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FRACTIONAL ELLIPTIC EQUATIONS WITH SIGN-CHANGING AND SINGULAR NONLINEARITY

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Abstract. In this article, we study the fractional Laplacian equation with singular nonlinearity

$$\begin{gathered}(-\Delta)^s u = a(x)u^{-q} + \lambda b(x)u^p \quad \text{in } \Omega,\\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega,\end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, $\lambda > 0$. Using variational methods, we show existence and multiplicity of positive solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $n > 2s$ and $s \in (0,1)$. We consider the fractional elliptic problem with singular nonlinearity

$$\begin{gathered}(-\Delta)^s u = a(x)u^{-q} + \lambda b(x)u^p \quad \text{in } \Omega,\\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega.\end{gathered} \tag{1.1}$$

We use the following assumptions on a and b:

- (A1) $a : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $0 < a \in L^{\frac{2_s^*}{2_s^*-1+q}}(\Omega)$.
- (A2) $b : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a sign-changing function such that $b^+ \not\equiv 0$ and $b(x) \in L^{\frac{2_s^*}{2_s^*-1-p}}(\Omega)$.

Here $\lambda > 0$ is a parameter, $0 < q < 1 < p < 2_s^* - 1$, with $2_s^* = \frac{2n}{n-2s}$, known as fractional critical Sobolev exponent and where $(-\Delta)^s$ is the fractional Laplacian operator in Ω with zero Dirichlet boundary values on $\partial\Omega$.

To define the fractional Laplacian operator $(-\Delta)^s$ in Ω, let $\{\lambda_k, \phi_k\}$ be the eigenvalues and the corresponding eigenfunctions of $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$

$$(-\Delta)^s \phi_k = \lambda_k \phi_k \text{ in } \Omega, \quad \phi_k = 0 \text{ on } \partial\Omega.$$

normalized by $\|\phi_k\|_{L^2(\Omega)} = 1$. Then one can define the fractional Laplacian $(-\Delta)^s$ for $s \in (0,1)$ by

$$(-\Delta)^s u = \sum_{k=1}^{\infty} \lambda_k^s c_k \phi_k,$$

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which clearly maps

$$H_0^s(\Omega) := \Big\{u = \sum_{k=1}^{\infty} c_k\phi_k \in L^2(\Omega) : \|u\|_{H_0^s(\Omega)} = \Big(\sum_{k=1}^{\infty} \lambda_k^s c_k^2\Big)^{1/2} < \infty\Big\}$$

into $L^2(\Omega)$. Moreover $H_0^s(\Omega)$ is a Hilbert space endowed with an inner product

$$\Big\langle \sum_{k=1}^{\infty} c_k\phi_k, \sum_{k=1}^{\infty} d_k\phi_k\Big\rangle_{H_0^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^s c_k d_k\phi_k, \quad \text{if } \sum_{k=1}^{\infty} c_k\phi_k,\ \sum_{k=1}^{\infty} d_k\phi_k \in H_0^s(\Omega).$$

Definition 1.1. A function $u \in H_0^s(\Omega)$ such that $u(x) > 0$ in Ω is a solution of (1.1) such that for every function $v \in H_0^s(\Omega)$, it holds

$$\int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}v dx = \int_\Omega a(x)u^{-q}v dx + \lambda \int_\Omega b(x)u^p v dx.$$

Associated with (1.1), we consider the energy functional for $u \in H_0^s(\Omega)$, $u > 0$ in Ω such that

$$I_\lambda(u) = \int_\Omega |(-\Delta)^{s/2}u|^2 dx - \frac{1}{1-q}\int_\Omega a(x)|u|^{1-q}dx - \frac{\lambda}{p+1}\int_\Omega b(x)|u|^{p+1}dx.$$

The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusions in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, we refer to [3, 14] and reference therein.

Recently the study of existence, multiplicity of solutions for fractional elliptic equations attracted a lot of interest by many researchers. Among the works dealing with fractional elliptic equations we cite [6, 9, 21, 22, 23, 24, 25, 26] and references therein, with no attempt to provide a complete list. Caffarelli and Silvestre [8] gave a new formulation of fractional Laplacian through Dirichlet-Neumann maps. This formulation transforms problems involving the fractional Laplacian into a local problem which allows one to use the variational methods.

On the other hand, there are some works where multiplicity results are shown using the structure of associated Nehari manifold. In [15, 16] authors studied subcritical problems and in [28] the authors obtained the existence of multiplicity for critical growth nonlinearity. In the case of the square root of Laplacian, the multiplicity results for sublinear and superlinear type of nonlinearity with sign-changing weight functions are studied in [7, 27].

In the local setting, $s = 1$, the paper by Crandall, Robinowitz and Tartar [12] is the starting point on semilinear problem with singular nonlinearity. There is a large body of literature on singular problems, see [1, 2, 11, 12, 17, 18, 19, 20] and reference therein. Recently, Chen and Chen in [10] studied the following problem with singular nonlinearity

$$-\Delta u - \frac{\lambda}{|x|^2} = h(x)u^{-q} + \mu W(x)u^p \text{ in } \Omega\setminus\{0\}, \quad u > 0 \text{ in } \Omega\setminus\{0\}, \quad u = 0 \text{ on } \partial\Omega,$$

where $0 \in \Omega \subset \mathbb{R}^n (n \geq 3)$ is a bounded smooth domain with smooth boundary, $0 < \lambda < \frac{(n-2)^2}{4}$ and $0 < q < 1 < p < \frac{n+2}{n-2}$. Also h, W both are continuous functions in $\overline{\Omega}$ with $h > 0$ and W is sign-changing. By variational methods, they showed that there exists T_λ such that for $\mu \in (0, T_\lambda)$ the above problem has two positive solutions.

In case of the fractional Laplacian, Fang [13] proved the existence of a solution of the singular problem

$$(-\Delta)^s w = w^{-p},\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

with $0 < p < 1$, using the method of sub and super solution. Recently, Barrios, Peral and et al [4] extend the result of [13]. They studied the existence result for the singular problem

$$(-\Delta)^s u = \lambda \frac{f(x)}{u^\gamma} + M u^p,\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where Ω is a bounded smooth domain of $\mathbb{R}^n$, $n > 2s$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $p > 1$ and $f \in L^m(\Omega)$, $m \geq 1$ is a nonnegative function. For $M = 0$, they proved the existence of solution for every $\gamma > 0$ and $\lambda > 0$. For $M = 1$ and $f \equiv 1$, they showed that there exist Λ such that it has a solution for every $0 < \lambda < \Lambda$, and have no solution for $\lambda > \Lambda$. Here the authors first studied the uniform estimates of solutions $\{u_n\}$ of the regularized problems

$$(-\Delta)^s u = \lambda \frac{f(x)}{(u + \frac{1}{n})^\gamma} + u^p,\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \tag{1.2}$$

Then they obtained the solutions by taking limit in the regularized problem (1.2).

As far as we know, there is no work related to fractional Laplacian for singular nonlinearity and sign-changing weight functions. So, in this paper, we study the multiplicity results for problem (1.1) for $0 < q < 1 < p < 2_s^* - 1$ and $\lambda > 0$. This work is motivated by the work of Chen and Chen in [10]. Due to the singularity of problem, it is not easy to deal the problem (1.1) as the associated functional is not differentiable even in sense of Gâteaux and the strong maximum principle is not applicable to show the positivity of solutions. Moreover one can not directly extend all the results from Laplacian case to fractional Laplacian, due to the non-local behavior of the operator and the bounded support of the test function is not preserved. To overcome these difficulties, we first use the Cafferelli and Silvestre [9] approach to convert the problem (1.1) into the local problem. Then we use the variational technique to study the local problem as in [10]. In this paper, the proofs of some Lemmas follow the similar lines as in [10] but for completeness, we give the details.

The article is organized as follows: In section 2 we present some preliminaries on extension problem and necessary weighted trace inequalities required for variational settings. We also state our main results. In section 3, we study the decomposition of Nehari manifold and local charts using the fibering maps. In Section 4, we show the existence of a nontrivial solutions and show how these solutions arise out of nature of Nehari manifold.

We will use the following notation throughout this paper: The same symbol $\|\cdot\|$ denotes the norms in the three spaces $L^{\frac{2_s^*}{2_s^*-1-q}}(\Omega)$, $L^{\frac{2_s^*}{2_s^*-1-p}}(\Omega)$, and $H^s_{0,L}(\mathcal{C}_\Omega)$ defined by (2.1). Also $S := k_s S(s,n)$ where $S(s,n)$ is the best constant of Sobolev embedding (see (2.2)).

2. Preliminaries and main results

In this section we give some definitions and functional settings. At the end of this section, we state our main results. To state our main result, we introduce some notation and basic preliminaries results. Denote the upper half-space in $\mathbb{R}^{n+1}$ by

$$\mathbb{R}^{n+1}_+ := \{z = (x, y) = (x_1, x_2, \dots, x_n, y) \in \mathbb{R}^{n+1} | y > 0\},$$

the half cylinder standing on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ by $\mathcal{C}_\Omega := \Omega \times (0, \infty) \subset \mathbb{R}^{n+1}_+$ and its lateral boundary is denoted by $\partial_L \mathcal{C}_\Omega = \partial\Omega \times [0, \infty)$. Define the function space $H^s_{0,L}(\mathcal{C}_\Omega)$ as the completion of $C^\infty_{0,L}(\mathcal{C}_\Omega) = \{w \in C^\infty(\overline{\mathcal{C}_\Omega}) : w = 0 \text{ on } \partial_L \mathcal{C}_\Omega\}$ under the norm

$$\|w\|_{H^s_{0,L}(\mathcal{C}_\Omega)} = \Big(k_s \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla w(x, y)|^2 \, dx \, dy\Big)^{1/2}, \tag{2.1}$$

where $k_s := \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$ is a normalization constant. Then it is a Hilbert space endowed with the inner product

$$\langle w, v\rangle_{H^s_{0,L}(\mathcal{C}_\Omega)} = k_s \int_{\Omega\times\{0\}} y^{1-2s} \nabla w \nabla v \, dx \, dy.$$

If Ω is a smooth bounded domain then it is verified that (see [8, Proposition 2.1], [6, Proposition 2.1], [26, section 2])

$$H^s_0(\Omega) := \{u = \operatorname{tr}|_{\Omega\times\{0\}} w : w \in H^s_{0,L}(\mathcal{C}_\Omega)\}$$

and there exists a constant $C > 0$ such that

$$\|w(\cdot, 0)\|_{H^s_0(\Omega)} \le C\|w\|_{H^s_{0,L}(\mathcal{C}_\Omega)} \text{ for all } w \in H^s_{0,L}(\mathcal{C}_\Omega).$$

Now we define the extension operator and fractional Laplacian for functions in $H^s_0(\Omega)$.

