we study two bifurcation problems, where we obtain the main results in this paper.

2. Preliminaries

Unless otherwise stated Ω is supposed to be a bounded, smooth and strictly convex domain in $\mathbb{R}^n$. Let us recall the Monge-Ampére operator $\mathcal{M} : C^2(\Omega) \to C(\Omega)$, $\mathcal{M}[u] = \det D^2 u$. Since it is n-homogeneous, the eigenvalue problem for a single Monge-Ampére equation with Dirichlet boundary condition is described as

$$\begin{gathered}
\det D^2 u = |\lambda u|^n, \quad x \in \Omega, \\
u = 0, \quad x \in \partial\Omega.
\end{gathered}$$

It is known that there exists a unique positive $\lambda = \lambda_0(\Omega)$ such that the above problem admits nonzero convex solutions. In the literature, $\lambda_0(\Omega)$ is called the first eigenvalue, or the principal eigenvalue, of the Monge-Ampére operator corresponding to Ω, see references [3, 4, 8]. As for systems, we have the following lemma.

Lemma 2.1 ([10, Corollary 1.5]). *The eigenvalue problem*

$$\begin{aligned} \det D^2 u &= |\lambda v|^n, \quad x \in \Omega, \\ \det D^2 v &= |\lambda u|^n, \quad x \in \Omega, \\ u = v &= 0, \quad x \in \partial\Omega \end{aligned} \tag{2.1}$$

admits nonzero convex solutions if and only if $|\lambda| = \lambda_0(\Omega)$.

Thanks to Lemma 2.1, we are able to study global bifurcation problems for some systems of Monge-Ampére equations. Before we do this in the next sections, let us make some preparations first. We begin with some notations and terminologies that will be used later.

As in reference [3], we will use the following terminologies. Let Z be a real Banach space with a cone $P \subset Z$. The cone P induces a partial order via $u \preceq v \Leftrightarrow v - u \in P$. Let $A_0 : Z \to Z$.

- A_0 is called homogeneous if it is positively homogeneous with degree 1.
- A_0 is monotone if it satisfies $x \preceq y \Rightarrow A_0(x) \preceq A_0(y)$.

Now we recall a result due to Trudinger. As a special case of Trudinger [7, Theorem 1.1], in the second paragraph on p. 1253, we have

Lemma 2.2. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$, $\psi \in C(\overline{\Omega})$ with $\psi \geq 0$, $\phi \in C(\overline{\Omega})$. Then there exists a unique admissible weak solution $u \in C^1(\Omega) \cap C(\overline{\Omega})$ of the equation*

$$\begin{aligned} \det D^2 u &= \psi, \quad x \in \Omega, \\ u &= \phi, \quad x \in \partial\Omega. \end{aligned}$$

Remark 2.3. The definition of admissible weak solution coincides with the Aleksandrov sense weak solution (please see [7, page 1252-1253]), thus the admissible weak solutions occurred in the rest of the paper are also Aleksandrov solutions. For the notion of Aleksandrov solution, see [2, Definition 1.1.1, Theorem 1.1.13 and Definition 1.2.1].

By Lemma 2.2, we can define a solution operator as follows. Denote $C(\overline{\Omega})$ to be the usual Banach space of continuous functions with sup-norm. Define a cone

$$K_0 := \{u \in C(\overline{\Omega}) : u(x) \leq 0, \forall x \in \Omega\},$$

and an operator

$$T : C(\overline{\Omega}) \to C(\overline{\Omega}), \quad T(f) = u,$$

where u is the unique admissible weak solution of

$$\begin{aligned} \det D^2 u &= |f(x)|, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{2.2}$$

Note the Monge-Ampére operator is n-hessian (see [1]), so the solution operator defined by (2.2) coincides with T_n, where $T_k (k = 1, 2, \cdots, n)$ are solution operators for k-hessian equations defined in Section 3.1 in [3]. By [3, Proposition 3.2], $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ is completely continuous.

Let us define another operator. Take the Banach space $E := C(\overline{\Omega}) \times C(\overline{\Omega})$, with norm $\|(u,v)\| := \|u\|_\infty + \|v\|_\infty$. Define a cone

$$K = \{(u,v) \in E : u(x) \le 0,\ v(x) \le 0,\ \forall x \in \Omega\}.$$

We remark that we don't distinguish the writing of norms in E and $C(\overline{\Omega})$, and both are denoted by $\|\cdot\|$ in the paper. Define

$$A : E \to E,\ A(u,v) = (w,z),$$

where (w,z) is the unique admissible weak solution pair of

$$\begin{gathered} \det D^2 w = |v|^n, \quad x \in \Omega, \\ \det D^2 z = |u|^n, \quad x \in \Omega, \\ w = z = 0, \quad x \in \partial\Omega. \end{gathered} \tag{2.3}$$

By the completely continuity of T it is easy to see A is also completely continuous from E to E. Now E is a real Banach space, and the cone K induced a partial order on E via $(u_1,v_1) \preceq (u_2,v_2) \Leftrightarrow (u_2-u_1, v_2-v_1) \in K$. It is readily checked that A is homogeneous; by [2, Lemma 1.4.6], the comparison principle, we see T is monotone, so is A, i.e., $(u_1,v_1) \preceq (u_2,v_2) \Rightarrow A(u_1,v_1) \preceq A(u_2,v_2)$.

Properties of the operators T and A can be summarized as follows.

Lemma 2.4. *$T : C(\overline{\Omega}) \to C(\overline{\Omega})$ is a completely continuous, monotone operator. $A : E \to E$ is a completely continuous, monotone operator; moreover, it is homogeneous.*

In this article, we take b_0 a fixed number such that $b_0 > \lambda_0$. We have the following crucial result essentially given by Jacobsen[3], and give a proof for completeness.

Lemma 2.5. *For Leray-Schauder degree, we have*

$$\deg(\mathrm{id} - b_0 A(\cdot,\cdot), B_r(0,0), 0) = 0, \quad \forall r > 0. \tag{2.4}$$

Proof. First we note, by Lemma 2.1, that the degree in (2.4) is well defined for any $r > 0$, and it is independent with the value of r.

