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MULTIPLE SOLUTIONS FOR BREZIS-NIRENBERG PROBLEMS WITH FRACTIONAL LAPLACIAN

HUI GUO

Abstract. In this article, we prove the multiplicity of nontrivial solutions to the critical problem with fractional Laplacian

$$\begin{gathered}(-\Delta)^{\alpha/2}u = |u|^{2^*_\alpha-2}u + \lambda u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where $0 < \alpha < 2$, $N > (1+\sqrt{2})\alpha$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. More precisely, for any $\lambda > 0$, this problem has at least $[(N+1)/2]$ pairs of nontrivial weak solutions.

1. Introduction

In recent decades, the study of nonlocal diffusion problems has attracted much attention from mathematicians, in particular, of equations involving fractional Laplace operator. As is known to us, the fractional Laplace operator appears in anomalous diffusion phenomena in several fields such as physics, biology and probability. Moreover, it can be viewed as the infinitesimal generator of a stable Lévy process. For more details and applications, one can see [1] and references therein.

In this article, we focus on the multiplicity of nontrivial solutions to the Brezis-Nirenberg problem involving fractional Laplacian

$$\begin{gathered}(-\Delta)^{\alpha/2}u = |u|^{2^*_\alpha-2}u + \lambda u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{1.1}$$

where $\alpha \in (0,2)$, $N > (1+\sqrt{2})\alpha$, Ω is a bounded domain in $\mathbb{R}^N$, $(-\Delta)^{\alpha/2}$ stands for the fractional Laplacian operator, and $2^*_\alpha = \frac{2N}{N-\alpha}$ is the critical Sobolev exponent.

It is well known that (1.1) has been widely studied when $\alpha = 2$. In a pioneering work [4], Brezis and Nirenberg proved that (1.1) possesses a positive solution for $\lambda \in (0,\lambda_1)$, where λ_1 denotes the eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary data. Devillanova and Solimini [11] proved that there exist infinitely many solutions of equation (1.1) when $N \geq 7$ and $\lambda > 0$. For $N \geq 4$ and $\lambda > 0$, Clapp and Weth [10] showed that (1.1) possesses finitely many solutions with energy less than $\frac{2}{N}S^{N/2}$. Later, based on these results, the authors in [9] showed that there are at least $[\frac{N+1}{2}]$ pairs of nontrivial solutions for $N \geq 5$ and $\lambda \geq \lambda_1$.Here $[a]$ is the least integer n such that $n \geq a$.

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When $\alpha \in (0,2)$, the operator $(-\Delta)^{\alpha/2}$ defined in a bounded domain Ω has several definitions, and these definitions are not necessarily equivalent to each other. In this article, we consider the fractional Laplace operator defined as in [3, 5] by the spectral decomposition of the Laplacian,

$$(-\Delta)^{\alpha/2}u = \sum_{j=1}^{\infty} \lambda_j^{\alpha/2} a_j e_j, \quad \text{for } u = \sum_{j=1}^{\infty} a_j e_j \quad \text{with } \sum_{j=1}^{\infty} a_j^2 \lambda_j^{\alpha/2} < \infty. \tag{1.2}$$

Here (λ_j, e_j) denote the eigenvalues and eigenfunctions of $-\Delta$ in Ω with zero Dirichlet boundary data, and then $(\lambda_j^{\alpha/2}, e_j)$ are the eigenvalues and eigenfunctions of $(-\Delta)^{\alpha/2}$ in Ω with zero Dirichlet boundary data. With this definition, many results on the existence and nonexistence of nontrivial solutions of the fractional Brezis-Nirenberg problem (1.1) has been obtained by using the formulation of the fractional Laplacian through Dirichlet-Neumann maps introduced in [6]. When $\alpha = 1$, Cabre and Tan [5] proved that there is no solution when $\lambda = 0$ and Ω star-shaped domain. Later, Tan [15] obtained a positive solution if $\lambda \in (0, \lambda_1^{1/2})$.For general $\alpha \in (0,2)$, the authors [2] showed that problem (1.1) has no positive solution for $\lambda \geq \lambda_1^{\alpha/2}$, and has at least a positive solution for each $\lambda \in (0, \lambda_1^{\alpha/2})$. By using the general Nehari manifold method, Hua and Yu [12] obtained a nontrivial ground state solution for any $\lambda > 0$, provided $N > (1+\sqrt{2})\alpha$. For more related results, one may see [3, 12] and references therein. But to the best of our knowledge, there exist few results on the multiplicity of solutions for (1.1) with critical case.

Motivated by this, in this paper, we are devoted to the multiplicity of nontrivial solutions of (1.1) with any $\alpha \in (0,2), N > (1+\sqrt{2})\alpha$ and $\lambda > 0$.The first difficulty lies in that the fractional Laplacian operator $(-\Delta)^s$ is nonlocal. This nonlocal property makes some calculations difficult. To overcome this difficulty, we transform the nonlocal problem into a local problem by using the extension technique introduced by Caffarelli and Silvestre in [6]. More precisely, for any bounded domain Ω, define cylinder $\mathcal{C}_\Omega := \Omega \times (0,\infty) \subset \mathbb{R}_+^{N+1}$. If we denote the points in $\mathcal{C}_\Omega$ by (x,t), then for any $u \in H_0^{\alpha/2}(\Omega)$, the α-harmornic extension $U = E_\alpha(u)$ can be defined as the solution of

$$\begin{gathered} \operatorname{div}(t^{1-\alpha}\nabla U) = 0 \quad \text{in } \mathcal{C}_\Omega, \\ U = 0 \quad \text{on } \partial_L \mathcal{C}_\Omega, \\ U(x,0) = u(x) \quad \text{on } \Omega \times \{t=0\}, \end{gathered} \tag{1.3}$$

where $\partial_L \mathcal{C}_\Omega := \partial\Omega \times [0,+\infty)$. The relevance between U and the fractional Laplacian of the original functions u is through the formula

$$-\lim_{t\to 0^+} t^{1-\alpha} \frac{\partial U}{\partial y}(x,t) = \frac{1}{k_\alpha}(-\Delta)^{\alpha/2}u(x), \tag{1.4}$$

where k_α is a normalization constant and only depends on N and α. Therefore, after this extension, problem (1.1) can be transformed into an equivalent form

$$\begin{gathered} L_\alpha U = 0 \quad \text{in } \mathcal{C}_\Omega, \\ U = 0 \quad \text{on } \partial_L \mathcal{C}_\Omega \\ \frac{\partial U}{\partial \nu^\alpha} = |u|^{2_\alpha^*-2}u + \lambda u, \quad \text{in } \Omega \times \{t=0\}. \end{gathered} \tag{1.5}$$

Here

$$L_\alpha U := -\operatorname{div}(t^{1-\alpha}\nabla U), \quad \frac{\partial U}{\partial \nu^\alpha} := -k_\alpha \lim_{t\to 0^+} t^a \frac{\partial U}{\partial t}.$$

The second difficulty lies in that (1.1) is a critical problem. Hence, the corresponding energy functional does not satisfy the (PS) condition. To overcome this difficulty, one has to use $(PS)_c$ condition instead of (PS) condition. This idea has been widely used in the past decades, see [4]. For our paper, we shall use the global compactness results in fractional Sobolev space, see [16, Proposition 2.1].

The third difficulty lies in that the $\alpha-$*harmornic* extension function has no explicit expression. In order to find a critical value in some interval where the $(PS)_c$ condition holds, the usual way is to estimate some test functions. But for our problem, different from classical Laplace operator, our eigenfunctions in $\mathcal{C}_\Omega$ and test functions can not be written explicitly. To overcome this difficulty, we use the Poisson kernel, trace inequality and some asymptotic behavior of Bessel functions.

The fourth difficulty lies in how to find multiple solutions of problem (1.1). Following the ideas in [9], we can obtain the multiplicity of nontrivial solutions of (1.1) by using the Krasnoselskii genus. This article extends the multiplicity results in [9] from classical Laplace operator to the fractional case. Now we are ready to state our main result.

Theorem 1.1. *Let $\alpha \in (0,2), N > (1+\sqrt{2})\alpha$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Then, for any $\lambda > 0$, the problem* (1.1) *admits at least* $[(N+1)/2]$ *pairs of nontrivial solutions.*

This article is organized as follows. In section 2, we introduce a variational setting for problem (1.1), and present some preliminary results. In section 3, some useful estimates are obtained. In section 4, we are devoted to the proof of Theorem 1.1.