Definition 2.1. Given a function $u \in H^s_0(\Omega)$, we define its s-harmonic extension $w = E_s(u)$ to the cylinder $\mathcal{C}_\Omega$ as a solution of the problem

$$\begin{gathered} \operatorname{div}(y^{1-2s}\nabla w) = 0 \quad \text{in } \mathcal{C}_\Omega, \\ w = 0 \quad \text{on } \partial_L \mathcal{C}_\Omega, \\ w = u \quad \text{on } \Omega \times \{0\}. \end{gathered}$$

Definition 2.2. For any regular function $u \in H^s_0(\Omega)$, the fractional Laplacian $(-\Delta)^s$ acting on u is defined by

$$(-\Delta)^s u(x) = -k_s \lim_{y\to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y) \quad \text{for all } (x, y) \in \mathcal{C}_\Omega.$$

From [5] and [9], the map $E_s(\cdot)$ is an isometry between $H^s_0(\Omega)$ and $H^s_{0,L}(\mathcal{C}_\Omega)$. Furthermore, we have

(1)
$$\|(-\Delta)^s u\|_{H^{-s}(\Omega)} = \|u\|_{H^s_0(\Omega)} = \|E_s(u)\|_{H^s_{0,L}(\mathcal{C}_\Omega)},$$
where $H^{-s}(\Omega)$ denotes the dual space of $H^s_0(\Omega)$;

(2) For any $w \in H^s_{0,L}(\mathcal{C}_\Omega)$, there exists a constant C independent of w such that
$$\|\operatorname{tr}_\Omega w\|_{L^r(\Omega)} \le C\|w\|_{H^s_{0,L}(\mathcal{C}_\Omega)}$$
holds for every $r \in [2, \frac{2n}{n-2s}]$. Moreover, $H^s_{0,L}(\mathcal{C}_\Omega)$ is compactly embedded into $L^r(\Omega)$ for $r \in [2, \frac{2n}{n-2s})$.

Lemma 2.3. *For every $1 \le r \le \frac{2n}{n-2s}$ and every $w \in H^s_{0,L}(\mathcal{C}_\Omega)$, it holds*
$$\Big(\int_{\Omega\times\{0\}} |w(x,0)|^r dx\Big)^{2/r} \le Ck_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w(x,y)|^2\,dx\,dy,$$
where the constant C depends on r, s, n and $|\Omega|$.

Lemma 2.4. *For every $w \in H^s(\mathbb{R}^{n+1}_+)$, it holds*
$$S(s,n)\Big(\int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2s}}dx\Big)^{\frac{n-2s}{n}} \le \int_{\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla w(x,y)|^2\,dx\,dy, \tag{2.2}$$
where $u = \operatorname{tr}_\Omega w$. The constant $S(s,n)$ is known as the best constant and takes the value
$$S(s,n) = \frac{2\pi^s\Gamma(\frac{2-2s}{2})\Gamma(\frac{n+2s}{2})(\Gamma(\frac{n}{2}))^{\frac{2s}{n}}}{\Gamma(s)\Gamma(\frac{n-2s}{2})(\Gamma(n))^{\frac{2s}{n}}}\,.$$

Now we can transform the nonlocal problem (1.1) into the local problem
$$\begin{gathered} -\operatorname{div}(y^{1-2s}\nabla w) = 0 \quad \text{in } \mathcal{C}_\Omega := \Omega\times(0,\infty),\\ w = 0 \text{ on } \partial_L\mathcal{C}_\Omega, \quad w > 0 \text{ on } \Omega\times\{0\},\\ \frac{\partial w}{\partial v^{2s}} = a(x)w^{-q} + \lambda b(x)w^p \quad \text{on } \Omega\times\{0\}, \end{gathered} \tag{2.3}$$
where $\frac{\partial w}{\partial v^{2s}} := -k_s\lim_{y\to0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y)$, for all $x\in\Omega$.

Definition 2.5. A weak solution of (2.3) is a function $w \in H^s_{0,L}(\mathcal{C}_\Omega)$, $w > 0$ in $\Omega\times\{0\}$ such that for every $v \in H^s_{0,L}(\mathcal{C}_\Omega)$,
$$\begin{aligned} &k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\nabla w\nabla v\,dx\,dy\\ &= \int_{\Omega\times\{0\}} a(x)(w^{-q}v)(x,0)dx + \lambda\int_{\Omega\times\{0\}} b(x)(w^pv)(x,0)dx. \end{aligned}$$
If w satisfies (2.3), then $u = \operatorname{tr}_\Omega w = w(x,0) \in H^s_0(\Omega)$ is a weak solution to problem (1.1).

Let $c = 1-2s$, then the associated functional $J_\lambda : H^s_{0,L}(\mathcal{C}_\Omega) \to \mathbb{R}$ to the problem (2.3) is
$$\begin{aligned} J_\lambda(w) = \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^c|\nabla w|^2\,dx\,dy - \frac{1}{1-q}\int_{\Omega\times\{0\}} a(x)|w|^{1-q}dx\\ - \frac{\lambda}{p+1}\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}dx. \end{aligned}$$
Now for $w \in H^s_{0,L}(\mathcal{C}_\Omega)$, we define the fiber map $\phi_w : \mathbb{R}^+ \to \mathbb{R}$ as
$$\begin{aligned} \phi_w(t) = J_\lambda(tw) = \frac{t^2}{2}\|w\|^2 - \frac{1}{1-q}\int_{\Omega\times\{0\}} a(x)|tw|^{1-q}dx\\ - \frac{\lambda t^{p+1}}{p+1}\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}dx. \end{aligned}$$
Then
$$\phi_w'(t) = t\|w\|^2 - t^{-q}\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}dx - \lambda t^p\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}dx,$$

$$\begin{aligned}\phi_w''(t) &= \|w\|^2 + qt^{-q-1}\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}dx\\ &\quad - p\lambda t^{p-1}\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}dx.\end{aligned}$$

It is easy to see that the energy functional J_λ is not bounded below on the space $H^s_{0,L}(\mathcal{C}_\Omega)$. But we will show that it is bounded below on an appropriate subset of $H^s_{0,L}(\mathcal{C}_\Omega)$ and a minimizer on subsets of this set gives rise to solutions of (2.3). In order to obtain the existence results, we define

$$\begin{aligned}\mathcal{N}_\lambda :&= \{w \in H^s_{0,L}(\mathcal{C}_\Omega) : \langle J_\lambda'(w), w\rangle = 0\}\\ &= \Big\{w \in H^s_{0,L}(\mathcal{C}_\Omega) : \|w\|^2 = \int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}dx\\ &\quad + \lambda\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}dx\Big\}.\end{aligned}$$

Note that $w \in \mathcal{N}_\lambda$ if w is a solution of (2.3). Also one can easily see that $tw \in \mathcal{N}_\lambda$ if and only if $\phi_w'(t) = 0$ and in particular, $w \in \mathcal{N}_\lambda$ if and only if $\phi_w'(1) = 0$. In order to obtain our result, we decompose $\mathcal{N}_\lambda$ with $\mathcal{N}_\lambda^\pm$, $\mathcal{N}_\lambda^0$ as follows:

$$\begin{aligned}\mathcal{N}_\lambda^\pm &= \{w \in \mathcal{N}_\lambda : \phi_w''(1) \gtrless 0\}\\ &= \big\{w \in \mathcal{N}_\lambda : (1+q)\|w\|^2 \gtrless \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx\big\},\\ \mathcal{N}_\lambda^0 &= \{w \in \mathcal{N}_\lambda : \phi_w''(1) = 0\}\\ &= \big\{w \in \mathcal{N}_\lambda : (1+q)\|w\|^2 = \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx\big\}.\end{aligned}$$

Inspired by [9] and [10], we show that how variational methods can be used to established some existence and multiplicity results. Our results are as follows:

Theorem 2.6. *Suppose that $\lambda \in (0,\Lambda)$, where*

$$\Lambda := \frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}}\frac{1}{\|b\|}\Big(\frac{S^{p+q}}{\|a\|^{p-1}}\Big)^{1/(1+q)}.$$

Then problem (2.3) has at least two solutions $w_0 \in \mathcal{N}_\lambda^+$, $W_0 \in \mathcal{N}_\lambda^-$ with $\|W_0\| > \|w_0\|$. Moreover, $u_0(x) = w_0(\cdot,0) \in H^s_0(\Omega)$ and $U_0(x) = W_0(\cdot,0) \in H^s_0(\Omega)$ are positive solutions of the problem (1.1).

Next, we obtain the blow up behavior of the solution $W_\epsilon \in \mathcal{N}_\lambda^-$ of problem (2.3) with $p = 1+\epsilon$ as $\epsilon \to 0^+$.

Theorem 2.7. *let $W_\epsilon \in \mathcal{N}_\lambda^-$ be the solution of problem (2.3) with $p = 1+\epsilon$, where $\lambda \in (0,\Lambda)$, then*

$$\|W_\epsilon\| > C_\epsilon\Big(\frac{\Lambda}{\lambda}\Big)^{1/\epsilon},$$

where

$$C_\epsilon = \Big(1 + \frac{1+q}{\epsilon}\Big)^{1/(1+q)}\|a\|^{1/(q+1)}\Big(\frac{1}{\sqrt{S}}\Big)^{\frac{1-q}{1+q}} \to \infty \quad \text{as } \epsilon \to 0^+.$$

Namely, W_ϵ blows up faster than exponentially with respect to ϵ.

We remark that if w is a positive solution of the problem

$$(-\Delta)^s u = a(x)u^{-q} + \lambda b(x)u^p \quad \text{in } \Omega$$
$$u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Then one can easily see that $v = \lambda^{1/(p-1)}u$ is a positive solution of the problem

$$\begin{gathered} (-\Delta)^s v = \lambda^{\frac{1+q}{p-1}} a(x)v^{-q} + b(x)v^p \quad \text{in } \Omega \\ v > 0 \text{ in } \Omega, \quad v = 0 \text{ in } \partial\Omega. \end{gathered} \tag{2.4}$$

That is, the problem (2.4) has two positive solution for $\lambda \in (0, \Lambda^{p-1})$.

3. Fibering map analysis

In this section, we show that $\mathcal{N}_\lambda^\pm$ is nonempty and $\mathcal{N}_\lambda^0 = \{0\}$. Moreover, J_λ is bounded below and coercive.

