Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 148, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE RESULTS FOR ANISOTROPIC DISCRETE BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we prove the existence of nontrivial weak solutions for a class of discrete boundary value problems. The main tools used here are the variational principle and critical point theory.

1. Introduction and Preliminaries

In this article, we are interested in the existence of solutions for the discrete boundary value problem

$$\begin{gathered} -\Delta(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda f(k,u(k)), \quad k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{gathered} \tag{1.1}$$

where $T \geq 2$ is a positive integer; $\mathbb{Z}[a,b]$ denotes the discrete interval $\{a, a+1, \dots, b\}$ with a and b are integers such that $a < b$; $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator; λ is a positive parameter; $f : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to $t \in \mathbb{R}$, and $k \in \mathbb{Z}[1,T]$. For the function $p : \mathbb{Z}[0,T] \to [2,\infty)$ denote

$$p^- := \min_{k\in\mathbb{Z}[0,T]} p(k) \leq p(k) \leq \max_{k\in\mathbb{Z}[0,T]} p(k) =: p^+ < \infty.$$

In the previous decades, the nonlinear difference equations have been intensively used for the mathematical modelling of various problems in different disciplines of science, such as computer science, mechanical engineering, control systems, artificial or biological neural networks and economics. This mades nonlinear difference equations very attractive to many authors, and hence, many paper have been devoted to the relative field by using a various methods such as fixed points theorems, topological methods and variational methods. For the recent progress in discrete problems, we refer the readers to the interesting book by Agarwal [1] and the papers [8, 17].

In [2, 3, 5, 6, 8, 9, 15, 19], the authors used different methods to study the existence and multiplicity of solutions for the discrete boundary value problem of the type

$$\begin{gathered} -\Delta\big(\phi_p(\Delta u(k-1))\big) = f(k,u(k)), \quad k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{gathered} \tag{1.2}$$

2010 *Mathematics Subject Classification.* 47A75, 35B38, 35P30, 34L05, 34L30.
Key words and phrases. Anisotropic discrete boundary value problems;
multiple solutions; variational methods; critical point theory.

Submitted March 23, 2016. Published June 16, 2016.

where $\phi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$. In [17], Mihăilescu et al. studied the eigenvalue problem for the anisotropic discrete boundary-value problem

$$\begin{gathered} -\Delta\big(\phi_{p(k-1)}(\Delta u(k-1))\big) = \lambda|\Delta u(k-1)|^{q(k-1)-2}\Delta u(k-1), \quad k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{gathered} \tag{1.3}$$

where $\phi_{p(k-1)}(s) = |s|^{p(k-1)-2}s$, $p : \mathbb{Z}[0,T] \to [2,+\infty)$, $q : \mathbb{Z}[1,T] \to [2,+\infty)$ and λ is a positive parameter. For the recent papers involving anisotropic discrete boundary value problems, we refer to recent works [4, 10, 11, 14, 16] and references therein. Motivated by the papers mentioned above, we study problem (1.1) and obtain the existence of nontrivial weak solutions by employing variational principle and critical point theory argued in [7].

Let us define the function space

$$W = \{u : \mathbb{Z}[0,T+1] \to \mathbb{R} \text{ such that } u(0) = u(T+1) = 0\}.$$

Then, W is a T-dimensional Hilbert space with the inner product

$$(u,v) = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \quad \forall u,v \in W,$$

while the corresponding norm is given by

$$\|u\|_W = \Big(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\Big)^{1/2}.$$

We can also define the following norm on W since W is finite-dimensional,

$$|u|_m = \Big(\sum_{k=1}^{T} |u(k)|^m\Big)^{1/m}, \quad \forall u \in W,\ m \geq 2.$$

Now, we recall some auxiliary results which we use through the paper.