2. Preliminaries

According to the definition of (1.2), the operator $(-\Delta)^{\alpha/2}$ is well defined on the space

$$H_0^{\alpha/2}(\Omega) = \big\{u = \sum_{j=1}^{\infty} a_j e_j \in L^2(\Omega) : \|u\|_{H_0^{\alpha/2}(\Omega)} = (\sum_{j=1}^{\infty} a_j^2 \lambda_j^{\alpha/2})^{1/2} < +\infty\big\}.$$

For each $u \in H_0^{\alpha/2}(\Omega)$, the corresponding extension function $U := E_\alpha(u)$ as a solution to (1.3), belongs to the space

$$\begin{aligned} X_0^\alpha(\mathcal{C}_\Omega) = \Big\{ & U \in L^2(\mathcal{C}_\Omega) : U = 0 \text{ on } \partial_L \mathcal{C}_\Omega, \\ & \|U\|_{X_0^\alpha(\mathcal{C}_\Omega)} = \Big(k_\alpha \int_{\mathcal{C}_\Omega} t^{1-\alpha} |\nabla U|^2 \, dx\, dt\Big)^{1/2} < \infty \Big\} \end{aligned}$$

with inner product

$$(U,V)_{X_0^\alpha(\mathcal{C}_\Omega)} := k_\alpha \int_{\mathcal{C}_\Omega} t^{1-\alpha} \nabla U \cdot \nabla V \, dx\, dt.$$

Clearly, we have

$$\|E_\alpha(u)\|_{X_0^\alpha(\mathcal{C}_\Omega)} = \|u\|_{H_0^{\alpha/2}(\Omega)}, \quad \forall\, u \in H_0^{\alpha/2}(\Omega). \tag{2.1}$$

Note that (1.5) is equivalent to (1.1) by extension technique (see [6]). Thus in this paper, we shall focus our attention on looking for weak solutions of (1.5) in

$X_0^\alpha(\mathcal{C}_\Omega)$. First, consider the energy functional associated to (1.5)

$$I(U)=\frac{k_\alpha}{2}\int_{\mathcal{C}_\Omega} t^{1-\alpha}|\nabla U(x,t)|^2\,dx\,dt-\frac{\lambda}{2}\int_\Omega |U(x,0)|^2dx-\frac{1}{2_\alpha^*}\int_\Omega |U(x,0)|^{2_\alpha^*}dx.$$

It is well known that for any critical point U of I in $X_0^\alpha(\mathcal{C}_\Omega)$, the function $u:=U(\cdot,0)$ defined in the sense of traces, belongs to $H_0^{\alpha/2}(\Omega)$ and thus is a solution to problem (1.1). The inverse is also true.

Next, to use Krasnoselskii genus, we consider a new functional as in [9],

$$\begin{aligned} J(U) :=&\frac{k_\alpha \int_{\mathcal{C}_\Omega} t^{1-\alpha}|\nabla U(x,t)|^2\,dx\,dt-\lambda\int_\Omega |u|^2dx}{\left(\int_\Omega |u|^{2_\alpha^*}dx\right)^{2/2_\alpha^*}} \\ =&k_\alpha\int_{\mathcal{C}_\Omega} t^{1-\alpha}|\nabla U(x,t)|^2\,dx\,dt-\lambda\int_\Omega |u|^2dx \end{aligned} \tag{2.2}$$

defined on

$$M:=\{U\in X_0^\alpha(\mathcal{C}_\Omega):\|U(x,0)\|_{L^{2_\alpha^*}(\Omega)}=1\}.$$

It is easy to check $J\in C^1(M,\mathbb{R})$, and $U\in M$ is a critical point of J with $J(U)=\beta>0$, if and only if $\tilde U=\beta^{\frac{1}{2_\alpha^*-2}}U$ is a critical point of I with $I(\tilde U)=\frac{\alpha}{2N}\beta^{N/\alpha}>0$. Similarly, (U_n) is a $(PS)_\beta$ sequence for J if and only if the sequence $(\tilde U_n)$ is a $(PS)_{\tilde\beta}$ sequence for I with $\tilde\beta=\frac{\alpha}{2N}\beta^{\frac{N}{\alpha}}$, where $\tilde U_n:=\beta^{\frac{1}{2_\alpha^*-2}}U_n$. Here we say a sequence (U_n) in M is a $(PS)_\beta$ sequence for J if

$$J(U_n)\to\beta\quad\text{and}\quad \|J'(U_n)\|\to 0,\quad \text{as } m\to\infty.$$

Let

$$w_\epsilon(x)=\Big(\frac{\epsilon}{\epsilon^2+|x|^2}\Big)^{\frac{N-\alpha}{2}},\quad \forall\epsilon>0,\ x\in\mathbb{R}^N, \tag{2.3}$$

be the extremal function of Sobolev trace inequality

$$\int_{\mathbb{R}_+^{N+1}} t^{1-\alpha}|\nabla U(x,t)|^2\,dx\,dt\ge S_{\alpha,N}\Big(\int_{\mathbb{R}^N}|U(x,0)|^{2_\alpha^*}dx\Big)^{2/2_\alpha^*}.$$

According to [8] and [14], after a translation, w_ϵ is the unique positive solution of

$$(-\Delta)^{\alpha/2}u=|u|^{2_\alpha^*-2}u\quad \text{in } \dot H^{\alpha/2}(\mathbb{R}^N), \tag{2.4}$$

and hence

$$\frac{1}{k_\alpha}\|W_\epsilon\|^2_{X_0^\alpha(\mathbb{R}_+^{N+1})}=\|w_\epsilon\|^{2_\alpha^*}_{L^{2_\alpha^*}(\mathbb{R}^N)}=(k_\alpha S_{\alpha,N})^{N/\alpha}. \tag{2.5}$$

It is well known that when $\Omega=\mathbb{R}^N$, the α-harmonic extension has an explicit expression in term of the Poisson kernel (see [6])

$$U(x,t)=P_t^\alpha * u(x)=C_{N,\alpha}t^\alpha\int_{\mathbb{R}^N}\frac{u(y)}{(|x-y|^2+t^2)^{\frac{N+\alpha}{2}}}dy,\quad \forall u\in H_0^{\alpha/2}(\mathbb{R}^N). \tag{2.6}$$

where $C_{N,\alpha}$ is a constant. So the α-harmonic extension of w_ϵ can be written as

$$W_\epsilon(x,t)=P_t^\alpha * w_\epsilon(x)=C_{N,\alpha}t^\alpha\int_{\mathbb{R}^N}\frac{w_\epsilon(y)}{(|x-y|^2+t^2)^{\frac{N+\alpha}{2}}}dy. \tag{2.7}$$

One can see [2, Remark 2.2] and references therein for more details. Let

$$\mathfrak{M}:=\{W_\epsilon(\cdot-(y,0)):\epsilon>0,y\in\mathbb{R}^N\}.$$

Then, we have the following compactness lemma.

Lemma 2.1. *Let (U_n) be a $(PS)_{\beta_j}$ sequence for functional J. Up to a subsequence, the following conclusions hold.*
(a) If $\beta_j \in (0, k_\alpha S_{\alpha,N})$, then (U_n) converges in M and β_j is a critical value of J.
(b) If $\beta_j \in (k_\alpha S_{\alpha,N}, 2^{\alpha/N} k_\alpha S_{\alpha,N})$, then one of the following cases follows:
(b.1) *(U_n) converges in M and β_j is a critical value of J;*
(b.2) *There exists a critical point $u \in M$ of J with*

$$J(U) = \Big(\beta_j^{N/\alpha} - (k_\alpha S_{\alpha,N})^{N/\alpha}\Big)^{\alpha/N} \in (0, k_\alpha S_{\alpha,N}).$$

(c) If $\beta_j = k_\alpha S_{\alpha,N}$, then one of the following cases holds:
(c.1) *(U_n) converges in M and β_j is a critical value of J;*
(c.2) $\operatorname{dist}(\beta_j^{\frac{1}{2^*_\alpha - 2}} U_n, \mathfrak{M}) \to 0$ *or* $\operatorname{dist}(\beta_j^{\frac{1}{2^*_\alpha - 2}} U_n, -\mathfrak{M}) \to 0$.