We argue by contradiction. Let (u_0,v_0) be a nonzero solution pair of (2.1) corresponding to λ_0, then we have

$$\lambda_0 A(u_0,v_0) = (u_0,v_0). \tag{2.5}$$

Fix $\overline{r} > 0$. By the continuity of Leray-Schauder degree, we can choose $\epsilon > 0$ small, such that

$$\deg(\mathrm{id} - b_0 A(\cdot,\cdot), B_{\overline{r}}(0,0), 0) = \deg(\mathrm{id} - b_0 A((\cdot,\cdot) + \epsilon(u_0,v_0)), B_{\overline{r}}(0,0), 0).$$

When (2.4) not true, we obtain

$$\deg(\mathrm{id} - b_0 A((\cdot,\cdot) + \epsilon(u_0,v_0)), B_{\overline{r}}(0,0), 0) \ne 0,$$

which implies the existence of $(\overline{u},\overline{v}) \in B_{\overline{r}}(0,0)$ such that

$$(\overline{u},\overline{v}) = b_0 A((\overline{u},\overline{v}) + \epsilon(u_0,v_0)). \tag{2.6}$$

Recall the partial order induced by K in E, we have $(\overline{u},\overline{v}) \preceq (\overline{u},\overline{v}) + \epsilon(u_0,v_0)$. Since A is monotone, we obtain

$$A(\overline{u},\overline{v}) \preceq A((\overline{u},\overline{v}) + \epsilon(u_0,v_0)). \tag{2.7}$$

Equations (2.6) and (2.7) give us

$$A(\overline{u},\overline{v}) \preceq \frac{(\overline{u},\overline{v})}{b_0}. \tag{2.8}$$

On the other hand, from $\epsilon(u_0,v_0) \preceq (\overline{u},\overline{v})+\epsilon(u_0,v_0)$, we have

$$A(\epsilon(u_0,v_0)) \preceq A((\overline{u},\overline{v})+\epsilon(u_0,v_0));$$

using (2.6) again, we reach

$$b_0 A(\epsilon(u_0,v_0)) \preceq (\overline{u},\overline{v}). \tag{2.9}$$

By (2.5), (2.9) and the homogeneity of A,

$$\frac{b_0\epsilon(u_0,v_0)}{\lambda_0} \preceq (\overline{u},\overline{v}). \tag{2.10}$$

Now operate A on both sides of (2.10), we have

$$\frac{b_0\epsilon A(u_0,v_0)}{\lambda_0} \preceq A(\overline{u},\overline{v}). \tag{2.11}$$

Combining (2.11) with (2.5) and (2.8), we deduce

$$\frac{b_0^2\epsilon(u_0,v_0)}{\lambda_0^2} \preceq (\overline{u},\overline{v}). \tag{2.12}$$

Noticing (2.10) and (2.12), one can prove by induction that

$$\frac{b_0^n\epsilon(u_0,v_0)}{\lambda_0^n} \preceq (\overline{u},\overline{v}), \quad \forall n\in\mathbb{N}.$$

So

$$(u_0,v_0) \preceq \Big(\frac{\lambda_0}{b_0}\Big)^n\cdot\frac{(\overline{u},\overline{v})}{\epsilon}, \quad \forall n\in\mathbb{N}.$$

Letting $n\to\infty$, from $b_0>\lambda_0>0$ we obtain $(u_0,v_0)\preceq(0,0)$. Thus $-(u_0,v_0)\in K$, giving $(u_0,v_0)\in K\cap(-K)=\{(0,0)\}$, a contradiction with $(u_0,v_0)\neq(0,0)$. This finishes the proof of the lemma. □

3. Global bifurcation

Our basic assumption on f and g is

(A1) $f,g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}_+:=[0,+\infty)$ are continuous.

We seek nontrivial solutions to (1.1). The approach used is motivated by [3]. More precisely, we embed (1.1) into the one-parameter family of problems

$$\begin{gathered}\det D^2u=|\lambda v|^n+f(u,v),\quad x\in\Omega,\\ \det D^2v=|\lambda u|^n+g(u,v),\quad x\in\Omega,\\ u=v=0,\quad x\in\partial\Omega,\end{gathered} \tag{3.1}$$

and consider the behavior of global bifurcation or global asymptotic bifurcation continuum. By continuum we shall mean a closed connected set.

We associated to (3.1) the solution operator $H:\mathbb{R}\times E\to E, H(\lambda,(u,v))=(w,z)$, where (w,z) is the unique solution pair of

$$\begin{gathered}\det D^2w=|\lambda v|^n+f(u,v),\quad x\in\Omega,\\ \det D^2z=|\lambda u|^n+g(u,v),\quad x\in\Omega,\\ w=z=0,\quad x\in\partial\Omega.\end{gathered} \tag{3.2}$$

Using the operator T defined in Section 2, we can write $H = (H_1, H_2)$, where $H_1(\lambda,(u,v)) = T(|\lambda v|^n + f(u,v))$, $H_2(\lambda,(u,v)) = T(|\lambda u|^n + g(u,v))$. By assumption (A1) and the completely continuity of T (see Lemma 2.4), it is easy to check H_1 and H_2 are both completely continuous. So H is completely continuous. Define $h : \mathbb{R} \times E \to E, h(\lambda,(u,v)) = (u,v) - H(\lambda,(u,v))$ and consider the equation

$$h(\lambda,(u,v)) = 0. \tag{3.3}$$

We see that λ, u and v satisfy (3.1) if and only if $(\lambda,(u,v))$ is a solution of (3.3).

Note that (3.1) can be seen as a perturbation of the eigenvalue problem (2.1). For our purpose in this section, the perturbation terms also need to satisfy:

(A2) $f(s,t) = o((|s|+|t|)^n), g(s,t) = o((|s|+|t|)^n)$, as $|s|+|t| \to 0$;

(A3) either $\frac{f(s,t)}{(|s|+|t|)^n} \to \infty$ or $\frac{g(s,t)}{(|s|+|t|)^n} \to \infty$, as $|s|+|t| \to \infty$.

Under assumptions of (A1) and (A2), one has $f(0,0) = g(0,0) = 0$, thus (3.3) admits trivial solution branch $\mathbb{R} \times (0,0)$. In order to obtain a nontrivial branch of solutions to (3.3), we need the following bifurcation theorem of Krasnosell'ski-Rabinowitz type.