Proposition 1.1 ([10, 17])**.** *(i) Let $u \in W$ and $\|u\|_W > 1$. Then*

$$\sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} \geq \frac{1}{p^+(\sqrt{T})^{p^- -2}}\|u\|_W^{p^-} - T.$$

(ii) Let $u \in W$ and $\|u\|_W < 1$. Then

$$\sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} \geq \frac{1}{p^+(\sqrt{T})^{2-p^+}}\|u\|_W^{p^+}.$$

(iii) For any $m \geq 2$, there exist positive constants c_m such that

$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall u \in W.$$

(iv) For any $m \geq 2$, we have

$$(T+1)^{\frac{2-m}{2}}\|u\|_W^m \leq \sum_{k=1}^{T+1} |\Delta u(k-1)|^m \leq (T+1)\|u\|_W^m, \quad \forall u \in W.$$

(v) For any $m \geq 2$, we have
$$2^m \sum_{k=1}^{T} |u(k)|^m \geq \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall u \in W.$$

The key arguments in our paper are the following results given in [7].

Proposition 1.2. *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{x \in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $0 < \Phi(\overline{x}) < r$, such that:*

1. $\frac{1}{r} \sup_{\Phi(x) \leq r} \Psi(x) \leq \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,
2. *for each $\lambda \in \Lambda_r := \big(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)}\big)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(P.S.)^{[r]}$ condition.*

Then, for each $\lambda \in \Lambda_r$, there is $x_{0,\lambda} \in \Phi^{-1}((0,r))$ such that $I'_\lambda(x_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(x_{0,\lambda}) \leq I_\lambda(x)$ for all $x \in \Phi^{-1}((0,r))$.

Proposition 1.3. *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that for each*
$$\lambda \in \Big(0, \frac{r}{\sup_{u \in \Phi^{-1}((-\infty,r))} \Psi(u)}\Big),$$
the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(P.S.)$ condition and it is unbounded from below. Then for each
$$\lambda \in \Big(0, \frac{r}{\sup_{u \in \Phi^{-1}((-\infty,r))} \Psi(u)}\Big),$$
the functional I_λ admits two distinct critical points.

Proposition 1.4. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\inf_{x \in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exists $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that:*

1. $\frac{1}{r} \sup_{\Phi(x) \leq r} \Psi(x) \leq \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,
2. *for each*
$$\lambda \in \Lambda_r := \Big(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)}\Big),$$
the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional I_λ has at least three distinct critical points.

Let us proceed with setting problem (1.1) in the variational structure. To this end, let us define the functionals $\Phi, \Psi : W \to \mathbb{R}$ as follows:
$$\Phi(u) = \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)},$$
$$\Psi(u) = \sum_{k=1}^{T} F(k, u(k)),$$

where $F(k,s)=\int_0^s f(k,t)dt$, $(k,s)\in\mathbb{Z}[1,T]\times\mathbb{R}$.

The functionals Φ and Ψ are well-defined and continuously Gâteaux differentiable where their derivatives are

$$\Phi'(u)\varphi=\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta\varphi(k-1),$$

$$\Psi'(u)\varphi=\sum_{k=1}^{T}f(k,u(k))\varphi(k),$$

for all $u,\varphi\in W$.

Then the functional $I_\lambda : W\to\mathbb{R}$ corresponding to problem (1.1) is

$$I_\lambda(u):=\Phi(u)-\lambda\Psi(u).$$

The functional I_λ is also well defined on W and $I_\lambda\in C^1(W,\mathbb{R})$ with the derivative

$$I'_\lambda(u)\varphi=\Phi'(u)\varphi-\lambda\Psi'(u)\varphi,$$

for all $u,\varphi\in W$.

We want to remark that since problem (1.1) is defined in a finite-dimensional Hilbert space W, it is not difficult to verify that the functionals Φ, Ψ and I_λ satisfy the regularity assumptions mentioned above (see, e.g., [13]).

Definition 1.5. We say that $u\in W$ is a weak solution of problem (1.1) if

$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta\varphi(k-1)-\lambda\sum_{k=1}^{T}f(k,u(k))\varphi(k)=0 \tag{1.4}$$

for all $\varphi\in W$, where (1.4) is called the weak form of problem (1.1).

From the above definition it is obvious that the weak solutions of problem (1.1) are in fact the critical points of I_λ.