Proof. By using the standard argument, this lemma follows directly from the global compactness result in fractional Sobolev space [16, Theorem 1.3]. □

In the following, we write $\lambda_0 = 0$ for $k = 0$. It is easy to see that for each $\lambda > 0$, there exists $k \ge 0$ such that $\lambda_k^{\alpha/2} \le \lambda < \lambda_{k+1}^{\alpha/2}$. Then, define

$$H^- := \operatorname{span}\{e_1, \dots, e_k\}, \quad H^+ := \overline{\operatorname{span}\{e_1, \dots, e_k\}^\perp}.$$

Clearly, $H^- = \emptyset$ for $0 < \lambda < \lambda_1^{\alpha/2}$. Let $\mathcal{E} := \{A \subset M : A \text{ is closed and symmetric}\}$. For any integer $j \ge k+1$, we define $\Sigma_j = \{A \in \mathcal{E} : \gamma(A) \ge j\}$, where γ denotes the usual Krasnoselskii genus, and consider

$$\beta_j := \inf_{A \in \Sigma_j} \sup_{U \in A} J(U).$$

Note that for each $A \in \Sigma_j$, $\gamma(A) > k$ and $A \cap \big\{U \in H^+ : \|U(\cdot, 0)\|_{L^{2^*_\alpha}(\Omega)} = 1\big\} \neq \emptyset$.Thus

$$\beta_j > 0 \quad \text{for any integer } j \ge k+1. \tag{2.8}$$

By using similar argument as in [9, Lemma 2.2], we can find a $(PS)_{\beta_j}$ sequence (U_n) for J. Moreover, we have the following lemma. Set $K^\beta := \{U \in \mathfrak{M} : J'(U) = 0 \text{ and } J(U) = \beta\}$.

Lemma 2.2. *If $0 < \beta_j = \beta_{j+1} < 2^{\alpha/N} k_\alpha S_{\alpha,N}$, then K^{β_j} is infinite.*

Proof. By using standard arguments as in the proof of [9, Lemma 2.4], we can obtain the result. So we omit the proof. □

3. Some estimates

In this section, we set $\mathbb{B}_r^+(z_0) := \{z \in \overline{\mathbb{R}^{N+1}_+} : |z - z_0| < r\}$ and $B_r(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < r\}$, and denote $r_{xt} := |(x,t)| = (|x|^2 + |t|^2)^{1/2}$.For simplicity, we shall write $\mathbb{B}_r^+$ and B_r instead of $\mathbb{B}_r^+(0)$ and $B_r(0)$, respectively.

Recall that $(\lambda_i^{\alpha/2}, e_i)$ are the eigenvalues and eigenfunctions of $(-\Delta)^{\alpha/2}$ with zero Dirichlet boundary data. Let E_i denote the α-harmonic extension of e_i, i.e., E_i is the solution of

$$\begin{gathered} -\operatorname{div}(t^{1-\alpha} E_i) = 0 \quad \text{in } \mathcal{C}_\Omega, \\ E_i(x,t) = 0 \quad \text{on } \partial_L \mathcal{C}_\Omega, \\ -k_\alpha \lim_{t \to 0^+} t^{1-\alpha} \frac{\partial E_i}{\partial t}(x,t) = \lambda_i^{\alpha/2} e_i(x) \quad \text{on } \Omega \times \{0\}. \end{gathered} \tag{3.1}$$

Then, we have the following Lemma.

Lemma 3.1. *There exists $C > 0$ such that*

$$\sup_{(x,t)\in\mathcal{C}_\Omega} E_i(x,t) \le C \quad \text{for all } i = 1,\dots,k.$$

Proof. For each $i = 1,\dots,k$, it follows from [3, Lemma 3.3] that

$$E_i = E_\alpha(e_i) = e_i(x)\psi(\lambda_i^{1/2}t),$$

where ψ is continuous and satisfies the following asymptotic behavior

$$\psi(s) \sim 1 - c_1 s^\alpha \quad \text{as } s \to 0,$$
$$\psi(s) \sim c_2 s^{\frac{\alpha-1}{2}} e^{-s} \text{ as } s \to \infty,$$

where $c_1 = \frac{2^{1-\alpha}\Gamma(1-\alpha/2)}{\alpha\Gamma(\alpha/2)}$, $c_2 = \frac{2^{\frac{1-\alpha}{2}}\pi^{1/2}}{\Gamma(\alpha/2)}$, (see[3] and [13] for more details). Clearly, ψ is bounded. Since $e_i \in C^\infty(\Omega)$, we conclude that there exists $C > 0$ such that

$$\sup_{\mathcal{C}_\Omega} e_i(x)\psi(\lambda_i^{1/2}t) \le C \quad \text{uniformly for } i = 1,\dots,k.$$

We completed the proof. □

Without loss of generality, we assume $0 \in \Omega$. Then, we have $\mathbb{B}^+_{2/m} \subset \mathcal{C}_\Omega$ for m large enough. Let

$$E_i^m(x,t) := \zeta_{2/m}(x,t)E_i(x,t), \tag{3.2}$$

where $\zeta_\eta(x,t) := \bar\zeta(\frac{r_{xt}}{\eta})$ for any $\eta > 0$, and $\bar\zeta$ is defined by

$$\bar\zeta(s) = \begin{cases} 0 & \text{if } s \in [0,\frac12), \\ 2s-1 & \text{if } s \in [\frac12, 1), \\ 1 & \text{if } s \in [1,+\infty). \end{cases} \tag{3.3}$$

Clearly,

$$|\nabla\zeta_{2/m}(x,t)| \le m, \quad E_i^m(x,0) = \zeta_{2/m}(x,0)e_i, \quad \operatorname{supp} E_i^m \subset \mathcal{C}_\Omega \backslash \overline{\mathbb{B}^+_{\frac1m}}. \tag{3.4}$$

In the following, we denote $\zeta_0 = 1$ for $\eta = 0$ and $A_m := \{(x,t) \in \overline{\mathcal{C}_\Omega} : r_{xt} \in (\frac1m, \frac2m)\}$.

Lemma 3.2. $\|E_i^m - E_i\|_{X_0^\alpha(\mathcal{C}_\Omega)} \to 0$ *as* $m \to +\infty$.

Proof. Note that

$$\int_\Omega e_i^2 dx = 1 \quad \text{and} \quad \int_{\mathcal{C}_\Omega} t^{1-\alpha}|\nabla E_i|^2\,dx\,dt = \lambda_i^{\alpha/2}. \tag{3.5}$$

This, combined with Lemma 3.1 and (3.4), implies that

$$\begin{aligned} \int_{\mathcal{C}_\Omega} t^{1-\alpha}|\nabla\zeta_{2/m}|^2|E_i|^2\,dx\,dt &= \int_{A_m} t^{1-\alpha}|\nabla\zeta_{2/m}|^2|E_i|^2\,dx\,dt \\ &\le Cm^2\int_{A_m} t^{1-\alpha}\,dx\,dt \\ &\le Cm^{\alpha-N} \to 0 \text{ as } m \to \infty, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\Big|\int_{\mathcal{C}_\Omega} t^{1-\alpha}(\zeta_{2/m}-1)E_i\nabla\zeta_{2/m}\cdot\nabla E_i\,dx\,dt\Big|\\
&\le \int_{A_m} t^{1-\alpha}|(\zeta_{2/m}-1)||E_i||\nabla\zeta_{2/m}\cdot\nabla E_i|\,dx\,dt\\
&\le Cm\int_{A_m} t^{1-\alpha}|\nabla E_i|\,dx\,dt\\
&\le Cm\Big(\int_{A_m} t^{1-\alpha}\,dx\,dt\Big)^{1/2}\Big(\int_{A_m} t^{1-\alpha}|\nabla E_i|^2\Big)^{1/2}\,dx\,dt\\
&\le Cm^{\frac{\alpha-N}{2}}\to 0 \text{ as } m\to\infty.
\end{aligned}\tag{3.7}$$