Theorem 3.1 (global bifurcation, [6]). *Let Y be a Banach space, let $F : \mathbb{R}\times Y \to Y$ be completely continuous, such that $F(\lambda,0) = 0$, for all $\lambda \in \mathbb{R}$. Suppose there exist constants $a, b \in \mathbb{R}$, with $a < b$, such that $(a,0),(b,0)$ are not bifurcation points for the equation*

$$y - F(\lambda,y) = 0.$$

Furthermore, assume for Leray-Schauder degree that

$$\deg(\mathrm{id} - F(a,\cdot), B_r(0), 0) \neq \deg(\mathrm{id} - F(b,\cdot), B_r(0), 0),$$

where $B_r(0) = \{y \in E : \|y\| < r\}$ is an isolating neighborhood of the trivial solution for both constants a and b. Let

$$\mathcal{S} = \overline{\{(\lambda,y) : y - F(\lambda,y) = 0, y \neq 0\}} \cup ([a,b] \times \{0\}),$$

and let $\mathcal{C}$ be the component of $\mathcal{S}$ containing $[a,b] \times \{0\}$. Then either

(1) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times Y$, or*

(2) *$\mathcal{C} \cap [(\mathbb{R}\backslash[a,b]) \times \{0\}] \neq \emptyset$.*

We shall apply Theorem 3.1 to the Banach space E and the operator H after we collect some lemmas.

Lemma 3.2. *Assuming* (A1) *and* (A2)*, a necessary condition for $(\mu,(0,0))$ to be a bifurcation point of* (3.3) *is that $|\mu| = \lambda_0$.*

Proof. Suppose $(\mu,(0,0))$ is a bifurcation point for (3.3). Then there exists a sequence $(\lambda_k,(u_k,v_k)) \to (\mu,(0,0))$ such that $\|u_k\| + \|v_k\| \neq 0$ for all k, and $h(\lambda_k,(u_k,v_k)) = 0$, i.e.,

$$\begin{aligned} \det D^2 u_k &= |\lambda_k v_k|^n + f(u_k,v_k), \quad x \in \Omega, \\ \det D^2 v_k &= |\lambda_k u_k|^n + g(u_k,v_k), \quad x \in \Omega, \\ u_k &= v_k = 0, \quad x \in \partial\Omega. \end{aligned} \tag{3.4}$$

Divide each equation in (3.4) by $(\|u_k\| + \|v_k\|)^n$, and denote

$$\widetilde{u}_k = \frac{u_k}{\|u_k\| + \|v_k\|}, \quad \widetilde{v}_k = \frac{v_k}{\|u_k\| + \|v_k\|},$$

$$\widetilde{f}_k = \frac{f(u_k, v_k)}{(\|u_k\| + \|v_k\|)^n}, \quad \widetilde{g}_k = \frac{g(u_k, v_k)}{(\|u_k\| + \|v_k\|)^n},$$

we obtain

$$\begin{gathered}
\det D^2\widetilde{u}_k = |\lambda_k \widetilde{v}_k|^n + \widetilde{f}_k, \quad x \in \Omega, \\
\det D^2\widetilde{v}_k = |\lambda_k \widetilde{u}_k|^n + \widetilde{g}_k, \quad x \in \Omega, \\
\widetilde{u}_k = \widetilde{v}_k = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.5}$$

This system can be rewritten as

$$\begin{aligned}
\widetilde{u}_k &= T(|\lambda_k \widetilde{v}_k|^n + \widetilde{f}_k), \\
\widetilde{v}_k &= T(|\lambda_k \widetilde{u}_k|^n + \widetilde{g}_k).
\end{aligned} \tag{3.6}$$

Note $\|u_k\| + \|v_k\| \neq 0$, and u_k, v_k are both convex functions with zero boundary data, we have $|u_k(x)| + |v_k(x)| \neq 0$ for any $x \in \Omega$. Thus, for $x \in \Omega$,

$$0 \leq \widetilde{f}_k = \frac{f(u_k, v_k)}{(|u_k| + |v_k|)^n} \cdot \Big(\frac{|u_k| + |v_k|}{\|u_k\| + \|v_k\|}\Big)^n \leq \frac{f(u_k, v_k)}{(|u_k| + |v_k|)^n}.$$

Noticing $(u_k, v_k) \to (0, 0)$ in $C(\overline{\Omega}) \times C(\overline{\Omega})$, we deduce from the above inequalities and (A2) that $\widetilde{f}_k(x) \to 0$, uniformly for $x \in \Omega$, as $k \to \infty$. Combining this with the facts $\|\widetilde{v}_k\| \leq 1$ and $\lambda_k \to \mu$, we see $\{|\lambda_k \widetilde{v}_k|^n + \widetilde{f}_k\}$ is bounded in $C(\overline{\Omega})$. Hence, by (3.6) and the compactness of T, we obtain a sub-sequence of $\{\widetilde{u}_k\}$, still denoted $\{\widetilde{u}_k\}$, such that $\widetilde{u}_k \to u_*$ for some $u_* \in C(\overline{\Omega})$. Similarly, one can prove $\widetilde{g}_k(x) \to 0$, uniformly for $x \in \Omega$, as $k \to \infty$, and there exists a sub-sequence of $\{\widetilde{v}_k\}$, still denoted $\{\widetilde{v}_k\}$, such that $\widetilde{v}_k \to v_*$ for some $v_* \in C(\overline{\Omega})$. By the continuity of T, we infer from (3.6)

$$\begin{aligned}
u_* &= T(|\mu v_*|^n), \\
v_* &= T(|\mu u_*|^n).
\end{aligned} \tag{3.7}$$

We claim $(u_*, v_*) \neq (0, 0)$. Indeed,

$$\|u_*\| = \lim_{k\to\infty} \frac{\|u_k\|}{\|u_k\| + \|v_k\|}, \quad \|v_*\| = \lim_{k\to\infty} \frac{\|v_k\|}{\|u_k\| + \|v_k\|},$$

which yield

$$\|u_*\| + \|v_*\| = \lim_{k\to\infty} \frac{\|u_k\| + \|v_k\|}{\|u_k\| + \|v_k\|} = 1.$$

Now, by (3.7) and Lemma 2.1, we reach the conclusion $|\mu| = \lambda_0$. □

Lemma 3.3. *Assume* (A1) *and* (A2) *hold, then there exists $r > 0$, sufficiently small, such that*

1. $\deg(\mathrm{id} - H(0, (\cdot, \cdot)), B_r(0, 0), 0) = 1$,
2. $\deg(\mathrm{id} - H(b_0, (\cdot, \cdot)), B_r(0, 0), 0) = 0$.