Lemma 3.1. *Let $\lambda \in (0, \Lambda)$. Then for each $w \in H^s_{0,L}(\mathcal{C}_\Omega)$ with*

$$\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}dx > 0,$$

we have the following:

(i) *$\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx \le 0$, then there exists a unique $t_1 < t_{\max}$ such that $t_1 w \in \mathcal{N}_\lambda^+$ and $J_\lambda(t_1 w) = \inf_{t>0} J_\lambda(tw)$;*

(ii) *$\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx > 0$, then there exists a unique t_1 and t_2 with $0 < t_1 < t_{\max} < t_2$ such that $t_1 w \in \mathcal{N}_\lambda^+$, $t_2 w \in \mathcal{N}_\lambda^-$ and $J_\lambda(t_1 w) = \inf_{0\le t\le t_{\max}} J_\lambda(tw)$, $J_\lambda(t_2 w) = \sup_{t\ge t_1} J_\lambda(tw)$.*

Proof. For $t > 0$, we define

$$\psi_w(t) = t^{1-p}\|w\|^2 - t^{-p-q}\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx - \lambda\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx.$$

One can easily see that $\psi_w(t) \to -\infty$ as $t \to 0^+$. Now

$$\psi_w'(t) = (1-p)t^{-p}\|w\|^2 + (p+q)t^{-p-q-1}\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx.$$

$$\begin{aligned} \psi_w''(t) &= -p(1-p)t^{-p-1}\|w\|^2 \\ &\quad - (p+q)(p+q+1)t^{-p-q-2}\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx. \end{aligned}$$

Then $\psi_w'(t) = 0$ if and only if $t = t_{\max} := [\frac{(p-1)\|w\|^2}{(p+q)\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx}]^{-1/(1+q)}$. Also

$$\begin{aligned} \psi_w''(t_{\max}) &= p(p-1)\Big[\frac{(p-1)\|w\|^2}{(p+q)\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx}\Big]^{\frac{p+1}{q+1}}\|w\|^2 \\ &\quad - (p+q)(p+q+1)\Big[\frac{(p-1)\|w\|^2}{(p+q)\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx}\Big]^{\frac{p+q+2}{q+1}} \\ &\quad \times \int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx \\ &= -\|w\|^2(p-1)(1+q)\Big[\frac{(p-1)\|w\|^2}{(p+q)\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx}\Big]^{\frac{p+1}{q+1}} < 0. \end{aligned}$$

Thus ψ_w achieves its maximum at $t = t_{\max}$. Now using the Hölder's inequality and Sobolev inequality (2.2), we obtain

$$\begin{aligned} &\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx \\ &\le \Big[\int_{\Omega\times\{0\}} |a(x)|^{\frac{2_s^*}{2_s^*-1+q}}\Big]^{\frac{2_s^*+q-1}{2_s^*}} \Big[\int_{\Omega\times\{0\}} |w(x,0)|^{2_s^*}dx\Big]^{\frac{1-q}{2_s^*}} \\ &\le \|a\|\Big(\frac{\|w\|}{\sqrt{S}}\Big)^{1-q}. \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx \\ &\le \Big[\int_{\Omega\times\{0\}} |b(x)|^{\frac{2_s^*}{2_s^*-1-p}}\Big]^{\frac{2_s^*-p-1}{2_s^*}} \Big[\int_{\Omega\times\{0\}} |w(x,0)|^{2_s^*}dx\Big]^{\frac{p+1}{2_s^*}} \\ &\le \|b\|\Big(\frac{\|w\|}{\sqrt{S}}\Big)^{p+1}. \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we obtain

$$\begin{aligned} &\psi_w(t_{\max}) \\ &= \frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}} \frac{\|w\|^{\frac{2(p+q)}{1+q}}}{[\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx]^{\frac{p-1}{1+q}}} - \lambda\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx \\ &\ge \Big[\frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}} \Big(\frac{(\sqrt{S})^{(1-q)}}{\|a\|}\Big)^{\frac{p-1}{1+q}} - \lambda\|b\|\Big(\frac{1}{\sqrt{S}}\Big)^{p+1}\Big]\|w\|^{p+1} \\ &\equiv E_\lambda\|w\|^{p+1}, \end{aligned} \tag{3.3}$$

where

$$E_\lambda := \Big[\frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}} \Big(\frac{(\sqrt{S})^{(1-q)}}{\|a\|}\Big)^{\frac{p-1}{1+q}} - \lambda\|b\|\Big(\frac{1}{\sqrt{S}}\Big)^{p+1}\Big].$$

Then we see that $E_\lambda = 0$ if and only if $\lambda = \Lambda$, where

$$\Lambda := \frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}} \frac{1}{\|b\|}\Big(\frac{S^{p+q}}{\|a\|^{p-1}}\Big)^{1/(1+q)}.$$

Thus for $\lambda \in (0,\Lambda)$, we have $E_\lambda > 0$, and therefore it follows from (3.3) that $\psi_w(t_{\max}) > 0$.

(i) If $\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx \ge 0$, then

$$\psi_w(t) \to -\lambda\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx < 0$$

as $t \to \infty$. Consequently, $\psi_w(t)$ has exactly two points $0 < t_1 < t_{\max} < t_2$ such that

$$\psi_w(t_1) = 0 = \psi_w(t_2) \text{ and } \psi_w'(t_1) > 0 > \psi_w'(t_2).$$

From the definition of $\mathcal{N}_\lambda^\pm$, we get $t_1w \in \mathcal{N}_\lambda^+$ and $t_2w \in \mathcal{N}_\lambda^-$. Now $\phi_w'(t) = t^p\psi_w(t)$. Thus $\phi_w'(t) < 0$ in $(0,t_1)$, $\phi_w'(t) > 0$ in (t_1,t_2) and $\phi_w'(t) < 0$ in (t_2,∞). Hence $J_\lambda(t_1w) = \inf_{0\le t\le t_{\max}} J_\lambda(tw)$, $J_\lambda(t_2w) = \sup_{t\ge t_1} J_\lambda(tw)$.

(ii) If $\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx < 0$ and

$$\psi_w(t) \to -\lambda \int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx > 0$$

as $t \to \infty$. Consequently, $\psi_w(t)$ has exactly one point $0 < t_1 < t_{\max}$ such that

$$\psi_w(t_1) = 0 \quad \text{and} \quad \psi_w'(t_1) > 0.$$

Now $\phi_w'(t) = t^p \psi_w(t)$. Then $\phi_w'(t) < 0$ in $(0, t_1)$, $\phi_w'(t) > 0$ in (t_1, ∞). Thus $J_\lambda(t_1 w) = \inf_{t\geq 0} J_\lambda(tw)$. Hence, it follows that $t_1 w \in \mathcal{N}_\lambda^+$. □

Corollary 3.2. *Suppose that $\lambda \in (0, \Lambda)$, then $\mathcal{N}_\lambda^\pm \neq \emptyset$.*

Proof. From (A1) and (A2), we can choose $w \in H_{0,L}^s(\mathcal{C}_\Omega) \setminus \{0\}$ such that

$$\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx > 0 \quad \text{and} \quad \int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx > 0.$$

Then by (ii) of Lemma 3.1, there exists a unique t_1 and t_2 such that $t_1 w \in \mathcal{N}_\lambda^+$, $t_2 w \in \mathcal{N}_\lambda^-$. In conclusion, $\mathcal{N}_\lambda^\pm \neq \emptyset$. □

Lemma 3.3. *For $\lambda \in (0, \Lambda)$, we have $\mathcal{N}_\lambda^0 = \{0\}$.*

Proof. We prove this by contradiction. Assume that there exists $0 \not\equiv w \in \mathcal{N}_\lambda^0$. Then it follows from $w \in \mathcal{N}_\lambda^0$ that

$$(1+q)\|w\|^2 = \lambda(p+q) \int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx$$

and consequently

$$\begin{aligned} 0 &= \|w\|^2 - \int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx - \lambda \int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx \\ &= \frac{p-1}{p+q}\|w\|^2 - \int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx. \end{aligned}$$

Therefore, as $\lambda \in (0, \Lambda)$ and $w \not\equiv 0$, we use similar arguments as those in (3.3) to obain

$$\begin{aligned} 0 &< E_\lambda \|w\|^{p+1} \\ &\leq \frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}} \frac{\|w\|^{\frac{2(p+q)}{1+q}}}{\big[\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx\big]^{\frac{p-1}{1+q}}} \\ &\quad - \lambda \int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx \\ &= \frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}} \frac{\|w\|^{\frac{2(p+q)}{1+q}}}{\big(\frac{p-1}{p+q}\|w\|^2\big)^{\frac{p-1}{1+q}}} - \frac{1+q}{p+q}\|w\|^2 = 0, \end{aligned}$$

a contradiction. Hence $w \equiv 0$. That is, $\mathcal{N}_\lambda^0 = \{0\}$. □

We note that Λ is also related to a gap structure in $\mathcal{N}_\lambda$:

Lemma 3.4. *Suppose that $\lambda \in (0,\Lambda)$, then there exist a gap structure in $\mathcal{N}_\lambda$:*
$$\|W\| > A_\lambda > A_0 > \|w\| \quad \text{for all } w \in \mathcal{N}_\lambda^+,\ W \in \mathcal{N}_\lambda^-,$$
where
$$A_\lambda = \Big[\frac{1+q}{\lambda(p+q)\|b\|}(\sqrt{S})^{p+1}\Big]^{1/(p-1)} \quad \text{and} \quad A_0 = \Big[\frac{p+q}{p-1}\|a\|\Big(\frac{1}{\sqrt{S}}\Big)^{1-q}\Big]^{1/(q+1)}.$$

Proof. If $w \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, then
$$\begin{aligned} 0 &< (1+q)\|w\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}dx \\ &= (1+q)\|w\|^2 - (p+q)\Big[\|w\|^2 - \int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}dx\Big] \\ &= (1-p)\|w\|^2 + (p+q)\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}dx. \end{aligned}$$
Hence it follows from (3.1) that
$$(p-1)\|w\|^2 < (p+q)\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}dx \le (p+q)\|a\|\Big(\frac{\|w\|}{\sqrt{S}}\Big)^{1-q}$$
which yields
$$\|w\| < \Big[\frac{p+q}{p-1}\|a\|\Big(\frac{1}{\sqrt{S}}\Big)^{1-q}\Big]^{1/(q+1)} \equiv A_0.$$
If $W \in \mathcal{N}_\lambda^-$, then it follows from (3.2) that
$$(1+q)\|W\|^2 < \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|W(x,0)|^{p+1}dx \le \lambda(p+q)\|b\|\Big(\frac{\|W\|}{\sqrt{S}}\Big)^{p+1}$$
which yields
$$\|W\| > \Big[\frac{1+q}{\lambda(p+q)\|b\|}(\sqrt{S})^{p+1}\Big]^{1/(p-1)} \equiv A_\lambda.$$
Now we show that $A_\lambda = A_0$ if and only if $\lambda = \Lambda$.
$$\lambda = \Lambda = \frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}}\frac{1}{\|b\|}\Big(\frac{S^{p+q}}{\|a\|^{p-1}}\Big)^{1/(1+q)}$$
if and only if
$$\begin{aligned} A_\lambda &= \lambda^{-1/(p-1)}\Big(\frac{1+q}{p+q}\Big)^{1/(p-1)}\Big(\frac{1}{\|b\|}\Big)^{1/(p-1)}\Big(\sqrt{S}\Big)^{\frac{p+1}{p-1}} \\ &= \Big(\frac{p+q}{p-1}\Big)^{1/(1+q)}\|a\|^{1/(q+1)}\Big(\sqrt{S}\Big)^{-\frac{2(p+q)}{(1+q)(p-1)}+\frac{p+1}{p-1}} \\ &= \Big[\frac{(p+q)\|a\|}{(p-1)(\sqrt{S})^{1-q}}\Big]^{1/(q+1)} \equiv A_0. \end{aligned}$$
Thus for all $\lambda \in (0,\Lambda)$, we can conclude that
$$\|W\| > A_\lambda > A_0 > \|w\| \quad \text{for all } w \in \mathcal{N}_\lambda^+, W \in \mathcal{N}_\lambda^-.$$
This completes the proof. □