We also use the following helpful notation:

$$\begin{gathered}\beta^{p_*}=\begin{cases}\beta^{p^+} & \text{if } \beta>1\\ \beta^{p^-} & \text{if } 0<\beta<1,\end{cases}\qquad \eta^{1/p_*}=\begin{cases}\eta^{1/p^-} & \text{if } \eta>1\\ \eta^{1/p^+} & \text{if } 0<\eta<1,\end{cases}\\ \delta^{(q/p)_*}=\begin{cases}\delta^{q^+/p^-} & \text{if } \delta>1\\ \delta^{q^-/p^+} & \text{if } 0<\delta<1\,.\end{cases}\end{gathered} \tag{1.5}$$

2. Existence of one solution

We sue the following assumptions:

(A1) There exist $C>0$ and a function $q:\mathbb{Z}[1,T]\to[2,+\infty)$ such that for all $(k,t)\in\mathbb{Z}[1,T]\times\mathbb{R}$,

$$|f(k,t)|\le C\big(1+|t|^{q(k)-1}\big).$$

(A2) There exist $r,a,b,l>0$ with

$$b<(\frac{p^-}{2})^{1/p^+}\frac{a}{(T+1)^{\frac{(p^+-2)}{2p^-}}(p^+)^{\frac{1}{p^-}}}$$

such that

$$\frac{l}{r}(T+(T+1)^{(p^+-2)q^+/2p^-}(p^+)^{q^+/p^-}r^{(q/p)_*}) < b_p\sum_{k=1}^{T}F(k,b),$$

where

$$b_p = \Big(\frac{b^{p(0)}}{p(0)}+\frac{b^{p(T)}}{p(T)}\Big)^{-1}.$$

Theorem 2.1. *Assume* (A1) *and* (A2) *are satisfied. Then for each*

$$\lambda \in \Lambda_{r,b} := \Big(\frac{1}{b_p\sum_{k=1}^{T}F(k,b)}, \frac{r}{l(T+(T+1)^{(p^+-2)q^+/2p^-}(p^+)^{q^+/p^-}r^{(q/p)_*})}\Big),$$

problem (1.1) *admits at least one nontrivial weak solution.*

Proof. We will apply Proposition 1.2. We know that Φ and Ψ are well-defined and continuously Gâteaux differentiable. Moreover, from the definitions of Φ and Ψ we have

$$\inf_{u\in W}\Phi(u)=\Phi(0)=\Psi(0)=0.$$

Let us define the function $\overline{u}:\mathbb{Z}[0,T+1]\to\mathbb{R}$ belonging to W by the formula

$$\overline{u}(k)=\begin{cases} b & \text{if } k\in\mathbb{Z}[1,T],\\ 0 & \text{if } k=0, k=T+1.\end{cases}$$

Then, we deduce that

$$\Phi(\overline{u})=\frac{b^{p(0)}}{p(0)}+\frac{b^{p(T)}}{p(T)}$$

which implies

$$\Phi(\overline{u})\le\frac{2}{p^-}b^{p_*}.$$

Moreover, we have

$$\frac{\Psi(\overline{u})}{\Phi(\overline{u})}=\frac{\sum_{k=1}^{T}F(k,b)}{\frac{b^{p(0)}}{p(0)}+\frac{b^{p(T)}}{p(T)}}.$$

For each $u\in\Phi^{-1}((-\infty,r))$, from Proposition 1.1(iv) and 1.5, one has

$$\frac{1}{p^+}(T+1)^{(2-p^+)/2}\|u\|_W^{p_*}\le\frac{1}{p^+}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)}\le\Phi(u)\le r$$

$$\|u\|_W\le((T+1)^{(p^+-2)/2}p^+r)^{1/p_*}\le(T+1)^{(p^+-2)/2p^-}(p^+)^{1/p^-}r^{1/p_*}:=a$$

Then

$$r=\frac{a^{p_*}}{(T+1)^{\frac{(p^+-2)p_*}{2p^-}}(p^+)^{\frac{p_*}{p^-}}}. \tag{2.1}$$