Proof. First of all, by Lemma 3.2, $(0, (0, 0))$ and $(b_0, (0, 0))$ are not bifurcation points for (3.3), so one can take $r > 0$ sufficiently small, such that the degrees in assertions (1) and (2) are well defined.

Let $\tilde{b} \in \{0, b_0\}$. Define a homotopic mapping $F_{\tilde{b}} : [0, 1] \times E \to E, F_{\tilde{b}}(t, (u, v)) = (w, z)$, where (w, z) is the unique solution pair of

$$\begin{gathered}
\det D^2 w = |\tilde{b} v|^n + tf(u, v), \quad x \in \Omega, \\
\det D^2 z = |\tilde{b} u|^n + tg(u, v), \quad x \in \Omega, \\
w = z = 0, \quad x \in \partial\Omega.
\end{gathered}$$

By the complete continuity of T, $F_{\tilde b} : [0,1]\times E \to E$ is completely continuous. We claim when $r>0$ is sufficiently small, $\deg(\mathrm{id} - F_{\tilde b}(t,(\cdot,\cdot)), B_r(0,0), 0)$ is well defined for all $t\in[0,1]$. If this were not true, then there exist $\{t_m\}\subset[0,1]$ with $t_m \to t_0 \in [0,1]$, and $\{(u_m,v_m)\}\subset E$ with $\|(u_m,v_m)\| = r_m > 0$, $r_m \to 0$, such that $(u_m,v_m) = F_{\tilde b}(t_m,(u_m,v_m))$, i.e.,

$$\begin{gathered}
\det D^2 u_m = |\tilde b v_m|^n + t_m f(u_m,v_m), \quad x\in\Omega,\\
\det D^2 v_m = |\tilde b u_m|^n + t_m g(u_m,v_m), \quad x\in\Omega,\\
u_m = v_m = 0, \quad x\in\partial\Omega.
\end{gathered}$$

By mimicking the rest proof after (3.4) in Lemma 3.2, one reaches again $|\tilde b| = \lambda_0$, a contradiction with $\tilde b\in\{0,b_0\}$. So for $r>0$ sufficiently small we have

$$(u,v)\neq F_{\tilde b}(t,(u,v)), \quad \forall (u,v)\in\partial B_r(0,0),\ \forall t\in[0,1].$$

This implies $F_{\tilde b}$ is a degree-preserving homotopic mapping. We distinguish the following two cases.

Case $\tilde b = 0$. For $r>0$ small, we have

$$\begin{aligned}
\deg(\mathrm{id} - F_0(1,(\cdot,\cdot)), B_r(0,0), 0) &= \deg(\mathrm{id} - F_0(0,(\cdot,\cdot)), B_r(0,0), 0)\\
&= \deg(\mathrm{id}, B_r(0,0), 0) = 1.
\end{aligned}$$

Since $F_0(1,(\cdot,\cdot)) = H(0,(\cdot,\cdot))$, assertion (1) is valid.

Case $\tilde b = b_0$. For $r>0$ small, we have

$$\begin{aligned}
\deg(\mathrm{id} - H(b_0,(\cdot,\cdot)), B_r(0,0), 0) &= \deg(\mathrm{id} - F_{b_0}(1,(\cdot,\cdot)), B_r(0,0), 0)\\
&= \deg(\mathrm{id} - F_{b_0}(0,(\cdot,\cdot)), B_r(0,0), 0)\\
&= \deg(\mathrm{id} - A(b_0(\cdot,\cdot)), B_r(0,0), 0)\\
&= \deg(\mathrm{id} - b_0 A(\cdot,\cdot), B_r(0,0), 0).
\end{aligned} \tag{3.8}$$

By Lemma 2.5 and (3.8), we see assertion (2) is also valid. □

Now let us recall a known blow-up result. Since the Monge-Ampére operator is n-hessian, we have a special case of Jacobsen [3, Lemma 5.1].

Lemma 3.4. *Let $\{v_m\}\subset C(\overline{\Omega})$ be a collection of admissible weak solutions to the Dirichlet problem*

$$\begin{gathered}
\det D^2 v_m = g_m, \quad x\in\Omega,\\
v_m = 0, \quad x\in\partial\Omega,
\end{gathered}$$

where $g_m : \Omega \to \mathbb{R}$ form a collection of nonnegative continuous functions. If $g_m(x)\to\infty$, uniformly on some compact sub-domain of Ω, then $\|v_m\|\to\infty$.

Using this lemma, we can establish some priori bounds concerning solutions of (3.3).

Lemma 3.5. *Under assumption* (A1)*, there exists $M_1>0$, such that any solution $(\lambda,(u,v))$ of* (3.3) *with $(u,v)\neq(0,0)$ must satisfy $|\lambda|\le M_1$.*

Proof. We argue by contradiction. If the conclusion of Lemma 3.5 is false, then there exists $\{(\lambda_k,(u_k,v_k))\}$, solving (3.3) for each k, such that $\|u_k\|+\|v_k\|>0$, and $|\lambda_k|\to\infty$ as $k\to\infty$. Let

$$\widetilde{u}_k := \frac{u_k}{\|u_k\|+\|v_k\|}, \quad \widetilde{v}_k := \frac{v_k}{\|u_k\|+\|v_k\|},$$

and let Ω' be a compact sub-domain of Ω. Since $\|\widetilde{u}_k\| + \|\widetilde{v}_k\| = 1$, we may assume, without loss of generality, that there exists a $\gamma > 0$ such that

$$\|\widetilde{v}_k\| \geq \gamma. \tag{3.9}$$

Moreover, by [3, Lemma 5.10], there exists $\eta > 0$ such that

$$|\widetilde{v}_k(x)| \geq \eta \|\widetilde{v}_k\|, \quad \forall x \in \Omega'. \tag{3.10}$$

As $(\lambda_k, (u_k, v_k))$ solves (3.3), we have

$$\begin{gathered}\det D^2 u_k = |\lambda_k v_k|^n + f(u_k, v_k), \quad x \in \Omega, \\ u_k = 0, \quad x \in \partial\Omega.\end{gathered}$$