In addition, according to the absolutely continuity of the integral, we obtain

$$\begin{aligned}
\int_{\mathcal{C}_\Omega} t^{1-\alpha}(\zeta_{2/m}-1)^2|\nabla E_i|^2\,dx\,dt &= \int_{\mathbb{B}^+_{2/m}} t^{1-\alpha}(\zeta_{2/m}-1)^2|\nabla E_i|^2\,dx\,dt\\
&\le \int_{\mathbb{B}^+_{2/m}} t^{1-\alpha}|\nabla E_i|^2\,dx\,dt\to 0\quad\text{as } m\to\infty.
\end{aligned}\tag{3.8}$$

Therefore, from (3.6), (3.7) and (3.8) it follows that

$$\begin{aligned}
&\|E_i^m-E_i\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}\\
&=\int_{\mathcal{C}_\Omega} t^{1-\alpha}|\nabla(\zeta_{2/m}E_i-E_i)|^2\,dx\,dt\\
&=\int_{\mathcal{C}_\Omega} t^{1-\alpha}[|\nabla\zeta_{2/m}|^2|E_i|^2+2(\zeta_{2/m}-1)E_i\nabla\zeta_{2/m}\cdot\nabla E_i+(\zeta_{2/m}-1)^2|\nabla E_i|^2]\,dx\,dt\\
&\to 0\quad\text{as } m\to\infty.
\end{aligned}$$

The proof is complete. □

Define

$$H_m^- := \operatorname{span}\{E_1^m,\dots,E_k^m\}\quad\text{for } k\ge 1.$$

Lemma 3.3. *Let $k\ge 1$. Then there exists $m_0>1$ such that for any $m\ge m_0$, it holds*

$$\max_{\{U\in H_m^-,\|u\|_{L^2(\Omega)}=1\}}\|U\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}\le\lambda_k^{\alpha/2}+C_1m^{\alpha-N},\tag{3.9}$$

where C_1 is a positive constant independent of m.

Proof. First, we denote $e_i^m(x):=E_i^m(x,0)$, then according to (3.2), $e_i^m(x)=\zeta_{2/m}(x,0)e_i(x)$. In what follows, we shall prove the following estimates:

$$\|E_i^m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}\le\lambda_i^{\alpha/2}+Cm^{\alpha-N},\quad i=1,2,\dots,\tag{3.10}$$

$$|(E_i^m,E_j^m)_{X_0^\alpha(\mathcal{C}_\Omega)}|\le Cm^{\alpha-N},\quad i,j=1,2,\dots,i\ne j,\tag{3.11}$$

$$|(e_i^m,e_j^m)_{L^2(\Omega)}|\le Cm^{-N},\quad i,j=1,2,\dots,i\ne j,\tag{3.12}$$

$$\|e_i^m\|^2_{L^2(\Omega)}\ge 1-Cm^{-N},\quad i=1,2,\dots.\tag{3.13}$$

Indeed, by (3.2), we have

$$\begin{aligned}
&\|E_i^m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \|E_i\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}\\
&= \int_{\mathcal{C}_\Omega} t^{1-\alpha}(|\nabla E_i^m|^2 - |\nabla E_i|^2)\,dx\,dt\\
&= \int_{\mathcal{C}_\Omega} t^{1-\alpha}[(\zeta_{2/m}^2 - 1)|\nabla E_i|^2 + 2\zeta_{2/m}E_i\nabla\zeta_{2/m}\cdot\nabla E_i + |E_i|^2|\nabla\zeta_{2/m}|^2]\,dx\,dt.
\end{aligned} \tag{3.14}$$

On the other hand, multiplying (3.1) by $(\zeta_{2/m}^2 - 1)E_i$ and integrating by parts over $\mathcal{C}_\Omega$, we obtain

$$\int_{\mathcal{C}_\Omega} t^{1-\alpha}[(\zeta_{2/m}^2 - 1)|\nabla E_i|^2 + 2\zeta_{2/m}E_i\nabla\zeta_{2/m}\cdot\nabla E_i]\,dx\,dt = \lambda\int_\Omega(\zeta_{2/m}^2(x,0) - 1)e_i^2 dx. \tag{3.15}$$

Inserting (3.15) into (3.14), we conclude from (3.5) and Lemma 3.1 that

$$\begin{aligned}
\|E_i^m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} &\le \|E_i\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + \int_{\mathcal{C}_\Omega} t^{1-\alpha}|E_i|^2|\nabla\zeta_{2/m}|^2\,dx\,dt + \lambda\int_\Omega(\zeta_{2/m}^2(x,0) - 1)e_i^2 dx\\
&\le \|E_i\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + \int_{\mathcal{C}_\Omega} t^{1-\alpha}|E_i|^2|\nabla\zeta_{2/m}|^2\,dx\,dt\\
&\le \lambda_i^{\alpha/2} + Cm^2\int_{A_m} t^{1-\alpha}\,dx\,dt\\
&\le \lambda_i^{\alpha/2} + Cm^{\alpha-N}.
\end{aligned}$$

Hence (3.10) holds.

Observe that from (3.1),

$$\int_{\mathcal{C}_\Omega} t^{1-\alpha}\nabla E_i\cdot\nabla E_j\,dx\,dt = 0, \quad i \ne j.$$

Then we have

$$\begin{aligned}
(E_i^m, E_j^m)_{X_0^\alpha(\mathcal{C}_\Omega)} &= \int_{\mathcal{C}_\Omega} t^{1-\alpha}\nabla E_i^m\cdot\nabla E_j^m\,dx\,dt\\
&= \int_{\mathcal{C}_\Omega} t^{1-\alpha}[\zeta_{2/m}^2\nabla E_i\cdot\nabla E_j + \zeta_{2/m}E_j\nabla E_i\cdot\nabla\zeta_{2/m}\\
&\quad + \zeta_{2/m}E_i\nabla\zeta_{2/m}\cdot\nabla E_j + E_iE_j|\nabla\zeta_{2/m}|^2]\,dx\,dt\\
&= \int_{\mathcal{C}_\Omega} t^{1-\alpha}[(\zeta_{2/m}^2 - 1)\nabla E_i\cdot\nabla E_j + \zeta_{2/m}E_j\nabla\zeta_{2/m}\cdot\nabla E_i\\
&\quad + \zeta_{2/m}E_i\nabla\zeta_{2/m}\cdot\nabla E_j + |\nabla\zeta_{2/m}|^2E_iE_j]\,dx\,dt.
\end{aligned} \tag{3.16}$$

Multiplying both sides of (3.1) by $(\zeta_{2/m}^2 - 1)E_j$ and integrating by parts, we obtain

$$\int_{\mathcal{C}_\Omega} t^{1-\alpha}[(\zeta_{2/m}^2 - 1)\nabla E_i\cdot\nabla E_j + 2\zeta_{2/m}E_j\nabla\zeta_{2/m}\cdot\nabla E_i]\,dx\,dt = \lambda_i^{\alpha/2}\int_\Omega(\zeta_{2/m}^2(x,0) - 1)e_ie_j dx.$$

Similarly, we obtain

$$\int_{\mathcal{C}_\Omega} t^{1-\alpha}[(\zeta_{2/m}^2 - 1)\nabla E_i\cdot\nabla E_j + 2\zeta_{2/m}E_i\nabla\zeta_{2/m}\cdot\nabla E_j]\,dx\,dt = \lambda_j^{\alpha/2}\int_\Omega(\zeta_{2/m}^2(x,0) - 1)e_ie_j dx.$$

This, combined with (3.16), implies

$$
\begin{aligned}
&(E_i^m, E_j^m)_{X_0^\alpha(\mathcal{C}_\Omega)} \\
&= \int_{\mathcal{C}_\Omega} t^{1-\alpha}|\nabla\zeta_{2/m}|^2 E_iE_j\,dx\,dy + \frac{\lambda_i^{\alpha/2}+\lambda_j^{\alpha/2}}{2}\int_\Omega(\zeta_{2/m}^2(x,0)-1)e_ie_j\,dx.
\end{aligned} \tag{3.17}
$$

By (3.17) and Lemma 3.1, we have

$$
\begin{aligned}
&|(E_i^m, E_j^m)_{X_0^\alpha(\mathcal{C}_\Omega)}| \\
&\le Cm^2\int_{\mathbb{B}^+_{2/m}} t^{1-\alpha}|E_iE_j|\,dx\,dt + \frac{\lambda_i^{\alpha/2}+\lambda_j^{\alpha/2}}{2}\int_{B_{2/m}}|(\zeta_{2/m}^2(x,0)-1)||e_ie_j|dx \\
&\le Cm^2\int_{\mathbb{B}^+_{2/m}} t^{1-\alpha}\,dx\,dt + C\int_{B_{2/m}}dx \\
&\le Cm^2m^{\alpha-N-2}+Cm^{-N} \\
&\le Cm^{\alpha-N},
\end{aligned}
$$

which yields (3.11).