Since

$$b<\big(\frac{p^-}{2}\big)^{1/p^+}\frac{a}{(T+1)^{\frac{(p^+-2)}{2p^-}}(p^+)^{\frac{1}{p^-}}},$$

we obtain $\Phi(\overline{u})<r$. Moreover, from condition (A1), there exists a constant $l>0$ such that $|F(k,t)|\le l(1+|t|^{q(k)})$. Then it follows

$$\Psi(u)\le\sum_{k=1}^{T}l(1+|u(k)|^{q(k)})\le l(T+\|u\|_W^{q_*}),$$

$$\|u\|_W^{q_*} \le (T+1)^{(p^+-2)q^+/2p^-}(p^+)^{q^+/p^-} r^{(q/p)_*},$$

and hence

$$\frac{1}{r}\sup_{\Phi(u)\le r}\Psi(u) \le \frac{l}{r}\Big(T+(T+1)^{(p^+-2)q^+/2p^-}(p^+)^{q^+/p^-} r^{(q/p)_*}\Big). \tag{2.2}$$

Taking into account the condition (A2), we have

$$\frac{1}{r}\sup_{\Phi(u)\le r}\Psi(u) \le \frac{\Psi(\overline{u})}{\Phi(\overline{u})}.$$

In conclusion, Proposition 1.2(1) is verified.

For Proposition 1.2(2), as mentioned before, Φ and Ψ are well-defined and continuously Gâteaux differentiable. Further, from (A1), Ψ has a compact derivative. This ensures that the functional I_λ satisfies the $(P.S.)^{[r]}$ condition for each $r>0$. Hence Proposition 1.2(2) is verified as well.

Consequently, by Proposition 1.2, for each $\lambda \in \Lambda_{r,b}$, the functional I_λ admits at least one critical point which corresponds to the nontrivial weak solution of problem (1.1). □

Example 2.2. As an application of Theorem 2.1, we consider the following: Let $T=2$, $p(k-1)=2(k+1)$, $q(k)=k+1$, $b=1$, $a=7$ and $f(k,u)=|u(k)|^k$. Then, $p^-=p(0)=4$, $p^+=p(T)=6$, $q^-=2$, $q^+=3$, $F(k,u)=\frac{1}{k+1}|u(k)|^{k+1}$ and $l=\frac{1}{k+1}$, say $l=1/3$. Then, the all the assumptions requested in Theorem 2.1 hold. Finally, by simple computations, it results that for each $\lambda\in\Lambda_{r,b}\subseteq(1/2,\lambda_a)$ problem (1.1) admits at least one nontrivial weak solution, where the real constant λ_a depends on a and satisfies $\lambda_a \ge 31/50$.

3. Existence of two solutions

For the next theorem we use the assumption

(A3) There exist positive real numbers θ and t_0 such that $\theta>p^+$ and
$$0<\theta F(k,t)\le f(k,t)t \quad \forall (k,t)\in\mathbb{Z}[1,T]\times\mathbb{R},\ |t|\ge t_0\,.$$

Theorem 3.1. *Assume that* (A1) *and* (A3) *hold. Then for each*

$$\lambda\in\Lambda_r := \Big(0, \frac{r}{l(T+(T+1)^{(p^+-2)q^+/2p^-}(p^+)^{q^+/p^-} r^{(q/p)_*})}\Big),$$

problem (1.1) *admits at least two distinct weak solutions.*

Proof. We will apply Proposition 1.3. It is obvious that $\Phi(0)=\Psi(0)=0$. Moreover, Φ is bounded from below. Indeed, for $\|u\|_W<1$ and by Proposition 1.1(ii), it reads

$$\Phi(u)=\sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} \ge \frac{1}{p^+(\sqrt{T})^{2-p^+}}\|u\|_W^{p^+}.$$