Dividing the above equation by $(\|u_k\| + \|v_k\|)^n$, we obtain

$$\begin{gathered}\det D^2 \widetilde{u}_k = \psi_k, \quad x \in \Omega, \\ \widetilde{u}_k = 0, \quad x \in \partial\Omega,\end{gathered} \tag{3.11}$$

where

$$\psi_k := |\lambda_k \widetilde{v}_k|^n + \frac{f(u_k, v_k)}{(\|u_k\| + \|v_k\|)^n}. \tag{3.12}$$

Equations (3.12), (3.9) and (3.10) yield that, for $x \in \Omega'$,

$$\psi_k(x) \geq |\lambda_k \widetilde{v}_k|^n \geq |\lambda_k \eta\gamma|^n \to \infty, k \to \infty. \tag{3.13}$$

By Lemma 3.4, we deduce from (3.11) and (3.13) that $\|\widetilde{u}_k\| \to \infty$ as $k \to \infty$, a contradiction with $\|\widetilde{u}_k\| \leq 1$. □

Lemma 3.6. *Under assumptions* (A1) *and* (A3)*, there exists $M_2 > 0$, such that a solution $(\lambda, (u, v))$ of* (3.3) *must satisfy $\|u\| + \|v\| \leq M_2$.*

Proof. Without loss of generality, we assume the first alternative of (A3) holds, i.e.,

$$\frac{f(s,t)}{(|s| + |t|)^n} \to \infty, \quad \text{as } |s| + |t| \to \infty. \tag{3.14}$$

We argue by contradiction. If the conclusion of Lemma 3.6 is false, then there exists $\{(\lambda_k, (u_k, v_k))\}$, solving (3.3) for each k, such that $\|u_k\| + \|v_k\| > 0$, $\|u_k\| + \|v_k\| \to \infty$ as $k \to \infty$. Now we have

$$\begin{gathered}\det D^2 u_k = |\lambda_k v_k|^n + f(u_k, v_k), \quad x \in \Omega, \\ u_k = 0, \quad x \in \partial\Omega.\end{gathered}$$

Divide the above equation by $(\|u_k\| + \|v_k\|)^n$, and denote

$$\varphi_k(x) := \frac{|\lambda_k v_k(x)|^n + f(u_k(x), v_k(x))}{(\|u_k\| + \|v_k\|)^n},$$

we reach

$$\begin{gathered}\det D^2 \Big(\frac{u_k}{\|u_k\| + \|v_k\|}\Big) = \varphi_k, \quad x \in \Omega, \\ \frac{u_k}{\|u_k\| + \|v_k\|} = 0, \quad x \in \partial\Omega.\end{gathered} \tag{3.15}$$

Note that

$$\varphi_k(x) \geq \frac{f(u_k, v_k)}{(|u_k| + |v_k|)^n} \Big(\frac{|u_k| + |v_k|}{\|u_k\| + \|v_k\|}\Big)^n, \forall x \in \Omega. \tag{3.16}$$

Let Ω' be a compact sub-domain of Ω. By [3, Lemma 5.10], there exists $\delta > 0$, such that $|u_k(x)| \geq \delta \|u_k\|$ and $|v_k(x)| \geq \delta \|v_k\|$ for any $x \in \Omega'$. Thus $|u_k(x)| + |v_k(x)| \geq$

$\delta(\|u_k\|+\|v_k\|)$ for any $x\in\Omega'$. Since $\|u_k\|+\|v_k\|\to\infty$, we see it holds uniformly for $x\in\Omega'$ that $|u_k(x)|+|v_k(x)|\to\infty$. Using these facts and (3.14), we deduce from (3.16) that for $x\in\Omega'$, it holds uniformly

$$\varphi_k(x)\to\infty,\quad \text{as } k\to\infty. \tag{3.17}$$

By (3.15) and (3.17), we infer from Lemma 3.4 that

$$\frac{\|u_k\|}{\|u_k\|+\|v_k\|}\to\infty,\quad \text{as } k\to\infty,$$

which contradicts

$$\frac{\|u_k\|}{\|u_k\|+\|v_k\|}\le 1,\quad \forall k\in\mathbb{N}.$$

□

We are in a position to give the main results in this section. Recall that, by continuum we mean a closed connected set.

Theorem 3.7. *Under assumptions* (A1)–(A3), *there exists a bounded continuum of solutions to* (3.3) *bifurcating from* $(\lambda_0,(0,0))$ *in* $\mathbb{R}\times E$. *This continuum connects* $(\lambda_0,(0,0))$ *to* $(-\lambda_0,(0,0))$. *It is nontrivial in the sense that it intersects the trivial solution branch of* (3.3) *only at* $(\pm\lambda_0,(0,0))$.

Proof. Let us apply Theorem 3.1 to the Banach space E and the operator H. By Lemmas 3.2 and 3.3, we infer from Theorem 3.1 that there exists a nontrivial branch of solutions to (3.3), say $\overline{\mathcal{C}}$, bifurcating from $(\lambda_0,(0,0))$, and it holds $([0,b_0]\times(0,0))\cap\overline{\mathcal{C}}=(\lambda_0,(0,0))$. Furthermore, either

(1) $\overline{\mathcal{C}}$ is unbounded in $\mathbb{R}\times E$, or
(2) $\overline{\mathcal{C}}\cap[(\mathbb{R}\backslash[0,b_0])\times(0,0)]\neq\emptyset$.

By Lemmas 3.5 and 3.6, $\overline{\mathcal{C}}$ must be bounded in $\mathbb{R}\times E$, so $\overline{\mathcal{C}}$ must connect to another bifurcation point. By Lemma 3.2, $\overline{\mathcal{C}}$ connects $(\lambda_0,(0,0))$ to $(-\lambda_0,(0,0))$, and it cannot intersect the trivial solution branch of (3.3) at points other than $(\pm\lambda_0,(0,0))$. □

Theorem 3.8. *Assume the functions f and g satisfy* (A1)–(A3), *then* (3.1) *admits at least a nontrivial convex solution for all* $\lambda\in(-\lambda_0,\lambda_0)$. *In particular,* (1.1) *admits at least a nontrivial convex solution.*

Proof. By Theorem 3.7, there exists a bounded continuum of solutions to (3.3) that is nontrivial, and it connects $(-\lambda_0,(0,0))$ to $(\lambda_0,(0,0))$ in $\mathbb{R}\times E$. Since it is connected, for arbitrarily fixed $\widehat{\lambda}\in(-\lambda_0,\lambda_0)$, the continuum must cross $\lambda=\widehat{\lambda}$ at a point, say $(\widehat{\lambda},(u,v))$. By Lemma 3.2, $\widehat{\lambda}$ is not a bifurcation value, thus $(u,v)\neq(0,0)$, and it is a nontrivial convex solution for (3.1) with $\lambda=\widehat{\lambda}$. □

4. Global asymptotic bifurcation

Besides (A1), we also need the following assumptions on f and g:

(A4) either $\frac{f(s,t)}{(|s|+|t|)^n}\to\infty$, or $\frac{g(s,t)}{(|s|+|t|)^n}\to\infty$, as $|s|+|t|\to 0$;
(A5) $\frac{f(s,t)}{(|s|+|t|)^n}\to 0$, and $\frac{g(s,t)}{(|s|+|t|)^n}\to 0$, as $|s|+|t|\to\infty$.