Lemma 3.5. *Suppose that $\lambda \in (0,\Lambda)$, then $\mathcal{N}_\lambda^-$ is a closed set in $H^s_{0,L}(\mathcal{C}_\Omega)$-topology.*

Proof. Let $\{W_k\}$ be a sequence in $\mathcal{N}_\lambda^-$ with $W_k \to W_0$ in $H^s_{0,L}(\mathcal{C}_\Omega)$. Then we have

$$\begin{aligned}
\|W_0\|^2 &= \lim_{k\to\infty} \|W_k\|^2 \\
&= \lim_{k\to\infty} \Big[\int_{\Omega\times\{0\}} a(x)|W_k(x,0)|^{1-q}dx + \lambda \int_{\Omega\times\{0\}} b(x)|W_k(x,0)|^{p+1}dx\Big] \\
&= \int_{\Omega\times\{0\}} a(x)|W_0(x,0)|^{1-q}dx + \lambda \int_{\Omega\times\{0\}} b(x)|W_0(x,0)|^{p+1}dx
\end{aligned}$$

and

$$\begin{aligned}
&(1+q)\|W_0\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|W_0(x,0)|^{p+1}dx \\
&= \lim_{k\to\infty} \Big[(1+q)\|W_k\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|W_k(x,0)|^{p+1}dx\Big] \le 0.
\end{aligned}$$

That is, $W_0 \in \mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^0$. Since $\{W_k\} \subset \mathcal{N}_\lambda^-$, from Lemma 3.4 we have

$$\|W_0\| = \lim_{k\to\infty} \|W_k\| \ge A_0 > 0,$$

which imply, $W_0 \neq 0$. It follows from Lemma 3.1, that $W_0 \notin \mathcal{N}_\lambda^0$ for any $\lambda \in (0,\Lambda)$. Thus $W_0 \in \mathcal{N}_\lambda^-$. Hence, $\mathcal{N}_\lambda^-$ is a closed set in $H^s_{0,L}(\mathcal{C}_\Omega)$-topology for any $\lambda \in (0,\Lambda)$. □

Lemma 3.6. *Let $w \in \mathcal{N}_\lambda^\pm$. Then for any $\phi \in C^\infty_{0,L}(\mathcal{C}_\Omega)$, there exists a number $\epsilon > 0$ and a continuous function $f : B_\epsilon(0) := \{v \in H^s_{0,L}(\mathcal{C}_\Omega) : \|v\| < \epsilon\} \to \mathbb{R}^+$ such that*

$$f(v) > 0, \quad f(0) = 1, \quad f(v)(w + v\phi) \in \mathcal{N}_\lambda^\pm \quad \text{for all } v \in B_\epsilon(0).$$

Proof. We give the proof only for the case $w \in \mathcal{N}_\lambda^+$, the case $\mathcal{N}_\lambda^-$ may be preceded exactly. For any $\phi \in C^\infty_{0,L}(\mathcal{C}_\Omega)$, we define $F : H^s_{0,L}(\mathcal{C}_\Omega) \times \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$\begin{aligned}
F(v,r) = r^{1+q}\|w+v\phi\|^2 &- \int_{\Omega\times\{0\}} a(x)|(w+v\phi)(x,0)|^{1-q} \\
&- \lambda r^{p+q}\int_{\Omega\times\{0\}} b(x)|(w+v\phi)(x,0)|^{p+1}.
\end{aligned}$$

Since $w \in \mathcal{N}_\lambda^+ (\subset \mathcal{N}_\lambda)$, we have

$$F(0,1) = \|w\|^2 - \int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}dx - \lambda \int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}dx = 0,$$

and

$$\frac{\partial F}{\partial r}(0,1) = (1+q)\|w\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}dx > 0.$$

Applying the implicit function theorem at $(0,1)$, we have that there exists $\bar\epsilon > 0$ such that for $\|v\| < \bar\epsilon$, $v \in H^s_{0,L}(\mathcal{C}_\Omega)$, the equation $F(v,r) = 0$ has a unique continuous solution $r = f(v) > 0$. It follows from $F(0,1) = 0$ that $f(0) = 1$ and from $F(v, f(v)) = 0$ for $\|v\| < \bar\epsilon$, $v \in H^s_{0,L}(\mathcal{C}_\Omega)$ that

$$\begin{aligned}
0 = f^{1+q}(v)\|w+v\phi\|^2 &- \int_{\Omega\times\{0\}} a(x)|(w+v\phi)(x,0)|^{1-q} \\
&- \lambda f^{p+q}(v)\int_{\Omega\times\{0\}} b(x)|(w+v\phi)(x,0)|^{p+1}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\|f(v)(w+v\phi)\|^2 - \int_{\Omega\times\{0\}} a(x)|f(v)(w+v\phi)(x,0)|^{1-q} \\
&\quad - \lambda \int_{\Omega\times\{0\}} b(x)|f(v)(w+v\phi)(x,0)|^{p+1}\Big)/f^{1-q}(v);
\end{aligned}
$$

that is,

$$
f(v)(w+v\phi)\in\mathcal{N}_\lambda \quad \text{for all } v\in H^s_{0,L}(\mathcal{C}_\Omega), \|v\|<\tilde\epsilon.
$$

Since $\frac{\partial F}{\partial r}(0,1)>0$ and

$$
\begin{aligned}
&\frac{\partial F}{\partial r}(v,f(v))\\
&= (1+q)f^q(v)\|w+v\phi\|^2 - \lambda(p+q)f^{p+q-1}(v)\int_{\Omega\times\{0\}} b(x)|(w+v\phi)(x,0)|^{p+1}dx\\
&= \frac{(1+q)\|f(v)(w+v\phi)\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|f(v)(w+v\phi)(x,0)|^{p+1}dx}{f^{2-q}(v)},
\end{aligned}
$$

we can take $\epsilon>0$ possibly smaller ($\epsilon<\bar\epsilon$) such that for any $v\in H^s_{0,L}(\mathcal{C}_\Omega)$, $\|v\|<\epsilon$,

$$
(1+q)\|f(v)(w+v\phi)\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)|f(v)(w+v\phi)(x,0)|^{p+1}dx>0;
$$

that is,

$$
f(v)(w+v\phi)\in\mathcal{N}_\lambda^+ \quad \text{for all } v\in B_\epsilon(0).
$$

This completes the proof. □

Lemma 3.7. *J_λ is bounded below and coercive on $\mathcal{N}_\lambda$.*

Proof. For $w\in\mathcal{N}_\lambda$, from (3.1), we obtain

$$
\begin{aligned}
J_\lambda(w) &= \Big(\frac12-\frac1{p+1}\Big)\|w\|^2 - \Big(\frac1{1-q}-\frac1{p+1}\Big)\int_{\Omega\times\{0\}} a(x)|w(x,0)|^{1-q}\,dx\\
&\ge \Big(\frac12-\frac1{p+1}\Big)\|w\|^2 - \Big(\frac1{1-q}-\frac1{p+1}\Big)\|a\|\Big(\frac{\|w\|}{\sqrt S}\Big)^{1-q}.
\end{aligned}
\tag{3.4}
$$

Now consider the function $\rho:\mathbb{R}^+\to\mathbb{R}$ as $\rho(t)=\alpha t^2-\beta t^{1-q}$, where α, β are both positive constants. One can easily show that ρ is convex($\rho''(t)>0$ for all $t>0$) with $\rho(t)\to0$ as $t\to0$ and $\rho(t)\to\infty$ as $t\to\infty$. ρ achieves its minimum at $t_{\min}=[\frac{\beta(1-q)}{2\alpha}]^{1/(1+q)}$ and

$$
\rho(t_{\min}) = \alpha\Big[\frac{\beta(1-q)}{2\alpha}\Big]^{\frac{2}{1+q}} - \beta\Big[\frac{\beta(1-q)}{2\alpha}\Big]^{\frac{1-q}{1+q}} = -\frac{1+q}{2}\beta^{\frac{2}{1+q}}\Big(\frac{1-q}{2\alpha}\Big)^{\frac{1-q}{1+q}}.
$$

Applying $\rho(t)$ with $\alpha=\big(\frac12-\frac1{p+1}\big)$, $\beta=\big(\frac1{1-q}-\frac1{p+1}\big)\|a\|\big(\frac1{\sqrt S}\big)^{1-q}$ and $t=\|w\|$, $w\in\mathcal{N}_\lambda$, we obtain from (3.4) that

$$
\lim_{\|w\|\to\infty} J_\lambda(w)\ge \lim_{t\to\infty}\rho(t)=\infty.
$$

Thus J_λ is coercive on $\mathcal{N}_\lambda$. Moreover, it follows from (3.4) that

$$
J_\lambda(w)\ge\rho(t)\ge\rho(t_{\min}) \quad \text{(a constant)}, \tag{3.5}
$$

i.e.,

$$
J_\lambda(w)\ge -\frac{1+q}{2}\beta^{\frac{2}{1+q}}\Big(\frac{1-q}{2\alpha}\Big)^{\frac{1-q}{1+q}} = -\frac{1+q}{(1-q)(p+1)}\Big(\frac{(p+q)\|a\|}{2(\sqrt S)^{1-q}}\Big)^{\frac{2}{1+q}}\Big(\frac{1}{p-1}\Big)^{\frac{1-q}{1+q}}.
$$

Thus J_λ is bounded below on $\mathcal{N}_\lambda$. □

4. Existence of solutions in $\mathcal{N}_\lambda^{\pm}$

Now from Lemma 3.5, $\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$ and $\mathcal{N}_\lambda^-$ are two closed sets in $H^s_{0,L}(\mathcal{C}_\Omega)$ provided $\lambda \in (0,\Lambda)$. Consequently, the Ekeland variational principle can be applied to the problem of finding the infimum of J_λ on both $\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$ and $\mathcal{N}_\lambda^-$. First, consider $\{w_k\} \subset \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$ with the following properties:

$$J_\lambda(w_k) < \inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) + \frac{1}{k} \tag{4.1}$$

$$J_\lambda(w) \geq J_\lambda(w_k) - \frac{1}{k}\|w - w_k\|, \quad \forall w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0. \tag{4.2}$$

From $J_\lambda(|w|) = J_\lambda(w)$, we may assume that $w_k \geq 0$ on C_Ω.