Note that $\int_\Omega e_ie_jdx=0$ when $i\neq j$, then from Lemma 3.1 it follows that

$$
\begin{aligned}
|(e_i^m, e_j^m)_{L^2(\Omega)}| &= |\int_\Omega \zeta_m^2(x,0)e_ie_jdx| \\
&= |\int_\Omega(\zeta_m^2(x,0)-1)e_ie_jdx| \\
&\le |\int_{B_{2/m}} e_ie_jdx| \le Cm^{-N}.
\end{aligned}
$$

So (3.12) holds.

In view of (3.5) and Lemma 3.1, we obtain

$$
\begin{aligned}
\|e_i^m\|^2_{L^2(\Omega)} &= \int_\Omega e_i^2dx - \int_\Omega(1-\zeta_{2/m}^2(x,0))e_i^2dx \\
&\ge 1-\int_{B_{2/m}} e_i^2dx \\
&\ge 1-Cm^{-N},
\end{aligned}
$$

which implies (3.13).

Now, by using the above estimates (3.10)–(3.13), we are ready to prove (3.9). Let $U_m\in H_m^-$ with the trace $\|u_m\|_{L^2(\Omega)}=1$ such that

$$
\max_{\{U\in H_m^-,\|u\|_{L^2(\Omega)}=1\}}\|U\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|U_m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}. \tag{3.18}
$$

Then, there exist numbers $a_1^m,\dots,a_k^m$ such that $U_m=\sum_{i=1}^k a_i^mE_i^m$. Thus, we have

$$
\|U_m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \sum_{i=1}^k (a_i^m)^2\|E_i^m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + 2\sum_{1\le i<j\le k} a_i^ma_j^m(E_i^m,E_j^m)_{X_0^\alpha(\mathcal{C}_\Omega)},
$$

$$
1=\|u_m\|^2_{L^2(\Omega)} = \sum_{i=1}^k (a_i^m)^2\|e_i^m\|^2_{L^2(\Omega)} + 2\sum_{1\le i<j\le k} a_i^ma_j^m(e_i^m,e_j^m)_{L^2(\Omega)}.
$$

According to (3.12) and (3.13), there exists $m_0 > 1$ such that for $m \geq m_0$,

$$|(e_i^m, e_j^m)_{L^2(\Omega)}| \leq \frac{1}{4} \quad \text{when } i \neq j, \quad \text{and} \quad \|e_i^m\|^2_{L^2(\Omega)} \geq \frac{3}{4}.$$

Then, it holds

$$\begin{aligned}
1 &= \sum_{i=1}^k (a_i^m)^2 \|e_i^m\|^2_{L^2(\Omega)} + 2 \sum_{1\leq i<j\leq k} a_i^m a_j^m (e_i^m, e_j^m)_{L^2(\Omega)} \\
&\geq \sum_{i=1}^k (a_i^m)^2 \|e_i^m\|^2_{L^2(\Omega)} - 2 \sum_{1\leq i<j\leq k} |a_i^m||a_j^m||(e_i^m, e_j^m)_{L^2(\Omega)}| \\
&\geq \frac{3}{4} \sum_{i=1}^k (a_i^m)^2 - \frac{1}{4} \sum_{1\leq i<j\leq k} (|a_i^m|^2 + |a_j^m|^2) \\
&\geq \frac{1}{4} \sum_{i=1}^k (a_i^m)^2,
\end{aligned}$$

which implies

$$|a_i^m| \text{ are uniformly bounded for } m \geq m_0. \tag{3.19}$$

By (3.12), (3.13) and (3.19), we conclude

$$\begin{aligned}
1 &\geq \sum_{i=1}^k (a_i^m)^2 \|e_i^m\|^2_{L^2(\Omega)} - 2 \sum_{1\leq i<j\leq k} |a_i^m||a_j^m||(e_i^m, e_j^m)_{L^2(\Omega)}| \\
&\geq \sum_{i=1}^k (a_i^m)^2 \|e_i^m\|^2_{L^2(\Omega)} - C \sum_{1\leq i<j\leq k} |(e_i^m, e_j^m)_{L^2(\Omega)}| \\
&\geq \sum_{i=1}^k (a_i^m)^2 \|e_i^m\|^2_{L^2(\Omega)} - Cm^{-1-N} \\
&\geq \sum_{i=1}^k (a_i^m)^2 - C \sum_{i=1}^k (a_i^m)^2 m^{-1-N} - Cm^{-1-N} \\
&\geq \sum_{i=1}^k (a_i^m)^2 - Cm^{-1-N}.
\end{aligned}$$

This, combined with (3.10),(3.11) and (3.19), implies that

$$\begin{aligned}
\|U_m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} &= \sum_{i=1}^k (a_i^m)^2 \|E_i^m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + 2 \sum_{1\leq i<j\leq k} a_i^m a_j^m (E_i^m, E_j^m)_{X_0^\alpha(\mathcal{C}_\Omega)} \\
&\leq \lambda_k^{\alpha/2} \sum_{i=1}^k (a_i^m)^2 + Cm^{\alpha-N} + Cm^{\alpha-N} \qquad (3.20) \\
&\leq \lambda_k^{\alpha/2} + Cm^{\alpha-N} + Cm^{\alpha-N} \\
&\leq \lambda_k^{\alpha/2} + C_1 m^{\alpha-N}
\end{aligned}$$

for some $C_1 > 0$. Therefore, (3.18) and (3.20) yield the proof. □

Based on the estimate in Lemma 3.3, we have the following lemma.

Lemma 3.4. *Suppose $k \geq 1$ and $\lambda \geq \lambda_k^{\alpha/2}$. Then for any $m \geq m_0$, it holds*

$$\sup_{U \in H_m^-} I(U) \leq C_2 m^{\frac{N(\alpha-N)}{\alpha}},$$

where C_2 is a positive number independent of m.

Proof. In view of Lemmas 3.2 and 3.3, there exists some constant $C_2 > 0$ such that for any $m \geq m_0$ and $U \in H_m^-$,

$$\begin{aligned} I(U) &\leq \frac{\lambda_k^{\alpha/2} - \lambda}{2} \int_\Omega |u|^2 dx + \frac{C_1 m^{\alpha-N}}{2} \int_\Omega |u|^2 dx - \frac{1}{2_\alpha^*} \int_\Omega |u|^{2_\alpha^*} dx \\ &\leq \frac{C_1 m^{\alpha-N}}{2} \int_\Omega |u|^2 dx - \frac{1}{2_\alpha^*} \int_\Omega |u|^{2_\alpha^*} dx \\ &\leq C m^{\alpha-N} \|u\|^2_{L^{2_\alpha^*}(\Omega)} - \frac{1}{2_\alpha^*} \|u\|^{2_\alpha^*}_{L^{2_\alpha^*}(\Omega)} \\ &\leq \max_{t \geq 0} (C m^{\alpha-N} t^2 - \frac{1}{2_\alpha^*} t^{2_\alpha^*}) \\ &\leq C_2 m^{\frac{N(\alpha-N)}{\alpha}}. \end{aligned}$$

Thus the proof is complete. □

In what follows, we shall introduce a lemma that describes the property of W_1 defined in (2.7). This lemma plays a key role in our estimates in this section. Here, we write $W_1^{(\alpha)}$ instead of W_1 to emphasize the dependence on the parameter α.