In this section, we study global asymptotic bifurcation problems for (3.3). Our analysis is based on the theorem below.

Theorem 4.1 (Global asymptotic bifurcation, [6])**.** *Let Y be a Banach space, let $F:\mathbb{R}\times Y\to Y$ be completely continuous. Suppose there exist constants $a,b\in\mathbb{R}$, with $a<b$, such that solutions of*

$$y-F(\lambda,y)=0 \tag{4.1}$$

are a priori bounded in Y for $\lambda=a$ and $\lambda=b$; i.e., there exists a constant $M>0$ such that

$$F(a,y)\neq y\neq F(b,y),$$

for all $y\in Y$ with $\|y\|\geq M$. Furthermore, assume that

$$\deg(\mathrm{id}-F(a,\cdot),B_R(0),0)\neq\deg(\mathrm{id}-F(b,\cdot),B_R(0),0),$$

for $R>M$. Then there exists at least one continuum $\mathcal{C}$ of solutions to (4.1) *that is unbounded in $[a,b]\times Y$ and either*

(1) *$\mathcal{C}$ in unbounded in the λ direction, or else,*
(2) *there exists an interval $[c,d]$ such that $(a,b)\cap(c,d)=\emptyset$, and $\mathcal{C}$ bifurcates from infinity in $[c,d]\times Y$.*

In Theorem 4.1, to say $\mathcal{C}$ bifurcates from infinity in $[c,d]\times Y$, we mean there exist $\nu\in[c,d]$ and a sequence $\{(\lambda_k,y_k)\}\subseteq\mathcal{C}$, such that $\lambda_k\to\nu$ and $\|y_k\|_Y\to\infty$ as $k\to\infty$. We shall apply Theorem 4.1 to the Banach space E and the operator H after we collect some lemmas.

Lemma 4.2. *Under assumptions* (A1) *and* (A5)*, a necessary condition for μ to be an asymptotic bifurcation value of* (3.3) *is $|\mu|=\lambda_0$.*

Proof. Suppose μ is an asymptotic bifurcation value for (3.3), i.e., there exists a sequence $\{(\lambda_k,(u_k,v_k))\}$ such that $\|u_k\|+\|v_k\|\to\infty,\lambda_k\to\mu$ as $k\to\infty$, and it satisfies $h(\lambda_k,(u_k,v_k))=0,\forall k\in\mathbb{N}$. Thus we have

$$\begin{gathered}\det D^2u_k=|\lambda_kv_k|^n+f(u_k,v_k),\quad x\in\Omega,\\ \det D^2v_k=|\lambda_ku_k|^n+g(u_k,v_k),\quad x\in\Omega,\\ u_k=v_k=0,\quad x\in\partial\Omega.\end{gathered} \tag{4.2}$$

Divide (4.2) by $(\|u_k\|+\|v_k\|)^n$, and denote

$$\begin{gathered}\overline{u}_k=\frac{u_k}{\|u_k\|+\|v_k\|},\quad \overline{v}_k=\frac{v_k}{\|u_k\|+\|v_k\|},\\ \overline{f}_k=\frac{f(u_k,v_k)}{(\|u_k\|+\|v_k\|)^n},\quad \overline{g}_k=\frac{g(u_k,v_k)}{(\|u_k\|+\|v_k\|)^n},\end{gathered}$$

we obtain

$$\begin{gathered}\det D^2\overline{u}_k=|\lambda_k\overline{v}_k|^n+\overline{f}_k,\quad x\in\Omega,\\ \det D^2\overline{v}_k=|\lambda_k\overline{u}_k|^n+\overline{g}_k,\quad x\in\Omega,\\ \overline{u}_k=\overline{v}_k=0,\quad x\in\partial\Omega.\end{gathered} \tag{4.3}$$

This system can be rewritten as

$$\begin{gathered}\overline{u}_k=T(|\lambda_k\overline{v}_k|^n+\overline{f}_k),\\ \overline{v}_k=T(|\lambda_k\overline{u}_k|^n+\overline{g}_k).\end{gathered} \tag{4.4}$$

We claim that $\overline{f}_k(x) \to 0$, uniformly for $x \in \Omega$, as $k \to \infty$. Indeed, by (A5), for each $\epsilon > 0$, there exists $M_0 > 0$, such that if $|(s,t)| := |s| + |t| > M_0$, then

$$\frac{f(s,t)}{(|s| + |t|)^n} < \epsilon. \tag{4.5}$$

For this $M_0 > 0$, denote $f_0 := \max_{|(s,t)| \le M_0} f(s,t)$, then for large k,

$$\frac{f_0}{(\|u_k\| + \|v_k\|)^n} < \epsilon. \tag{4.6}$$

By (4.5), (4.6) and (A1), we deduce that for k sufficiently large,

$$0 \le \overline{f}_k(x) < \epsilon, \quad \forall x \in \Omega.$$

So our claim holds.

Similarly, $\overline{g}_k(x) \to 0$, uniformly for $x \in \Omega$, as $k \to \infty$. By mimicking the counterpart in the proof of Lemma 3.2, one is ready to reach $|\mu| = \lambda_0$. We omit the details. □

Recall b_0 is a fixed number such that $b_0 > \lambda_0$.

Lemma 4.3. *Assume* (A1) *and* (A5). *Then there exists $M > 0$, such that for $R > M$,*

(1) $\deg(\mathrm{id} - H(0, (\cdot,\cdot)), B_R(0,0), 0) = 1$;
(2) $\deg(\mathrm{id} - H(b_0, (\cdot,\cdot)), B_R(0,0), 0) = 0$.