Lemma 4.1. *Show that the sequence $\{w_k\}$ is bounded in $\mathcal{N}_\lambda$. Moreover, there exists $0 \neq w_0 \in H^s_{0,L}(\mathcal{C}_\Omega)$ such that $w_k \rightharpoonup w_0$ weakly in $H^s_{0,L}(\mathcal{C}_\Omega)$.*

Proof. By equations (3.5) and (4.1), we have

$$at^2 - bt^{1-q} = \rho(t) \leq J_\lambda(w) < \inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) + \frac{1}{k} \leq C_5,$$

for sufficiently large k and a suitable positive constant. Hence putting $t = w_k$ in the above equation, we obtain $\{w_k\}$ is bounded.

Let $\{w_k\}$ is bounded in $H^s_{0,L}(\mathcal{C}_\Omega)$. Then, there exists a subsequence of $\{w_k\}_k$, still denoted by $\{w_k\}_k$ and $w_0 \in H^s_{0,L}(\mathcal{C}_\Omega)$ such that $w_k \rightharpoonup w_0$ weakly in $H^s_{0,L}(\mathcal{C}_\Omega)$, $w_k(\cdot,0) \to w_0(\cdot,0)$ strongly in $L^p(\Omega)$ for $1 \leq p < 2^*_s$ and $w_k(\cdot,0) \to w_0(\cdot,0)$ a.e. in Ω.

For any $w \in \mathcal{N}_\lambda^+$, we have from $0 < q < 1 < p$ that

$$\begin{aligned}
J_\lambda(w) &= \Big(\frac{1}{2} - \frac{1}{1-q}\Big)\|w\|^2 + \Big(\frac{1}{1-q} - \frac{1}{p+1}\Big)\lambda \int_{\Omega\times\{0\}} b(x)|w(x,0)|^{p+1}\,dx \\
&< \Big(\frac{1}{2} - \frac{1}{1-q}\Big)\|w\|^2 + \Big(\frac{1}{1-q} - \frac{1}{p+1}\Big)\frac{1+q}{p+q}\|w\|^2 \\
&= \Big(\frac{1}{p+1} - \frac{1}{2}\Big)\frac{1+q}{1-q}\|w\|^2 < 0,
\end{aligned}$$

which means that $\inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$. Now for $\lambda \in (0,\Lambda)$, we know from Lemma 3.1, that $\mathcal{N}_\lambda^0 = \{0\}$. Together, these imply that $w_k \in \mathcal{N}_\lambda^+$ for k large and

$$\inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) \leq \inf_{w \in \mathcal{N}_\lambda^+} J_\lambda(w) < 0.$$

Therefore, by weak lower semi-continuity of norm,

$$J_\lambda(w_0) \leq \liminf_{k\to\infty} J_\lambda(w_k) = \inf_{\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda < 0,$$

that is, $w_0 \geq 0$, $w_0 \not\equiv 0$. □

Lemma 4.2. *Suppose $w_k \in \mathcal{N}_\lambda^+$ such that $w_k \rightharpoonup w_0$ weakly in $H^s_{0,L}(\mathcal{C}_\Omega)$. Then for $\lambda \in (0,\Lambda)$,*

$$(1+q)\int_{\Omega\times\{0\}} a(x) w_0^{1-q}(x,0)dx - \lambda(p-1)\int_{\Omega\times\{0\}} b(x) w_0^{p+1}(x,0)dx > 0. \tag{4.3}$$

Moreover, there exists a constant $C_2 > 0$ *such that*

$$(1+q)\|w_k\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)\,dx \geq C_2 > 0. \tag{4.4}$$

Proof. For $\{w_k\} \subset \mathcal{N}_\lambda^+ (\subset \mathcal{N}_\lambda)$, since

$$\begin{aligned}
&(1+q)\int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx - \lambda(p-1)\int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx \\
&= \lim_{k\to\infty}\Big[(1+q)\int_{\Omega\times\{0\}} a(x)w_k^{1-q}(x,0)\,dx - \lambda(p-1)\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)\,dx\Big] \\
&= \lim_{k\to\infty}\Big[(1+q)\|w_k\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)\,dx\Big] \geq 0,
\end{aligned}$$

we can argue by a contradiction and assume that

$$(1+q)\int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx - \lambda(p-1)\int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx = 0. \tag{4.5}$$

Since $w_k \in \mathcal{N}_\lambda$, from the weak lower semi continuity of norm and (4.5) we have

$$\begin{aligned}
0 &= \lim_{k\to\infty}\Big[\|w_k\|^2 - \int_{\Omega\times\{0\}} a(x)w_k^{1-q}(x,0)\,dx - \lambda\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)\,dx\Big] \\
&\geq \|w_0\|^2 - \int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx - \lambda\int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx \\
&= \begin{cases} \|w_0\|^2 - \lambda\frac{p+q}{1+q}\int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx \\ \|w_0\|^2 - \frac{p+q}{p-1}\int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx. \end{cases}
\end{aligned}$$

Thus for any $\lambda \in (0,\Lambda)$ and $w_0 \not\equiv 0$, by similar arguments as those in (3.3) we have that

$$\begin{aligned}
&0 < E_\lambda\|w_0\|^{p+1} \\
&\leq \frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}} \frac{\|w_0\|^{\frac{2(p+q)}{1+q}}}{\big[\int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx\big]^{\frac{p-1}{1+q}}} - \lambda\int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx \\
&= \frac{1+q}{p+q}\Big(\frac{p-1}{p+q}\Big)^{\frac{p-1}{1+q}} \frac{\|w_0\|^{\frac{2(p+q)}{1+q}}}{\big(\frac{p-1}{p+q}\|w_0\|^2\big)^{\frac{p-1}{1+q}}} - \frac{1+q}{p+q}\|w_0\|^2 = 0,
\end{aligned}$$

which is clearly impossible. Now by (4.3), we have that

$$(1+q)\int_{\Omega\times\{0\}} a(x)w_k^{1-q}(x,0)\,dx - \lambda(p-1)\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)\,dx \geq C_2$$

for sufficiently large k and a suitable positive constant C_2. This, together with the fact that $w_k \in \mathcal{N}_\lambda$ we obtain equation (4.4). $\square$

Fix $\phi \in C_{0,L}^\infty(\mathcal{C}_\Omega)$ with $\phi \geq 0$. Then we apply Lemma 3.6 with $w = w_k \in \mathcal{N}_\lambda^+$ (k large enough such that $\frac{(1-q)C_1}{k} < C_2$), we obtain a sequence of functions $f_k : B\epsilon_k(0) \to \mathbb{R}$ such that $f_k(0) = 1$ and $f_k(w)(w_k + w\phi) \in \mathcal{N}_\lambda^+$ for all $w \in B_{\epsilon_k}(0)$. It follows from $w_k \in \mathcal{N}_\lambda$ and $f_k(w)(w_k + w\phi) \in \mathcal{N}_\lambda$ that

$$\|w_k\|^2 - \int_{\Omega\times\{0\}} a(x)w_k^{1-q}(x,0)\,dx - \lambda\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)\,dx = 0 \tag{4.6}$$

and

$$\begin{aligned} & f_k^2(w)\|w_k+w\phi\|^2 - f_k^{1-q}(w)\int_{\Omega\times\{0\}} a(x)(w_k+w\phi)^{1-q}(x,0)dx \\ & - \lambda f_k^{p+1}(w)\int_{\Omega\times\{0\}} b(x)(w_k+w\phi)^{p+1}(x,0)dx = 0. \end{aligned} \tag{4.7}$$

Choose $0<\rho<\epsilon_k$ and $w=\rho v$ with $\|v\|<1$. Then we find $f_k(w)$ such that $f_k(0)=1$ and $f_k(w)(w_k+w\phi)\in\mathcal{N}_\lambda^+$ for all $w\in B_\rho(0)$.

Lemma 4.3. *For $\lambda\in(0,\Lambda)$ we have $|\langle f_k'(0),v\rangle|$ is finite for every $0\le v\in H^s_{0,L}(\mathcal{C}_\Omega)$ with $\|v\|\le 1$.*

Proof. By (4.6) and (4.7) we have

$$\begin{aligned} 0 &= [f_k^2(w)-1]\|w_k+w\phi\|^2 + \|w_k+w\phi\|^2 - \|w_k\|^2 \\ &\quad - [f_k^{1-q}(w)-1]\int_{\Omega\times\{0\}} a(x)(w_k+w\phi)^{1-q}(x,0)dx \\ &\quad - \int_{\Omega\times\{0\}} a(x)[((w_k+w\phi)^{1-q}-w_k^{1-q})(x,0)]dx \\ &\quad - \lambda[f_k^{p+1}(w)-1]\int_{\Omega\times\{0\}} b(x)(w_k+w\phi)^{p+1}(x,0)dx \\ &\quad - \lambda\int_{\Omega\times\{0\}} b(x)[((w_k+w\phi)^{p+1}-w_k^{p+1})(x,0)]dx \\ &\le [f_k^2(\rho v)-1]\|w_k+\rho v\phi\|^2 + \|w_k+\rho v\phi\|^2 - \|w_k\|^2 \\ &\quad - [f_k^{1-q}(\rho v)-1]\int_{\Omega\times\{0\}} a(x)(w_k+\rho v\phi)^{1-q}(x,0) \\ &\quad - \lambda[f_k^{p+1}(\rho v)-1]\int_{\Omega\times\{0\}} b(x)(w_k+\rho v\phi)^{p+1}(x,0) \\ &\quad - \lambda\int_{\Omega\times\{0\}} b(x)[((w_k+\rho v\phi)^{p+1}-w_k^{p+1})(x,0)] \end{aligned}$$

Dividing by $\rho>0$ and passing to the limit $\rho\to 0$, we derive that

$$\begin{aligned} 0 &\le 2\langle f_k'(0),v\rangle\|w_k\|^2 + 2k_s\int_{\mathcal{C}_\Omega} y^c\nabla w_k\nabla(v\phi)\,dx\,dy \\ &\quad - (1-q)\langle f_k'(0),v\rangle\int_{\Omega\times\{0\}} a(x)w_k^{1-q}(x,0)dx \\ &\quad - \lambda(p+1)\langle f_k'(0),v\rangle\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)dx \\ &\quad - \lambda(p+1)\int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx \\ &= \langle f_k'(0),v\rangle\Big[2\|w_k\|^2 - (1-q)\int_{\Omega\times\{0\}} a(x)w_k^{1-q}(x,0)dx \\ &\quad - \lambda(p+1)\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)dx\Big] \\ &\quad + 2k_s\int_{\mathcal{C}_\Omega} y^c\nabla w_k\nabla(v\phi)\,dx\,dy - \lambda(p+1)\int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx \end{aligned}$$

$$
\begin{aligned}
&= \langle f_k'(0), v\rangle \Big[(1+q)\|w_k\|^2 \\
&\quad - \lambda(p+q)\int_{\Omega\times\{0\}} b(x) w_k^{p+1}(x,0)dx\Big] \\
&\quad + 2k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_k \nabla(v\phi)\,dx\,dy - \lambda(p+1)\int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx.
\end{aligned} \tag{4.8}
$$

From this inequality and (4.4) we know that $\langle f_k'(0), v\rangle \neq -\infty$. Now we show that $\langle f_k'(0), v\rangle \neq +\infty$. Arguing by contradiction, we assume that $\langle f_k'(0), v\rangle = +\infty$. Now we note that

$$
\begin{aligned}
|f_k(\rho v) - 1|\|w_k\| + f_k(\rho v)\|\rho v\phi\| &\geq \|[f_k(\rho v) - 1]w_k + \rho v f_k(\rho v)\phi\| \\
&= \|f_k(\rho v)(w_k + \rho v\phi) - w_k\|
\end{aligned} \tag{4.9}
$$

and $f_k(\rho v) > f_k(0) = 1$ for sufficiently large k.