Let us show that I_λ is unbounded from below and satisfies the $(P.S.)$ condition. From condition (A3), there exists a constant $c>0$ such that $F(k,t)\ge c|t|^\theta$ for any $(k,t)\in\mathbb{Z}[1,T]\times\mathbb{R}$, $|t|\ge t_0$. Let $\|u\|_W>1$. Then, using Proposition 1.1(iv)-(v), it reads

$$I_\lambda(u)\le\frac{1}{p^-}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)}-\lambda\sum_{k=1}^{T}F(k,u(k))$$

$$
\begin{aligned}
&\le \frac{(T+1)}{p^-}\|u\|_W^{p^+} - \lambda c \sum_{k=1}^{T} |u(k)|^\theta \\
&\le \frac{(T+1)}{p^-}\|u\|_W^{p^+} - \lambda c 2^{-\theta} \sum_{k=1}^{T+1} |\Delta u(k-1)|^\theta \\
&\le \frac{(T+1)}{p^-}\|u\|_W^{p^+} - \lambda c 2^{-\theta} (T+1)^{(2-\theta)/2}\|u\|_W^\theta,
\end{aligned}
$$

from which we get

$$
\lim_{\|u\|_W \to \infty} I_\lambda(u) = -\infty.
$$

Therefore, I_λ is unbounded from below and anti-coercive. Additionally, since the space W is finite-dimensional, $(P.S.)$ condition follows immediately. Consequently, all assumptions of Proposition 1.3 are verified. Therefore, for each $\lambda \in \Lambda_r$, the functional I_λ admits two distinct critical points that are weak solutions of problem (1.1). □

We want to remark that from condition (A1), there exists a constant $C_1 > 0$ such that $|F(k,t)| \le C_1(1 + |t|^{q(k)})$, and from condition (A3), there exists a constant $C_2 > 0$ such that $F(k,t) \ge C_2|t|^\theta$ for any $(k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}$. So, we conclude that $C_2|t|^\theta \le C_1(1+|t|^{q(k)})$ which means that $q(k) \ge \theta$ for all $k \in \mathbb{Z}[1,T]$. Therefore, we have $p^+ < \theta \le q^-$ as a natural condition raised from (A1) and (A3).

Example 3.2. As an application of Theorem 3.1, we consider the function

$$
f(k,t) = \begin{cases} m + nq(k)t^{q(k)-1} & t \ge 0, \\ m - nq(k)(-t)^{q(k)-1} & t < 0, \end{cases}
$$

for each $(k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}$ where m, n are some positive constants. We also assume that

$$
t_0 > \max\Big\{ \Big(\frac{m(\theta-1)}{n(q^- - \theta)}\Big)^{\frac{1}{q(k)-1}}, \Big(\frac{m}{n}\Big)^{\frac{1}{q(k)-1}} \Big\}
$$

such that $q^- > \theta$. Then, condition (A1) is easily verified. Let us proceed for condition (A3). From the above definition of f, we get $F(k,t) = mt + n|t|^{q(k)}$. Since we have $t_0^{q(k)-1} > \frac{m}{n}$, for all $k \in \mathbb{Z}[1,T]$ and $|t| \ge t_0$ there holds

$$
F(k,t) \ge |t|\big(-m + n|t|^{q(k)-1}\big) \ge t_0(-m + nt_0^{q(k)-1}) > 0.
$$

Moreover, for all $k \in \mathbb{Z}[1,T]$ and $t < 0$, we have

$$
\begin{aligned}
tf(k,t) - \theta F(k,t) &= m(1-\theta)t + n(q(k)-\theta)|t|^{q(k)} \\
&= m(\theta-1)|t| + n(q(k)-\theta)|t|^{q(k)} > 0.
\end{aligned}
$$

Finally, thanks to the assumption $t_0^{q(k)-1} > \frac{m(\theta-1)}{n(q^- - \theta)}$, for all $k \in \mathbb{Z}[1,T]$ and $t \ge t_0$, we obtain

$$
\begin{aligned}
tf(k,t) - \theta F(k,t) &= t(n(q(k)-\theta)|t|^{q(k)-1} - m(\theta-1)) \\
&\ge t_0(n(q^- - \theta)r^{q(k)-1} - m(\theta-1)) > 0.
\end{aligned}
$$

Therefore condition (A3) holds as well.