Proof. By Lemma 4.2, there exists $M > 0$ such that for all $(u,v) \in E$ with $\|(u,v)\| \ge M$,

$$H(0,(u,v)) \ne (u,v) \ne H(b_0,(u,v)).$$

So when $R > M$, the degrees in the assertions are well defined and independent of R.

Let $\tilde b \in \{0, b_0\}$. Define a homotopic mapping $F_{\tilde b} : [0,1] \times E \to E$, be defined by $F_{\tilde b}(t,(u,v)) = (w,z)$, where (w,z) is the unique solution pair of

$$\begin{gathered}
\det D^2 w = |\tilde b v|^n + t f(u,v), \quad x \in \Omega, \\
\det D^2 z = |\tilde b u|^n + t g(u,v), \quad x \in \Omega, \\
w = z = 0, \quad x \in \partial\Omega.
\end{gathered}$$

By the complete continuity of T, one verifies $F_{\tilde b} : [0,1] \times E \to E$ is completely continuous. We point out that when $R > M$ is sufficient large, the function $\deg(\mathrm{id} - F_{\tilde b}(t,(\cdot,\cdot)), B_R(0,0), 0)$ is well defined for all $t \in [0,1]$. If this were not true, then there exist $\{t_m\} \subset [0,1]$ with $t_m \to t_0 \in [0,1]$, and $\{(u_m,v_m)\} \subset E$ with $\|(u_m,v_m)\| = R_m \to +\infty$, such that $(u_m,v_m) = F_{\tilde b}(t_m,(u_m,v_m))$, i.e.,

$$\begin{gathered}
\det D^2 u_m = |\tilde b v_m|^n + t_m f(u_m,v_m), \quad x \in \Omega, \\
\det D^2 v_m = |\tilde b u_m|^n + t_m g(u_m,v_m), \quad x \in \Omega, \\
u_m = v_m = 0, \quad x \in \partial\Omega.
\end{gathered}$$

Divide the above system by $(\|u_m\| + \|v_m\|)^n$, and then follow the arguments used in the proof of Lemma 4.2, one reaches again $|\tilde b| = \lambda_0$, a contradiction with $\tilde b \in \{0, b_0\}$. So when $R > M$ is sufficiently large,

$$(u,v) \ne F_{\tilde b}(t,(u,v)), \quad \forall (u,v) \in \partial B_R(0,0), \quad \forall t \in [0,1].$$

This implies $F_{\tilde{b}}$ is a degree-preserving homotopic mapping. We distinguish the following two cases.

Case $\tilde{b} = 0$. For $R > M$ large, we have

$$\begin{aligned}\deg(\mathrm{id} - F_0(1,(\cdot,\cdot)), B_R(0,0), 0) &= \deg(\mathrm{id} - F_0(0,(\cdot,\cdot)), B_R(0,0), 0)\\ &= \deg(\mathrm{id}, B_R(0,0), 0) = 1.\end{aligned}$$

Since $F_0(1,(\cdot,\cdot)) = H(0,(\cdot,\cdot))$, assertion (1) is valid.

Case $\tilde{b} = b_0$. For $R > M$ large, we have

$$\begin{aligned}\deg(\mathrm{id} - H(b_0,(\cdot,\cdot)), B_R(0,0), 0) &= \deg(\mathrm{id} - F_{b_0}(1,(\cdot,\cdot)), B_R(0,0), 0)\\ &= \deg(\mathrm{id} - F_{b_0}(0,(\cdot,\cdot)), B_R(0,0), 0)\\ &= \deg(\mathrm{id} - A(b_0(\cdot,\cdot)), B_R(0,0), 0)\\ &= \deg(\mathrm{id} - b_0 A(\cdot,\cdot), B_R(0,0), 0).\end{aligned}$$

By this equality and Lemma 2.5, we see assertion (2) is also valid. □

Lemma 4.4. *Assume* (A1) *and* (A4). *Then there exists $\varepsilon > 0$, such that any solution $(\lambda,(u,v))$ of* (3.3) *with $(u,v) \neq (0,0)$ must satisfy $\|(u,v)\| \geq \varepsilon$.*

Proof. Without loss of generality, we assume the first alternative of (A4) holds, i.e.,

$$\frac{f(s,t)}{(|s|+|t|)^n} \to \infty, \quad \text{as } |s|+|t| \to 0. \tag{4.7}$$

We argue by contradiction. If the conclusion of Lemma 4.4 is false, then there exists $\{(\lambda_k,(u_k,v_k))\}$, solving (3.3) for each k, such that $(u_k,v_k) \neq (0,0)$, $(u_k,v_k) \to (0,0)$ as $k \to \infty$. Now we have

$$\begin{gathered}\det D^2 u_k = |\lambda_k v_k|^n + f(u_k,v_k), \quad x \in \Omega,\\ u_k = 0, \quad x \in \partial\Omega.\end{gathered}$$

Divide the above equation by $(\|u_k\| + \|v_k\|)^n$, and denote

$$\zeta_k(x) := \frac{|\lambda_k v_k(x)|^n + f(u_k(x),v_k(x))}{(\|u_k\| + \|v_k\|)^n},$$

we reach

$$\begin{gathered}\det D^2\Big(\frac{u_k}{\|u_k\| + \|v_k\|}\Big) = \zeta_k, \quad x \in \Omega,\\ \frac{u_k}{\|u_k\| + \|v_k\|} = 0, \quad x \in \partial\Omega.\end{gathered} \tag{4.8}$$

Note that

$$\zeta_k(x) \geq \frac{f(u_k,v_k)}{(|u_k|+|v_k|)^n}\Big(\frac{|u_k|+|v_k|}{\|u_k\|+\|v_k\|}\Big)^n, \forall x \in \Omega. \tag{4.9}$$

Let Ω' be a compact sub-domain of Ω. By [3, Lemma 5.10], there exists $\widetilde{\delta} > 0$, such that $|u_k(x)| \geq \widetilde{\delta}\|u_k\|$ and $|v_k(x)| \geq \widetilde{\delta}\|v_k\|$, for any $x \in \Omega'$. Thus

$$|u_k(x)| + |v_k(x)| \geq \widetilde{\delta}(\|u_k\| + \|v_k\|), \quad \forall x \in \Omega'. \tag{4.10}$$

Note $(u_k,v_k) \to (0,0)$ in E, so for $x \in \Omega'$, it holds uniformly

$$|u_k(x)| + |v_k(x)| \to 0, \quad k \to \infty. \tag{4.11}$$

By (4.7), (4.10) and (4.11), it is easy to deduce from (4.9) that for $x \in \Omega'$, it holds uniformly

$$\zeta_k(x) \to \infty, \quad k \to \infty. \tag{4.12}$$

By (4.8) and (4.12), we infer from Lemma 3.4 that

$$\frac{\|u_k\|}{\|u_k\|+\|v_k\|}\to\infty,\quad k\to\infty,$$

which contradicts $\frac{\|u_k\|}{\|u_k\|+\|v_k\|}\le 1$ for all $k\in\mathbb{N}$. □

Now we can give the main results of this section.