From the definition of derivative $\langle f_k'(0), v\rangle$, applying equation (4.2) with $w = f_k(\rho v)(w_k + \rho v\phi) \in \mathcal{N}_\lambda^+$, we clearly have

$$
\begin{aligned}
&[f_k(\rho v) - 1]\frac{\|w_k\|}{k} + f_k(\rho v)\frac{\|\rho v\phi\|}{k} \\
&\geq \frac{1}{k}\|f_k(\rho v)(w_k + \rho v\phi) - w_k\| \\
&\geq J_\lambda(w_k) - J_\lambda(f_k(\rho v)(w_k + \rho v\phi)) \\
&= \Big(\frac{1}{2} - \frac{1}{1-q}\Big)\|w_k\|^2 + \lambda\Big(\frac{1}{1-q} - \frac{1}{p+1}\Big)\int_{\Omega\times\{0\}} b(x) w_k^{p+1}(x,0)dx \\
&\quad + \Big(\frac{1}{1-q} - \frac{1}{2}\Big) f_k^2(\rho v)\|w_k + \rho v\phi\|^2 \\
&\quad - \frac{\lambda(p+q)}{(1-q)(p+1)} f_k^{p+1}(\rho v)\int_{\Omega\times\{0\}} b(x)(w_k + \rho v\phi)^{p+1}(x,0)dx \\
&= \Big(\frac{1}{1-q} - \frac{1}{2}\Big)(\|w_k + \rho v\phi\|^2 - \|w_k\|^2) + \Big(\frac{1}{1-q} - \frac{1}{2}\Big)[f_k^2(\rho v) - 1]\|w_k + \rho v\phi\|^2 \\
&\quad - \lambda\Big(\frac{1}{1-q} - \frac{1}{p+1}\Big) f_k^{p+1}(\rho v)\int_{\Omega\times\{0\}} b(x)[((w_k + \rho v\phi)^{p+1} - w_k^{p+1})(x,0)]dx \\
&\quad - \lambda\Big(\frac{1}{1-q} - \frac{1}{p+1}\Big)[f_k^{p+1}(\rho v) - 1]\int_{\Omega\times\{0\}} b(x) w_k^{p+1}(x,0)dx.
\end{aligned}
$$

Dividing by $\rho > 0$ and passing to the limit as $\rho \to 0$, we can obtain

$$
\begin{aligned}
&\langle f_k'(0), v\rangle \frac{\|w_k\|}{k} + \frac{\|v\phi\|}{k} \\
&\geq \Big(\frac{1+q}{1-q}\Big) k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_k \nabla(v\phi)\,dx\,dy + \Big(\frac{1+q}{1-q}\Big)\langle f_k'(0), v\rangle \|w_k\|^2 \\
&\quad - \lambda\Big(\frac{p+q}{1-q}\Big)\langle f_k'(0), v\rangle \int_{\Omega\times\{0\}} b(x) w_k^{p+1}(x,0)dx \\
&\quad - \lambda\Big(\frac{p+q}{1-q}\Big)\int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx \\
&= \frac{\langle f_k'(0), v\rangle}{1-q}\Big[(1+q)\|w_k\|^2 - \lambda(p+q)\int_{\Omega\times\{0\}} b(x) w_k^{p+1}(x,0)\,dx\Big]
\end{aligned}
$$

$$+\Big(\frac{1+q}{1-q}\Big)k_s\int_{\mathcal{C}_\Omega} y^c\nabla w_k\nabla(v\phi)\,dx\,dy-\lambda\Big(\frac{p+q}{1-q}\Big)\int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx.$$

That is,

$$\begin{aligned}\frac{\|v\phi\|}{k}\geq \frac{\langle f_k'(0),v\rangle}{1-q}\Big[&(1+q)\|w_k\|^2-\lambda(p+q)\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)\,dx\\ &-\frac{(1-q)\|w_k\|}{k}\Big]+\Big(\frac{1+q}{1-q}\Big)k_s\int_{\mathcal{C}_\Omega} y^c\nabla w_k\nabla(v\phi)\,dx\,dy\\ &-\lambda\Big(\frac{p+q}{1-q}\Big)\int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx,\end{aligned}\tag{4.10}$$

which is impossible because $\langle f_k'(0),v\rangle=+\infty$ and

$$(1+q)\|w_k\|^2-\lambda(p+q)\int_{\Omega\times\{0\}} b(x)w_k^{p+1}(x,0)\,dx-\frac{(1-q)\|w_k\|}{k}\geq C_2-\frac{(1-q)C_1}{k}>0.$$

In conclusion, $|\langle f_k'(0),v\rangle|<+\infty$. Furthermore, (4.4) with $\|w_k\|\leq C_1$ and two inequalities (4.8) and (4.10) also imply that

$$|\langle f_k'(0),v\rangle|\leq C_3$$

for k sufficiently large and a suitable constant C_3. □

Lemma 4.4. *For each $0\leq\phi\in C_{0,L}^\infty(\mathcal{C}_\Omega)$ and for every $0\leq v\in H_{0,L}^s(\mathcal{C}_\Omega)$ with $\|v\|\leq 1$, we have $a(x)w_0^{-q}v\phi\in L^1(\Omega)$ and*

$$\begin{aligned}&k_s\int_{\mathcal{C}_\Omega} y^c\nabla w_0\nabla(v\phi)\,dx\,dy-\int_{\Omega\times\{0\}} a(x)(w_0^{-q}v\phi)(x,0)dx\\ &-\lambda\int_{\Omega\times\{0\}} b(x)(w_0^p v\phi)(x,0)dx\geq 0.\end{aligned}\tag{4.11}$$

Proof. Applying (4.9) and (4.2) again, we obtain

$$\begin{aligned}&[f_k(\rho v)-1]\frac{\|w_k\|}{k}+f_k(\rho v)\frac{\|\rho v\phi\|}{k}\\ &\geq J_\lambda(w_k)-J_\lambda(f_k(\rho v)(w_k+\rho v\phi))\\ &=-\frac{f_k^2(\rho v)-1}{2}\|w_k\|^2-\frac{f_k^2(\rho v)}{2}(\|w_k+\rho v\phi\|^2-\|w_k\|^2)\\ &\quad+\frac{f_k^{1-q}(\rho v)-1}{1-q}\int_{\Omega\times\{0\}} a(x)(w_k+\rho v\phi)^{1-q}(x,0)\\ &\quad+\frac{1}{1-q}\int_{\Omega\times\{0\}} a(x)[((w_k+\rho v\phi)^{1-q}-w_k^{1-q})(x,0)]\\ &\quad+\lambda\frac{f_k^{p+1}(\rho v)-1}{p+1}\int_{\Omega\times\{0\}} b(x)(w_k+\rho v\phi)^{p+1}(x,0)\\ &\quad+\frac{\lambda}{p+1}\int_{\Omega\times\{0\}} b(x)[((w_k+\rho v\phi)^{p+1}-w_k^{p+1})(x,0)].\end{aligned}$$

Dividing by $\rho>0$ and passing to the limit $\rho\to 0^+$, we obtain

$$|\langle f_k'(0),v\rangle|\frac{\|w_k\|}{k}+\frac{\|v\phi\|}{k}$$

$$
\begin{aligned}
&\geq -\langle f_k'(0), v\rangle \|w_k\|^2 - k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_k \nabla(v\phi)\,dx\,dy \\
&\quad + \langle f_k'(0), v\rangle \int_{\Omega\times\{0\}} a(x) w_k^{1-q}(x,0)dx \\
&\quad + \liminf_{\rho\to 0^+} \frac{1}{1-q}\int_{\Omega\times\{0\}} \frac{a(x)[((w_k+\rho v\phi)^{1-q} - w_k^{1-q})(x,0)]}{\rho} dx \\
&\quad + \lambda\langle f_k'(0), v\rangle \int_{\Omega\times\{0\}} b(x) w_k^{p+1}(x,0)dx + \lambda \int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx. \\
&= -\langle f_k'(0), v\rangle \Big[\|w_k\|^2 - \int_{\Omega\times\{0\}} a(x) w_k^{1-q}(x,0)\,dx - \lambda\int_{\Omega\times\{0\}} b(x) w_k^{p+1}(x,0)\,dx\Big] \\
&\quad - k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_k \nabla(v\phi)\,dx\,dy + \lambda \int_{\Omega\times\{0\}} b(x)(w_k^p\phi)(x,0)dx \\
&\quad + \liminf_{\rho\to 0^+} \frac{1}{1-q}\int_{\Omega\times\{0\}} \frac{a(x)[(w_k+\rho v\phi)^{1-q} - w_k^{1-q})(x,0)]}{\rho} dx \\
&= -k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_k \nabla(v\phi)\,dx\,dy + \lambda \int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx \\
&\quad + \liminf_{\rho\to 0^+} \frac{1}{1-q}\int_{\Omega\times\{0\}} \frac{a(x)[((w_k+\rho v\phi)^{1-q} - w_k^{1-q})(x,0)]}{\rho} dx,
\end{aligned}
$$