Lemma 3.5 ([2, Lemma 3.7])**.** *It holds*

$$|\nabla W_1^{(\alpha)}(x,t)| \leq \frac{C}{t} W_1^{(\alpha)}(x,t), \quad 0 < \alpha < 2, (x,t) \in \mathbb{R}_+^{N+1}, \tag{3.21}$$

$$|\nabla W_1^{(\alpha)}(x,t)| \leq C W_1^{(\alpha-1)}(x,t), \quad 1 < \alpha < 2, (x,t) \in \mathbb{R}_+^{N+1}. \tag{3.22}$$

Now, we define a cut-off function $\bar{\phi}(s) \in C^\infty(\mathbb{R}^+)$ with $0 \leq \bar{\phi}(s) \leq 1$, which is non-increasing and satisfies

$$\bar{\phi}(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 0 & \text{if } s \geq 1, \end{cases}$$

and $|\nabla \bar{\phi}|$ is bounded. For any $r > 0$, set

$$\phi_r(x,t) = \bar{\phi}(\frac{r_{xt}}{r})$$

then $|\nabla \phi_r| \leq C/r$ for some positive constant C independent of r. Let $0 < \epsilon < r < \frac{2}{m}$.According to (2.3) and (2.7), define

$$W_\epsilon^r(x,t) := \phi_r(x,t) W_\epsilon(x,t) \quad \text{and} \quad w_\epsilon^r(x) := W_\epsilon^r(x,0).$$

Obviously, $W_\epsilon^r \in X_0^\alpha(\mathcal{C}_\Omega)$ and $w_\epsilon^r(x) = \phi_r(x,0) w_\epsilon(x)$. Recalling that $\zeta_\eta(x,t) = \bar{\zeta}(\frac{r_{xt}}{\eta})$ defined in (3.3) and $\zeta_0 = 1$ for $\eta = 0$, we have the following lemma.

Lemma 3.6. *Let $0 \leq 2\eta < \epsilon < r$ and $\tilde{x} \in \Omega$. Then the following estimates hold: (a)*

$$\|\zeta_\eta(x - \tilde{x}, t) W_\epsilon^r\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}$$

$$\le \begin{cases} (k_\alpha S_{\alpha,N})^{N/\alpha} + C(\frac{\epsilon}{r})^{N-\alpha} + C(\frac{\eta}{\epsilon})^{N-\alpha}, & \text{if } \alpha \in (0,1), \\ (k_\alpha S_{\alpha,N})^{N/\alpha} + C(\frac{\epsilon}{r})^{N-1}|\log\frac{\epsilon}{r}| + C(\frac{\eta}{\epsilon})^{N-\alpha}, & \text{if } \alpha = 1, \\ (k_\alpha S_{\alpha,N})^{N/\alpha} + C(\frac{\epsilon}{r})^{N-\alpha} + C(\frac{\eta}{\epsilon})^{N-\alpha}, & \text{if } \alpha \in (1,2). \end{cases}$$

(b)

$$\int_\Omega |\zeta_\eta(x-\tilde{x},0)w_\epsilon^r(x)|^{2^*_\alpha}\,dx \ge (k_\alpha S_{\alpha,N})^{N/\alpha} - C(\frac{\epsilon}{r})^N - C(\frac{\eta}{\epsilon})^N.$$

Proof. Let $r<1$ and $C_r := \{(x,t)\in\mathbb{R}^{N+1}_+ : r/2 \le |(x-\tilde{x},t)| \le r\}$. According to (2.3), $w_1^{(\alpha)}(x) \le |x|^{\alpha-N}$. Then by(2.7), for any $(x,t)\in C_{r/\epsilon}$, we have

$$\begin{aligned} W_1^{(\alpha)}(x,t) &= \int_{|y|<\frac{r}{4\epsilon}} P_t^\alpha(x-y)w_1(y)dy + \int_{|y|>\frac{r}{4\epsilon}} P_t^\alpha(x-y)w_1(y)dy \\ &\le Ct^\alpha \int_{|y|<\frac{r}{4\epsilon}} \frac{w_1(y)}{(|x|^2+|t|^2-|y|^2)^{\frac{N+\alpha}{2}}}dy + C(\frac{\epsilon}{r})^{N-\alpha}\int_{\mathbb{R}^N} P_t^\alpha(y)dy \\ &\le Ct^\alpha \int_{|y|<\frac{r}{4\epsilon}} \frac{w_1(y)}{((\frac{r}{2\epsilon})^2-(\frac{r}{4\epsilon})^2)^{\frac{N+\alpha}{2}}}dy + C(\frac{\epsilon}{r})^{N-\alpha}\int_{\mathbb{R}^N} P_t^\alpha(y)dy \\ &\le C(\frac{\epsilon}{r})^{N+\alpha}t^\alpha \int_{|y|<\frac{r}{4\epsilon}} w_1(y)dy + C(\frac{\epsilon}{r})^{N-\alpha}\int_{\mathbb{R}^N} P_t^\alpha(y)dy \\ &\le C(\frac{\epsilon}{r})^{N+\alpha}t^\alpha \int_{|y|<\frac{r}{4\epsilon}} \frac{1}{|y|^{N-\alpha}}dy + C(\frac{\epsilon}{r})^{N-\alpha} \\ &\le C(\frac{\epsilon}{r})^{N}t^\alpha + C(\frac{\epsilon}{r})^{N-\alpha} \\ &\le C(\frac{\epsilon}{r})^{N-\alpha}. \end{aligned} \tag{3.23}$$

Moreover, by Lemma 3.5 and (3.23), we obtain

$$\begin{aligned} &\int_{C_{\frac{r}{\epsilon}}} t^{1-\alpha}|W_1^{(\alpha)}\nabla W_1^{(\alpha)}|\,dx\,dt \\ &\le \begin{cases} C(\frac{\epsilon}{r})^{2N-2\alpha}\int_{C_{r/\epsilon}} t^{-\alpha}\,dx\,dt \le C(\frac{\epsilon}{r})^{N-\alpha-1}, & \text{if } \alpha\in(0,1), \\ C(\frac{\epsilon}{r})^{2N-2}\int_{C_{r/\epsilon}} t^{-1}\,dx\,dt \le C(\frac{\epsilon}{r})^{N-2}|\log\frac{\epsilon}{r}|, & \text{if } \alpha=1, \\ C(\frac{\epsilon}{r})^{2N-\alpha+1}\int_{C_{r/\epsilon}} t^{1-\alpha}\,dx\,dt \le C(\frac{\epsilon}{r})^{N-\alpha-1}, & \text{if } \alpha\in(1,2). \end{cases} \end{aligned} \tag{3.24}$$

Note that $W_\epsilon(x,t) = \epsilon^{\frac{\alpha-N}{2}}W_1(\frac{x}{\epsilon},\frac{t}{\epsilon})$. Then for the case $\eta=0$, we have

$$\begin{aligned} &\int_{\mathcal{C}_\Omega} t^{1-\alpha}W_\epsilon\phi_r\nabla\phi_r\cdot\nabla W_\epsilon\,dx\,dt \\ &\le Cr^{-1}\int_{C_r} t^{1-\alpha}|W_\epsilon||\nabla W_\epsilon|\,dx\,dt \\ &= Cr^{-1}\epsilon\int_{C_{r/\epsilon}} t^{1-\alpha}|W_1(x,t)||\nabla W_1(x,t)|\,dx\,dt \\ &\le \begin{cases} C(\frac{\epsilon}{r})^{N-\alpha}, & \text{if } \alpha\in(0,1), \\ C(\frac{\epsilon}{r})^{N-1}|\log\frac{\epsilon}{r}|, & \text{if } \alpha=1, \\ C(\frac{\epsilon}{r})^{N-\alpha}, & \text{if } \alpha\in(1,2). \end{cases} \end{aligned} \tag{3.25}$$

Since $0 \le w_\epsilon(x) \le \epsilon^{\frac{N-\alpha}{2}}|x|^{\alpha-N}$ and the $\alpha-$extension of $|x|^{\alpha-N}$ is $r_{xt}^{\alpha-N}$, we conclude that $W_\epsilon(x,t) \le \epsilon^{\frac{N-\alpha}{2}} r_{xt}^{\alpha-N}$ and

$$\begin{aligned}\int_{\mathcal{C}_\Omega} t^{1-\alpha}|W_\epsilon\nabla\phi_r|^2\,dx\,dt &\le Cr^{-2}\int_{C_r} t^{1-\alpha}|W_\epsilon|^2\,dx\,dt\\ &\le C\frac{\epsilon^{N-\alpha}}{r^2}\int_{C_r} t^{1-\alpha} r_{xt}^{2(\alpha-N)}\,dx\,dt\\ &\le C\frac{\epsilon^{N-\alpha}}{r^{2N+2-2\alpha}}\int_{C_r} t^{1-\alpha}\,dx\,dt\\ &\le C\frac{\epsilon^{N-\alpha}}{r^{N-\alpha}}.\end{aligned} \tag{3.26}$$