Theorem 4.5. *Assume* (A1), (A4) *and* (A5). *Then there exists an unbounded continuum of nontrivial solutions to* (3.3) *in* $\mathbb{R}\times E$. *The continuum bifurcates from infinity at* $\mu=\pm\lambda_0$, *and it is bounded in the* λ *direction.*

Proof. Let us apply Theorem 4.1 to the Banach space E and the operator H. By Lemma 4.2, there exists $M>0$ such that for $(u,v)\in E$ with $\|(u,v)\|\ge M$, it holds $H(0,(u,v))\ne(u,v)\ne H(b_0,(u,v))$. By Lemma 4.3, we can choose $M>0$ large, so that

$$\deg(\mathrm{id}-H(0,\cdot),B_R(0),0)\ne\deg(\mathrm{id}-H(b_0,\cdot),B_R(0),0)$$

for $R>M$. We infer from Theorem 4.1 that there exists a continuum of solutions to (3.3), say $\widetilde{\mathcal{C}}$, that is unbounded in $[0,b_0]\times E$, which forces λ_0 to be an asymptotic bifurcation value by Lemma 4.2. Furthermore, either

1. $\widetilde{\mathcal{C}}$ is unbounded in the λ direction, or
2. there exist an interval $[c,d]$ such that $(0,b_0)\cap(c,d)=\emptyset$, and $\widetilde{\mathcal{C}}$ bifurcates from infinity in $[c,d]\times E$.

By Lemma 3.5, $\widetilde{\mathcal{C}}$ is bounded in the λ direction, so it must bifurcate from infinity at $-\lambda_0$ by Lemma 4.2. Since $\widetilde{\mathcal{C}}$ is connected and unbounded, we infer from Lemma 4.4 that

$$(u,v)\ne(0,0),\quad\forall(\lambda,(u,v))\in\widetilde{\mathcal{C}}. \tag{4.13}$$

□

Theorem 4.6. *Assume* (A1), (A4) *and* (A5). *Then* (3.1) *has a nontrivial convex solution for all* $\lambda\in(-\lambda_0,\lambda_0)$. *In particular,* (1.1) *admits a nontrivial convex solution.*

Proof. Note that the continuum $\widetilde{\mathcal{C}}$ obtained in the proof of Theorem 4.5 bifurcates from infinity at $\mu=\pm\lambda_0$. So by connectedness and (4.13), we see (3.1) has a nontrivial convex solution for all $\lambda\in(-\lambda_0,\lambda_0)$. □

Remark 4.7. We say that a solution (u,v) of (3.1) is a vector solution if $u\ne0$ and $v\ne0$. When $0<|\lambda|<\lambda_0$, the solutions for (3.1) obtained in Theorem 3.8 and 4.6 are vector solutions, which can be inferred from system (3.1) itself and the assumption (A1). Similarly, if f and g are such that

$$f(s,t)>0,\quad g(s,t)>0,\quad\forall(s,t)\ne(0,0),$$

then solutions for (1.1) obtained in Theorem 3.8 and Theorem 4.6 are also vector solutions.

To illustrate our results for problem (1.1), We present the following example: Let Ω be a bounded, smooth, and strictly convex domain in $\mathbb{R}^n$. If $0<p_1,p_2<n$ or $p_1,p_2>n$, then the system

$$\det D^2u_1=\lambda(-u_1-u_2)^{p_1},\quad x\in\Omega,$$
$$\det D^2u_2=\lambda(-u_1-u_2)^{p_2},\quad x\in\Omega,$$

$$u_1 = u_2 = 0, \quad x \in \partial\Omega$$

admits at least a nontrivial convex solution for any $\lambda > 0$.

Acknowledgements. This work was supported by Doctoral Starting up Foundation of Henan Normal university (Grant No. 5101019170149).

References

[1] L. Caffarelli, L. Nirenberg, J. Spruck; *The Dirichlet problem for nonlinear second order elliptic equations III: Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261-301.

[2] C. Gutierrez; *The Monge-Ampére Equation*, Birkhauser, Basel, 2000.

[3] J. Jacobsen; *Global bifurcation problems associated with K-Hessian operators*, Topol. Methods Nonlinear Anal. **14** (1999), 81-130.

[4] P. L. Lions; *Two remarks on Monge-Ampére equations*, Ann. Mat. Pura Appl. **142**(4) (1985), 263-275.

[5] L. Ma, B. Liu; *Symmetry results for classical solutions of Monge-Ampére systems on bounded planar domains*, J. Math. Anal. Appl. **369**(2) (2010), 678-685.

[6] K. Schmitt, R. Thompson; *Nonlinear analysis and differential equations: An introduction.* Univ. of Utah Lecture Notes, Univ. of Utah Press, Salt Lake City, 2004.

[7] N. S. Trudinger; *Weak solutions of hessian equations*, Communication in Partial Differential Equations. **22**(7&8) (1997), 1251-1261.

[8] K. Tso; *On a real Monge-Ampére functional*, Invent Math. **101** (1990), 425-448.

[9] H. Wang; *Convex solutions of systems arising from Monge-Ampére equations*, Electric Journal of Qualitative Theory of Differential Equations, Special Edition I. **26** (2009), 1-8.

[10] Z. Zhang, Z. Qi; *On a power-type coupled system of Monge-Ampere equations*, Topol. Methods Nonlinear Anal. **46**(2) (2015), 717-729.

Zexin Qi
School of Mathematics and Information Sciences, Henan Normal University, Xinxiang 453007, China
E-mail address: `qizedong@126.com`