Using above inequality, we have

$$
\liminf_{\rho\to 0^+} \int_{\Omega\times\{0\}} \frac{a(x)[((w_k+\rho v\phi)^{1-q} - w_k^{1-q})(x,0)]}{\rho} dx
$$

is finite. Now, since $a(x)[((w_k+v\phi)^{1-q} - w_k^{1-q})(x,0)] \geq 0$, then by Fatou's Lemma, we have

$$
\begin{aligned}
&\int_{\Omega\times\{0\}} a(x)(w_k^{-q} v\phi)(x,0)dx \\
&\leq \liminf_{\rho\to 0^+} \frac{1}{1-q}\int_{\Omega\times\{0\}} \frac{a(x)[((w_k+\rho v\phi)^{1-q} - w_k^{1-q})(x,0)]}{\rho} dx \\
&\leq \frac{|\langle f_k'(0), v\rangle| \|w_k\| + \|v\phi\|}{k} + k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_k \nabla(v\phi)\,dx\,dy \\
&\quad - \lambda \int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx \\
&\leq \frac{C_1C_3 + \|v\phi\|}{k} + k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_k \nabla(v\phi)\,dx\,dy - \lambda \int_{\Omega\times\{0\}} b(x)(w_k^p v\phi)(x,0)dx.
\end{aligned}
$$

Again using Fatou's Lemma and this inequality, we have

$$
\begin{aligned}
&\int_{\Omega\times\{0\}} a(x)(w_0^{-q} v\phi)(x,0)dx \\
&\leq \int_{\Omega\times\{0\}} \big[\liminf_{k\to\infty} a(x)(w_k^{-q} v\phi)(x,0)\big]dx \\
&\leq \liminf_{k\to\infty} \int_{\Omega\times\{0\}} a(x)(w_k^{-q} v\phi)(x,0)dx
\end{aligned}
$$

$$\leq k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla (v\phi)\,dx\,dy - \lambda \int_{\Omega\times\{0\}} b(x)(w_0^p v\phi)(x,0)dx < \infty,$$

which completes the proof. □

Corollary 4.5. *For every $0 \leq \phi \in H^s_{0,L}(\mathcal{C}_\Omega)$, we have $a(x)w_0^{-q}\phi \in L^1(\Omega)$, $w_0 > 0$ in $\mathcal{C}_\Omega$ and*

$$\begin{aligned} &k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla \phi\,dx\,dy - \int_{\Omega\times\{0\}} a(x)(w_0^{-q}\phi)(x,0)dx \\ &- \lambda \int_{\Omega\times\{0\}} b(x)(w_0^p\phi)(x,0)dx \geq 0. \end{aligned} \tag{4.12}$$

Proof. Choosing $v \in H^s_{0,L}(\mathcal{C}_\Omega)$ such that $v \geq 0$, $v \equiv l$ in the neighborhood of support of ϕ and $\|v\| \leq 1$, for some $l > 0$ is a constant. Then by the Lemma 4.4, we note that $\int_{\Omega\times\{0\}} a(x)(w_0^{-q}\phi)(x,0)dx < \infty$, for every $0 \leq \phi \in C^\infty_{0,L}(\mathcal{C}_\Omega)$, which guarantees that $w_0 > 0$ a.e. in $\Omega \times \{0\}$. Also by the strong maximum principle [9], we obtain $w_0 > 0$ in $\mathcal{C}_\Omega$. Putting this choice of v in (4.11) we have

$$\begin{aligned} &k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla \phi\,dx\,dy - \lambda \int_{\Omega\times\{0\}} b(x)(w_0^p\phi)(x,0)dx \\ &- \int_{\Omega\times\{0\}} a(x)(w_0^{-q}\phi)(x,0)dx \geq 0. \end{aligned}$$

for every $0 \leq \phi \in C^\infty_{0,L}(\mathcal{C}_\Omega)$. Hence by density argument, (4.12) holds for every $0 \leq \phi \in H^s_{0,L}(\mathcal{C}_\Omega)$, which completes the proof. □

Lemma 4.6. *We have $w_0 \in \mathcal{N}^+_\lambda$.*

Proof. Using (4.12) with $\phi = w_0$, we obtain

$$\|w_0\|^2 \geq \int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx + \lambda \int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx.$$

On the other hand, by the weak lower semi-continuity of the norm, we have

$$\begin{aligned} \|w_0\|^2 &\leq \liminf_{k\to\infty} \|w_k\|^2 \leq \limsup_{k\to\infty} \|w_k\|^2 \\ &= \int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx + \lambda \int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx. \end{aligned}$$

Thus

$$\|w_0\|^2 = \int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx + \lambda \int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx. \tag{4.13}$$

Consequently, $w_k \to w_0$ in $H^s_{0,L}(\mathcal{C}_\Omega)$ and $w_0 \in \mathcal{N}_\lambda$. Moreover, from (4.3) it follows that

$$\begin{aligned} &(1+q)\|w_0\|^2 - \lambda(p+q) \int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx \\ =&(1+q) \int_{\Omega\times\{0\}} a(x)w_0^{1-q}(x,0)dx - \lambda(p-1) \int_{\Omega\times\{0\}} b(x)w_0^{p+1}(x,0)dx > 0; \end{aligned}$$

that is, $w_0 \in \mathcal{N}^+_\lambda$. □

Lemma 4.7. *The function w_0 is a positive weak solution of problem (2.3).*

Proof. Suppose that $\phi \in H^s_{0,L}(\mathcal{C}_\Omega)$ and $\epsilon > 0$, we define $\Psi = (w_0 + \epsilon\phi)^+$. Let $\mathcal{C}_\Omega = \Gamma_1 \cap \Gamma_2$ with

$$\begin{gathered}\Gamma_1 := \big\{(x,y) \in \mathcal{C}_\Omega : (w_0 + \epsilon\phi)(x,y) > 0\big\},\\ \Gamma_2 := \big\{(x,y) \in \mathcal{C}_\Omega : (w_0 + \epsilon\phi)(x,y) \le 0\big\}.\end{gathered}$$

Let $\Omega \times \{0\} = \Omega_1 \times \Omega_2$ with

$$\begin{gathered}\Omega_1 := \{(x,0) \in \Omega \times \{0\} : (w_0 + \epsilon\phi)(x,0) > 0\},\\ \Omega_2 := \{(x,0) \in \Omega \times \{0\} : (w_0 + \epsilon\phi)(x,0) \le 0\}.\end{gathered}$$

Then $\Omega_1 \subset \Gamma_1$, $\Omega_2 \subset \Gamma_2$, $\Psi|_{\Gamma_1} = w_0 + \epsilon\phi$, $\Psi|_{\Gamma_2} = 0$, $\Psi|_{\Omega_1}(x,0) = (w_0 + \epsilon\phi)(x,0)$, and $\Psi|_{\Omega_2}(x,0) = 0$. Putting Ψ into (4.12) and using (4.13), we see that

$$\begin{aligned}
0 &\le k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla \Psi \,dx\,dy - \int_{\Omega\times\{0\}} a(x)(w_0^{-q}\Psi)(x,0)dx\\
&\quad - \lambda \int_{\Omega\times\{0\}} b(x)(w_0^p \Psi)(x,0)dx\\
&= k_s \int_{\Gamma_1} y^c \nabla w_0 \nabla (w_0 + \epsilon\phi)\,dx\,dy - \int_{\Omega_1} a(x)(w_0^{-q}(w_0+\epsilon\phi))(x,0)dx\\
&\quad - \lambda \int_{\Omega_1} b(x)(w_0^p(w_0+\epsilon\phi))(x,0)dx\\
&= k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla (w_0 + \epsilon\phi)\,dx\,dy - \int_{\Omega\times\{0\}} a(x)\left(w_0^{-q}(w_0+\epsilon\phi)\right)(x,0)dx\\
&\quad - \lambda \int_{\Omega\times\{0\}} b(x)\left(w_0^p(w_0+\epsilon\phi)\right)(x,0)dx - \Big[k_s \int_{\Gamma_2} y^c \nabla w_0 \nabla (w_0 + \epsilon\phi)\,dx\,dy\\
&\quad - \int_{\Omega_2} a(x)\left(w_0^{-q}(w_0+\epsilon\phi)\right)(x,0)dx - \lambda \int_{\Omega_2} b(x)\left(w_0^p(w_0+\epsilon\phi)\right)(x,0)dx\Big]\\
&= k_s \int_{\mathcal{C}_\Omega} y^c |\nabla w_0|^2\,dx\,dy - \int_{\Omega\times\{0\}} a(x) w_0^{1-q}(x,0)dx - \lambda \int_{\Omega\times\{0\}} b(x) w_0^{p+1}(x,0)dx\\
&\quad + \epsilon\Big[k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla \phi\,dx\,dy - \int_{\Omega\times\{0\}} a(x)(w_0^{-q}\phi)(x,0)dx\\
&\quad - \lambda \int_{\Omega\times\{0\}} b(x)(w_0^p\phi)(x,0)dx\Big]\\
&\quad - [k_s \int_{\Gamma_2} y^c |\nabla w_0|^2 - \int_{\Omega_2} a(x) w_0^{1-q}(x,0)dx - \lambda \int_{\Omega_2} b(x) w_0^{p+1}(x,0)dx\Big]\\
&\quad - \epsilon\Big[k_s \int_{\Gamma_2} y^c \nabla w_0 \nabla \phi\,dx\,dy - \int_{\Omega_2} a(x)(w_0^{-q}\phi)(x,0)dx\\
&\quad - \lambda \int_{\Omega_2} b(x)(w_0^p\phi)(x,0)dx\Big]\\
&\le \epsilon\Big[k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla \phi\,dx\,dy - \int_{\Omega\times\{0\}} a(x)(w_0^{-q}\phi)(x,0)dx\\
&\quad - \lambda \int_{\Omega\times\{0\}} b(x)(w_0^p\phi)(x,0)dx\Big]\\
&\quad - \epsilon k_s \int_{\Gamma_2} y^c \nabla w_0 \nabla \phi\,dx\,dy + \int_{\Omega_2} a(x) w_0^{-q}(w_0+\epsilon\phi)dx
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda \int_{\Omega_2} b(x)|\epsilon\phi|^{p+1}(x,0)dx + \epsilon\lambda \int_{\Omega_2} b(x)(w_0^p\phi)(x,0)dx \\
&\leq \epsilon\Big[k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla\phi \,dx\,dy - \int_{\Omega\times\{0\}} a(x)(w_0^{-q}\phi)(x,0)dx \\
&- \lambda \int_{\Omega\times\{0\}} b(x)(w_0^p\phi)(x,0)dx\Big] \\
&- \epsilon k_s \int_{\Gamma_2} y^c \nabla w_0 \nabla\phi \,dx\,dy + \epsilon\lambda\epsilon^p \|b\|_{L^{\frac{2_s^*}{2_s^*-p-1}}(\Omega_2)} \Big(\int_{\Omega_2} |\phi|^{2_s^*} dx\Big)^{\frac{p+1}{2_s^*}} \\
&+ \epsilon\lambda \int_{\Omega_2} b(x)(w_0^p\phi)(x,0)dx.
\end{aligned}
$$