From (3.25) and (3.26) it follows that

$$\begin{aligned}&\|W_\epsilon^r\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}\\ &= k_\alpha\int_{\mathcal{C}_\Omega} t^{1-\alpha}(|\phi_r\nabla W_\epsilon|^2 + |W_\epsilon\nabla\phi_r|^2 + 2W_\epsilon\phi_r\nabla\phi_r\cdot\nabla W_\epsilon)\,dx\,dt\\ &\le \|W_\epsilon\|^2_{\mathbb{R}^{N+1}_+} + k_\alpha\int_{\mathcal{C}_\Omega} t^{1-\alpha}|W_\epsilon\nabla\phi_r|^2 + 2k_\alpha\int_{\mathcal{C}_\Omega} t^{1-\alpha}W_\epsilon\phi_r\nabla\phi_r\cdot\nabla W_\epsilon\,dx\,dt\\ &\le \begin{cases}(k_\alpha S_{\alpha,N})^{N/\alpha} + C(\frac{\epsilon}{r})^{N-\alpha}, & \text{if } \alpha\in(0,1),\\ (k_\alpha S_{\alpha,N})^{N/\alpha} + C(\frac{\epsilon}{r})^{N-1} + C(\frac{\epsilon}{r})^{N-1}|\log\frac{\epsilon}{r}|, & \text{if } \alpha=1,\\ (k_\alpha S_{\alpha,N})^{N/\alpha} + C(\frac{\epsilon}{r})^{N-\alpha}, & \text{if } \alpha\in(1,2).\end{cases}\end{aligned} \tag{3.27}$$

In addition, since

$$\int_{\mathbb{R}^N\setminus B_r}|W_\epsilon(x)|^{2^*_\alpha}dx = C\int_r^\infty\big(\frac{\epsilon}{\epsilon^2+\rho^2}\big)^N\rho^{N-1}d\rho \le C\epsilon^N r^{-N},$$

from (2.5) we conclude that

$$\begin{aligned}\int_\Omega |w_\epsilon^r|^{2^*_\alpha}dx &\ge \int_{B(r/2)}|w_\epsilon|^{2^*_\alpha}dx\\ &= (k_\alpha S_{\alpha,N})^{N/\alpha} - \int_{\mathbb{R}^N\setminus B(r/2)}|w_\epsilon|^{2^*_\alpha}dx\\ &\ge (k_\alpha S_{\alpha,N})^{N/\alpha} - C(\frac{\epsilon}{r})^N.\end{aligned} \tag{3.28}$$

Now, we turn to the case $\eta > 0$. Since $w_\epsilon \le C\epsilon^{(\alpha-N)/2}$ and $|\nabla w_\epsilon| \le C\epsilon^{(\alpha-N-2)/2}$, from (2.7) we obtain

$$\begin{aligned}W_\epsilon(x,t) &\le C\epsilon^{(\alpha-N)/2}\int_{\mathbb{R}^N} t^\alpha\frac{1}{(|x-s|^2+t^2)^{\frac{N+\alpha}{2}}}ds\\ &= C\epsilon^{(\alpha-N)/2}\int_{\mathbb{R}^N}\frac{1}{(|s|^2+1)^{\frac{N+\alpha}{2}}}ds\\ &\le C\epsilon^{(\alpha-N)/2}\end{aligned} \tag{3.29}$$

and

$$|\nabla W_\epsilon(x,t)| = \int_{\mathbb{R}^N} P_t^\alpha(y)|\nabla w_\epsilon(x-y)|dy$$

$$\begin{aligned}&\le C\epsilon^{(\alpha-N-2)/2}\int_{\mathbb{R}^N}P_t^\alpha(y)dy\\&\le C\epsilon^{(\alpha-N-2)/2}.\end{aligned}$$

Furthermore,

$$\begin{aligned}|\nabla W_\epsilon^r(x,t)|&\le |W_\epsilon\nabla\phi_r|+\phi_r|\nabla W_\epsilon|\\&\le Cr^{-1}W_\epsilon+|\nabla W_\epsilon|\\&\le Cr^{-1}\epsilon^{(\alpha-N)/2}+\epsilon^{(\alpha-N)/2}\epsilon^{-1}\\&\le C\epsilon^{(\alpha-N-2)/2}.\end{aligned}\tag{3.30}$$

Since $W_\epsilon^r\le W_\epsilon$, by (3.29) and (3.30), we have

$$\begin{aligned}\int_{\mathcal{C}_\Omega}t^{1-\alpha}|W_\epsilon^r(x,t)\nabla\zeta_\eta(x-\tilde{x},t)|^2\,dx\,dt&\le C\eta^{-2}\int_{C_\eta}t^{1-\alpha}|W_\epsilon^r|^2\,dx\,dt\\&\le C\eta^{-2}\epsilon^{\alpha-N}\int_{C_\eta}t^{1-\alpha}\,dx\,dt\\&\le C\frac{\eta^{N-\alpha}}{\epsilon^{N-\alpha}}\end{aligned}\tag{3.31}$$

and

$$\begin{aligned}&\int_{\mathcal{C}_\Omega}t^{1-\alpha}\zeta_\eta\nabla\zeta_\eta(x-\tilde{x},t)\cdot W_\epsilon^r\nabla W_\epsilon^r(x,t)\,dx\,dt\\&\le C\eta^{-1}\int_{C_\eta}t^{1-\alpha}|W_\epsilon^r|\,|\nabla W_\epsilon^r|\,dx\,dt\\&\le C\eta^{-1}\epsilon^{\alpha-N-1}\int_{C_\eta}t^{1-\alpha}\,dx\,dt\\&\le C(\frac{\eta}{\epsilon})^{N-\alpha+1}.\end{aligned}\tag{3.32}$$

From (2.5), (3.31) and (3.32) it follows that

$$\begin{aligned}&\|\zeta_\eta(x-\tilde{x},t)W_\epsilon^r(x,t)\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}\\&=k_\alpha\int_{\mathcal{C}_\Omega}t^{1-\alpha}(|\zeta_\eta\nabla W_\epsilon^r|^2+|W_\epsilon^r\nabla\zeta_\eta|^2+2W_\epsilon^r\zeta_\eta\nabla\zeta_\eta\cdot\nabla W_\epsilon^r)\,dx\,dt\\&\le\|W_\epsilon^r\|^2_{\mathbb{R}_+^{N+1}}+k_\alpha\int_{\mathcal{C}_\Omega}t^{1-\alpha}|W_\epsilon^r\nabla\zeta_\eta|^2+2k_\alpha\int_{\mathcal{C}_\Omega}t^{1-\alpha}W_\epsilon^r\zeta_\eta\nabla\zeta_\eta\cdot\nabla W_\epsilon^r\,dx\,dt\\&\le\|W_\epsilon^r\|^2_{\mathbb{R}_+^{N+1}}+C(\frac{\eta}{\epsilon})^{N-\alpha},\end{aligned}\tag{3.33}$$

which, together with (3.27), implies (a).

In addition, by (3.28), we have

$$\begin{aligned}\int_\Omega|\zeta_\eta(x-\tilde{x},0)w_\epsilon^r|^{2_\alpha^*}dx&=\int_\Omega|w_\epsilon^r|^{2_\alpha^*}dx-\int_\Omega(1-\zeta_\eta^{2_\alpha^*}(x,0))|w_\epsilon^r|^{2_\alpha^*}dx\\&=\int_\Omega|w_\epsilon^r|^{2_\alpha^*}dx-\int_{|(x-\tilde{x},t)|\le\eta}|w_\epsilon|^{2_\alpha^*}dx\\&\ge(k_\alpha S_{\alpha,N})^{N/\alpha}-C(\frac{\epsilon}{r})^N-C\int_{|(x-\tilde{x},t)|\le\eta}\epsilon^{-N}dx\\&\ge(k_\alpha S_{\alpha,N})^{N/\alpha}-C(\frac{\epsilon}{r})^N-C(\frac{\eta}{\epsilon})^N,\end{aligned}\tag{3.34}$$

and then (b) follows. The proof is complete. □

Note that $N > (1+\sqrt{2})\alpha$, then $\frac{N(N-2)-\alpha^2}{\alpha^2} > \frac{\alpha}{N-2\alpha}$. Fix $\tilde{\theta} \in (\frac{\alpha}{N-2\alpha}, \frac{N(N-2)-\alpha^2}{\alpha^2})$, then we can define $r_1 := \frac{1}{6m}$ and $\epsilon_r := r^{\tilde{\theta}+1}$. Set

$$\tilde{W}_r(x,t) := \phi_r(x,t)W_{\epsilon_r}(x,t) \quad \text{and} \quad \tilde{w}_r(x) = \tilde{W}_r(x,0). \tag{3.35}$$

Obviously, $\tilde{W}_r$ and $\tilde{w}_r$ are continuous with respect to $r \in (0, r_1]$ in $X_0^\alpha(\mathcal{C}_\Omega)$. In addition, for any $0 < r \le r_1$ and $\eta \in [0, r^{2\tilde{\theta}+1}]$, the following result holds.