4. Existence of three solutions

For the next theorem we use the assumption

(A4) There exist $C_d, d > 0$ with
$$d > (\frac{p^+}{2})^{1/p^+} \frac{a}{(T+1)^{\frac{(p^+-2)}{2p^-}} (p^+)^{\frac{1}{p^-}}}$$
such that
$$\frac{l}{r}(T + (T+1)^{(p^+-2)q^+/2p^-} (p^+)^{q^+/p^-} r^{(q/p)_*}) < C_d d_p \sum_{k=1}^{T} F(k,d),$$
where $d_p = \big(\frac{d^{p(0)}}{p(0)} + \frac{d^{p(T)}}{p(T)}\big)^{-1}$.

Theorem 4.1. *Assume* (A1), (A4) *and* $q^+ < p^-$. *Then for each*
$$\lambda \in \Lambda_{r,d} := \Big(\frac{1}{C_d d_p \sum_{k=1}^{T} F(k,d)}, \frac{r}{l(T + (T+1)^{(p^+-2)q^+/2p^-} (p^+)^{q^+/p^-} r^{(q/p)_*})}\Big),$$
problem (1.1) *admits at least three distinct weak solutions.*

Proof. We will apply Proposition 1.4. We know that, Φ and Ψ are well-defined and continuously Gâteaux differentiable, and $\inf_{u\in W} \Phi(u) = \Phi(0) = \Psi(0) = 0$. The compactness of derivative of Ψ follows from the growth condition (A1). Since Φ is of class C^1 on the finite-dimensional Hilbert space W, to prove that Φ is weakly lower semicontinuous, it is sufficient to show the coercivity of Φ (see [12]). Indeed, let $u \in W$ such that $\|u\|_W \to +\infty$. Then, without loss of generality, we can assume that $\|u\|_W > 1$. From the definition of the functional Φ and Proposition 1.1(i), we deduce that
$$\Phi(u) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \geq \frac{1}{p^+(\sqrt{T})^{p^--2}} \|u\|_W^{p^-} - T.$$
So, $\Phi(u) \to +\infty$ as $\|u\|_W \to +\infty$ which means that Φ is coercive. We continue to show the existence of the inverse function $(\Phi')^{-1} : W^* \to W$. At first, we show the strict monotonicity of Φ'. For the case $u_1 \neq u_2 \in W$, we have
$$\begin{aligned} &(\Phi'(u_1) - \Phi'(u_2))(u_1 - u_2) \\ &\geq \sum_{k=1}^{T+1} (|\Delta u_1(k-1)|^{p(k-1)-2} \Delta u_1(k-1) - |\Delta u_2(k-1)|^{p(k-1)-2} \Delta u_2(k-1)) \\ &\quad \times (\Delta u_1(k-1) - \Delta u_2(k-1)) \end{aligned}$$
By the well-known inequality, for any $\zeta, \xi \in \mathbb{R}^N$,
$$(|\zeta|^{r-2}\zeta - |\xi|^{r-2}\xi)(\zeta - \xi) \geq C_r |\zeta - \xi|^r, \quad r \geq 2,\ C_r > 0,$$
we obtain
$$(\Phi'(u_1) - \Phi'(u_2))(u_1 - u_2) \geq c_3 \sum_{k=1}^{T+1} |\Delta u_1(k-1) - \Delta u_2(k-1)|^{p(k-1)-2} > 0,$$

where c_3 is a positive constant depends only on p. Therefore Φ' is strictly monotone, which ensures that Φ' is an injection. Moreover, by Proposition 1.1, we have

$$\Phi'(u)u = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq c_4 \min\left\{\|u\|_W^{p^-}, \|u\|_W^{p^+}\right\} - c_5,$$

where c_4, c_5 are positive constants and $u \in W$. So, $\Phi'(u) \to +\infty$ as $\|u\|_W \to +\infty$.