Since the measure of Γ_2 and Ω_2 tend to zero as $\epsilon \to 0$, it follows that

$$
\int_{\Gamma_2} y^c \nabla w_0 \nabla\phi \,dx\,dy \to 0
$$

as $\epsilon \to 0$, and similarly for

$$
\lambda\epsilon^p \|b\|_{L^{\frac{2_s^*}{2_s^*-p-1}}(\Omega_2)} \Big(\int_{\Omega_2} |\phi|^{2_s^*} dx\Big)^{\frac{p+1}{2_s^*}}
$$

and $\lambda \int_{\Omega_2} b(x)(w_0^p\phi)(x,0)dx$. Dividing by ϵ and letting $\epsilon \to 0$, we obtain

$$
\begin{aligned}
&k_s \int_{\mathcal{C}_\Omega} y^c \nabla w_0 \nabla\phi \,dx\,dy - \int_{\Omega\times\{0\}} a(x)(w_0^{-q}\phi)(x,0)dx \\
&- \lambda \int_{\Omega\times\{0\}} b(x)(w_0^p\phi)(x,0)dx \geq 0
\end{aligned}
$$

and since this holds equally well for $-\phi$, it follows that w_0 is indeed a positive weak solution of problem (2.3). □

Lemma 4.8. *There exists a minimizing sequence $\{W_k\}$ in $\mathcal{N}_\lambda^-$ such that $W_k \to W_0$ strongly in $\mathcal{N}_\lambda^-$. Moreover W_0 is a positive weak solution of* (2.3).

Proof. Using the Ekeland variational principle again, we may find a minimizing sequence $\{W_k\} \subset \mathcal{N}_\lambda^-$ for the minimizing problem $\inf_{\mathcal{N}_\lambda^-} J_\lambda$ such that for $W_k \in H^s_{0,L}(\mathcal{C}_\Omega)$, we have $W_k \rightharpoonup W_0$ weakly in $H^s_{0,L}(\mathcal{C}_\Omega)$ and pointwise a.e. in $\Omega \times \{0\}$. We can now repeat the argument used in Lemma 4.2 to derive that when $\lambda \in (0, \Lambda)$

$$
(1+q) \int_{\Omega\times\{0\}} a(x)|W_0(x,0)|^{1-q}dx - \lambda(p-1) \int_{\Omega\times\{0\}} b(x)|W_0(x,0)|^{p+1}dx < 0 \quad (4.14)
$$

which yields

$$
(1+q) \int_{\Omega\times\{0\}} a(x)|W_k(x,0)|^{1-q}dx - \lambda(p-1) \int_{\Omega\times\{0\}} b(x)|W_k(x,0)|^{p+1}dx \leq -C_4
$$

for k sufficiently large and a suitable positive constant C_4. At this point we may proceed exactly as in Lemmas 4.3, 4.4, 4.6, 4.7 and Corollary 4.5, and conclude that $0 < W_0 \in \mathcal{N}_\lambda$ is the required positive weak solution of problem (2.3). Moreover from (4.14) it follows that

$$
(1+q)\|W_0\|^2 - \lambda(p+q) \int_{\Omega\times\{0\}} b(x)W_0^{p+1}(x,0)dx
$$

$$
\begin{aligned}
&= (1+q)\Big[\int_{\Omega\times\{0\}} a(x)W_0^{1-q}(x,0)dx + \lambda\int_{\Omega\times\{0\}} b(x)W_0^{p+1}(x,0)dx\Big]\\
&\quad - \lambda(p+q)\int_{\Omega\times\{0\}} b(x)W_0^{p+1}(x,0)dx\\
&= (1+q)\int_{\Omega\times\{0\}} a(x)W_0^{1-q}(x,0)dx - \lambda(p-1)\int_{\Omega\times\{0\}} b(x)W_0^{p+1}(x,0)dx < 0,
\end{aligned}
$$

that is $W_0 \in \mathcal{N}_\lambda^-$. □

Proof of the Theorem 2.6. From Lemmas 4.7, 4.8 and 3.4, we can conclude that problem (2.3) has two positive weak solutions $w_0 \in \mathcal{N}_\lambda^+$, $W_0 \in \mathcal{N}_\lambda^-$ with $\|W_0\| > \|w_0\|$ for any $\lambda \in (0,\Lambda)$. Hence, $u_0(\cdot) = w_0(\cdot,0) \in H_0^s(\Omega)$ and $U_0(\cdot) = W_0(\cdot,0) \in H_0^s(\Omega)$ are positive solutions of the problem (1.1). □

Proof of the Corollary 2.7. For any $W \in \mathcal{N}_\lambda^-$, it follows from Lemma 3.4 that

$$
\|W\| > A_\lambda = \Lambda^{\frac{-1}{p-1}}\Big(\frac{1+q}{p+q}\Big)^{1/(p-1)}\Big(\frac{1}{\|b\|}\Big)^{1/(p-1)}\big(\sqrt{S}\big)^{\frac{p+1}{p-1}}\Big(\frac{\Lambda}{\lambda}\Big)^{1/(p-1)}.
$$

Thus by the definition of Λ, and using $\frac{2(p+q)}{(1+q)(p-1)} - \frac{p+1}{p-1} = \frac{1-q}{1+q}$, we obtain,

$$
\|W\| > \Big(1 + \frac{1+q}{p-1}\Big)^{1/(1+q)}\|a\|^{1/(q+1)}\Big(\frac{1}{\sqrt{S}}\Big)^{\frac{1-q}{1+q}}\Big(\frac{\Lambda}{\lambda}\Big)^{1/(p-1)}.
$$

Hence, let $W_\epsilon \in \mathcal{N}_\lambda^-$ be the solution of problem (2.3) with $p = 1+\epsilon$, where $\lambda \in (0,\Lambda)$, we have

$$
\|W\| > C_\epsilon\Big(\frac{\Lambda}{\lambda}\Big)^{1/\epsilon},
$$

where $C_\epsilon = \big(1 + \frac{1+q}{\epsilon}\big)^{1/(1+q)}\|a\|^{1/(q+1)}\big(\frac{1}{\sqrt{S}}\big)^{\frac{1-q}{1+q}} \to \infty$ as $\epsilon \to 0$. This completes the proof of Corollary 2.7. □

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References

[1] Adimurthi, J. Giacomoni; *Multiplicity of positive solutions for a singular and critical elliptic problem in* $\mathbb{R}^2$, Comm. in Contemporary Mathematics, 8 (5) (2006), 621-656.

[2] Ahmed Mohammed; *Positive solutions of the p-Laplace equation with singular nonlinearity*, J. Math. Anal. Appl., 352 (2009), 234-245.

[3] D. Applebaum; *Lèvy process-from probability to finance and quantum groups*, Notices Amer. Math. Soc., 51 (2004), 1336-1347.

[4] B. Barrios, I. D. Bonis, M. Medina, I. Peral; *Semilinear problems for the fractional Laplacian with a singular nonlinearity*, Open Math., 13 (2015), 390-407.

[5] B. Brändle, E. Colorado, A. de Pablo, U. Sànchez; *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh. Sect. A, 143 (2013), 39-71.

[6] A. Capella, J. Dávila, L. Dupaigne, Y. Sire; *Regularity of radial extremal solutions for some non-local semilinear equations*, Comm. Partial Differential Equations, 36 (2011), 1353-1384.

[7] X. Cabré, J. Tan; *Positive solutions for nonlinear problems involving the square root of the Laplacian*, Adv. Math., 224 (2010), 2052-2093.

[8] L. Caffarelli, L. Silvestre; *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, 32 (2007), 1245-1260.

[9] L. Silvestre; *Regularity of the obstacle problem for a fractional power of the Laplace operator,* Comm. Pure Appl. Math., 60 (2007) 67-112.

[10] Y. Chen, J. Chen; *Existence of multiple positive weak solutions and estimates for extremal values to a class of elliptic problems with Hardy term and singular nonlinearity*, J. Math. Anal. Appl., 429 (2015), 873-900.

[11] M. M. Coclite, G. Palmieri; *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations, 14 (10) (1989), 1315-1327.

[12] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, 2 (1977), 193-222.

[13] Y. Fang; *Existence, uniqueness of positive solution to a fractional laplacians with singular nonlinearity*, arXiv:1403.3149v1. (2014).

[14] A. Garroni, S. Müller; Γ-*limit of a phase-field model of dislocations*, SIAM J. Math. Anal., 36 (2005) 1943-1964.

[15] S. Goyal, K. Sreenadh; *Existence of multiple solutions of p-fractional Laplace operator with sign-changing weight function,* Adv. Nonlinear Anal., 4 (1) (2015) 37-58.

[16] S. Goyal, K. Sreenadh; *The Nehari manifold for non-local elliptic operator with concave-convex nonlinearities and sign-changing weight functions*, Proc. Indian Acad. Sci. Math. Sci., PMSC-D-13-00460.

[17] N. Hirano, C. Saccon, N. Shioji; *Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities*, Adv. Differential Equations, 9 (2) (2004) 197-220.

[18] N. Hirano, C. Saccon, N. Shioji; *Brezis-Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem,* J. Differential Equations, 245 (2008) 1997-2037.

[19] A. C. Lazer, P. J. McKenna; *On a singular nonlinear elliptic boundary-value problem*, Proc. of Amer. Math. Soc., 111 (3) (1991) 721-730.

[20] A. C. Lazer, P. J. McKenna; *On singular boundary value problems for the Monge-Ampère operator*, J. Math. Anal. Appl., 197 (1996) 341-362.

[21] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012) 225-236.

[22] R. Servadei, E. Valdinoci; *Mountain pass solutions for non-local elliptic operators,* J. Math. Anal. Appl., 389 (2012) 887-898.

[23] R. Servadei, E. Valdinoci; *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst., 33 (5) (2013) 2105-2137.

[24] R. Servadei, E. Valdinoci; *A Brezis-Nirenberg result for non-local critical equations in low dimension,* Commun. Pure Appl. Anal., 12 (6) (2013) 2445-2464.

[25] R. Servadei, E. Valdinoci; *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc., 367 (2015) 67-102.

[26] J. Tan; *Positive solutions for nonlocal elliptic problems*, Discrete Contin. Dyn. Syst., 33 (2013) 837-859.

[27] Xiaohui Yu; *The Nehari manifold for elliptic equation involving the square root of the Laplacian*, J. Differential Equations, 252 (2012) 1283-1308.

[28] J. Zhang, X. Liu, H. Jiao; *Multiplicity of positive solutions for a fractional Laplacian equations involving critical nonlinearity*, arXiv:1502.02222v1 [math.AP].

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