From the above information, and Minty-Browder theorem (see [20]), we obtain that Φ' is a surjection. As a consequence, Φ' has an inverse mapping $(\Phi')^{-1} : W^* \to W$. We now show that $(\Phi')^{-1}$ is continuous. To this end, let $(u_n^*), u^* \in W^*$ with $u_n^* \to u^*$, and let $(\Phi')^{-1}(u_n^*) = (u_n), (\Phi')^{-1}(u^*) = u$. Then, $\Phi(u_n) = u_n^*$ and $\Phi(u) = u^*$, which means that (u_n) is bounded in W. Hence there exists $u_0 \in W$ and a subsequence, again denoted by (u_n), such that $u_n \rightharpoonup u_0$ in W, and therefore $u_n \to u_0$ in W. Since the limit is unique, it follows that $u_n \to u$ in W. Therefore $(\Phi')^{-1}$ is continuous.

Next we verify Proposition 1.4(1). To do this, let us define the function $\overline{v} : \mathbb{Z}[0, T+1] \to \mathbb{R}$ belonging to W by the formula

$$\overline{v}(k) = \begin{cases} d & \text{if } k \in \mathbb{Z}[1,T], \\ 0 & \text{if } k = 0, k = T+1, \end{cases}$$

Then we deduce that

$$\Phi(\overline{v}) = \frac{d^{p(0)}}{p(0)} + \frac{d^{p(T)}}{p(T)}$$

which implies that

$$\Phi(\overline{v}) \geq \frac{2}{p^+} d^{p_*}.$$

Since $d > (\frac{p^+}{2})^{1/p^+} \frac{a}{(T+1)^{\frac{(p^+-2)}{2p^-}} (p^+)^{\frac{1}{p^-}}}$, we get $\Phi(\overline{v}) > r$, where r is as in (2.1). Moreover, we have

$$\frac{\Psi(\overline{v})}{\Phi(\overline{v})} = \frac{\sum_{k=1}^{T} F(k,d)}{\frac{d^{p(0)}}{p(0)} + \frac{d^{p(T)}}{p(T)}}.$$

For each $u \in \Phi^{-1}((-\infty, r))$, similarly to (2.2), we have

$$\frac{1}{r} \sup_{\Phi(u) \leq r} \Psi(u) \leq \frac{l}{r}(T + (T+1)^{(p^+-2)q^+/2p^-} (p^+)^{q^+/p^-} r^{(q/p)_*}).$$

Therefore, from condition (A4), it holds

$$\frac{1}{r} \sup_{\Phi(u) \leq r} \Psi(u) \leq \frac{\Psi(\overline{v})}{\Phi(\overline{v})}.$$

Hence, Proposition 1.4(1) is verified. Let us proceed with the coercivity of I_λ. Let $u \in W$ such that $\|u\|_W \to +\infty$. Then, without loss of generality, we can assume that $\|u\|_W > 1$. Then from Proposition 1.1(i) and condition (A1), it reads

$$\begin{aligned} I_\lambda(u) &= \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^{T} F(k, u(k)) \\ &\geq \frac{1}{p^+(\sqrt{T})^{p^- - 2}} \|u\|_W^{p^-} - T - \lambda l(T + \|u\|_W^{q^+}) \end{aligned}$$

$$\geq \frac{1}{p^+(\sqrt{T})^{p^- -2}}\|u\|_W^{p^-} - \lambda l\|u\|_W^{q^+} - T(1+\lambda l);$$

that is, I_λ is coercive. So, Proposition 1.4(2) is verified.

Consequently, the assumptions of Proposition 1.4 are verified. Therefore, for each $\lambda \in \Lambda_{r,d}$, the functional I_λ admits at least three distinct critical points that are weak solutions of problem (1.1). □

Example 4.2. As an application of Theorem 4.1, if we consider function f and the assumptions as given in Example 2.2, take $d = 4$ and $C_d \geq 42$, then the all the assumptions requested in Theorem 4.1 hold. Moreover, for each $\lambda \in \Lambda_{r,d} \subseteq (25/43, \lambda_a)$ problem (1.1) admits at least three nontrivial weak solutions, where the real constant λ_a depends on a and satisfies $\lambda_a \geq 31/50$.

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