Proposition 3.7. *There exist $C_3 > 0$ and $m_1 > m_0$ such that for any $m \ge m_1$ and $\tilde{x} \in \Omega$,*

$$\sup_{\tau \ge 0} I(\tau\zeta_\eta(x-\tilde{x},t)\tilde{W}_r(x,t)) \begin{cases} < \frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha} & \text{if } r \in (0, \frac{r_1}{2}], \\ \le S_m & \text{if } r \in [\frac{r_1}{2}, r_1], \end{cases}$$

where

$$S_m = \frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha} - C_3 m^{-(\tilde{\theta}+1)\alpha}$$

and $S_m + C_2 m^{\frac{N(\alpha-N)}{\alpha}} < \frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha}$. Here m_0 and C_2 are defined in Lemmas 3.3 and 3.4, respectively.

Proof. By Lemma 3.6 and (3.35), we have

$$\begin{aligned} &\|\zeta_\eta(x-\tilde{x},t)\tilde{W}_r(x,t)\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} \\ &\le (k_\alpha S_{\alpha,N})^{N/\alpha} + C(\frac{\epsilon_r}{r})^{N-\alpha} + C(\frac{\eta}{\epsilon_r})^{N-\alpha} \\ &\le \begin{cases} (k_\alpha S_{\alpha,N})^{N/\alpha} + Cr^{\tilde{\theta}(N-\alpha)}, & \text{if } \alpha \in (0,1)\cup(1,2), \\ (k_1 S_{1,N})^N + Cr^{\tilde{\theta}(N-1)}|\log r|, & \text{if } \alpha = 1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \int_\Omega |\zeta_\eta(x,0)\tilde{w}_r(x)|^{2^*_\alpha}dx &\ge (k_\alpha S_{\alpha,N})^{N/\alpha} - C(\frac{\epsilon_r}{r})^N - C(\frac{\eta}{\epsilon_r})^N \\ &\ge (k_\alpha S_{\alpha,N})^{N/\alpha} - Cr^{\tilde{\theta}N}. \end{aligned}$$

Then for $r \in [\frac{r_1}{2}, r_1]$, we have

$$\begin{aligned} &\|\zeta_\eta(x-\tilde{x},t)\tilde{W}_r(x,t)\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} \\ &\le \begin{cases} (k_\alpha S_{\alpha,N})^{N/\alpha} + Cm^{-\tilde{\theta}(N-\alpha)} & \text{if } \alpha \in (0,1)\cup(1,2), \\ (k_1 S_{1,N})^N + Cm^{-\tilde{\theta}(N-1)}\log m & \text{if } \alpha = 1, \end{cases} \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} \int_\Omega |\zeta_\eta(x,0)\tilde{W}_r(x,0)|^{2^*_\alpha}dx &\ge (k_\alpha S_{\alpha,N})^{N/\alpha} - C(\frac{\epsilon_r}{r})^N - C(\frac{\eta}{\epsilon_r})^N \\ &\ge (k_\alpha S_{\alpha,N})^{N/\alpha} - Cm^{-\tilde{\theta}N} \quad \text{if } r \in [\frac{r_1}{2}, r_1]. \end{aligned} \tag{3.37}$$

Note that $w_{\epsilon_r}(x)=(\frac{\epsilon_r}{\epsilon_r^2+|x|^2})^{\frac{N-\alpha}{2}}\geq(\frac{1}{2\epsilon_r})^{\frac{N-\alpha}{2}}$ for $|x|\leq\epsilon_r$. Then for $\eta=0$, we have

$$\begin{aligned}\int_\Omega|\tilde{w}_r(x)|^2dx&\geq\int_{B_{\epsilon_r}(0)}|w_{\epsilon_r}|^2dx\\&\geq C\int_{B_{\epsilon_r}(0)}\epsilon_r^{\alpha-N}dx\\&\geq C\epsilon_r^\alpha\\&=Cr^{(\tilde{\theta}+1)\alpha}\quad\big(\geq Cm^{-(\tilde{\theta}+1)\alpha}\text{ if }r\in[\frac{r_1}{2},r_1]\big).\end{aligned}\tag{3.38}$$

In addition, since $\tilde{w}_r(x)\leq C\epsilon_r^{\frac{\alpha-N}{2}}$, for any $\tilde{x}\in\Omega$ and $0<\eta\leq r^{2\tilde{\theta}+1}$, we obtain

$$\begin{aligned}\int_\Omega|\zeta_\eta(x-\tilde{x},0)\tilde{w}_r|^2dx&=\int_\Omega|\tilde{w}_r|^2dx-\int_\Omega(1-\zeta_\eta^2(x-\tilde{x},0))|\tilde{w}_r|^2dx\\&\geq C\epsilon_r^\alpha-\int_{|x-\tilde{x}|\leq\eta}\epsilon_r^{\alpha-N}dx\\&\geq C\epsilon_r^\alpha-\eta^N\epsilon_r^{\alpha-N}\\&\geq Cr^{(\tilde{\theta}+1)\alpha}\ \Big(\geq Cm^{-(\tilde{\theta}+1)\alpha}\text{ if }r\in[\frac{r_1}{2},r_1]\Big).\end{aligned}\tag{3.39}$$

Since $\tilde{\theta}\in(\frac{\alpha}{N-2\alpha},\frac{N(N-\alpha)-\alpha^2}{\alpha^2})$, we have $\tilde{\theta}(N-\alpha)>(\tilde{\theta}+1)\alpha>0$, and then there exists $r_0>0$ such that for any $0<r\leq r_0$, it holds

$$r^{\tilde{\theta}(N-\alpha)}<r^{(\tilde{\theta}+1)\alpha}\quad\text{and}\quad r^{\tilde{\theta}(N-\alpha)}|\log r|<r^{(\tilde{\theta}+1)\alpha}.\tag{3.40}$$

Thus for $\alpha\in(0,1)\cup(1,2)$ and $r\in[r_1/2,r_1]$ with $r_1<r_0$, by (3.36)-(3.40), we have

$$\begin{aligned}&I(\tau\zeta_\eta(x-\tilde{x},t)\tilde{W}_r(x,t))\\&=\frac{\tau^2}{2}[k_\alpha\int_{\mathcal{C}_\Omega}t^{1-\alpha}|\zeta_\eta(x-\tilde{x},t)\tilde{W}_r(x,t)|^2\,dx\,dt\\&\quad-\lambda\int_\Omega|\zeta_\eta(x-\tilde{x},0)\tilde{w}_r(x)|^2dx]-\frac{\tau^{2_\alpha^*}}{2_\alpha^*}\int_\Omega|\tilde{w}_r(x)|^{2_\alpha^*}dx\\&\leq\max_{\tau\geq0}\frac{\tau^2}{2}((k_\alpha S_{\alpha,N})^{N/\alpha}+Cr^{\tilde{\theta}(N-\alpha)}-Cr^{(\tilde{\theta}+1)\alpha})\\&\quad-\frac{\tau^{2_\alpha^*}}{2_\alpha^*}((k_\alpha S_{\alpha,N})^{N/\alpha}-Cr^{\tilde{\theta}N})\\&\leq\frac{\alpha}{2N}((k_\alpha S_{\alpha,N})^{N/\alpha}-Cr^{(\tilde{\theta}+1)\alpha})\Big(\frac{(k_\alpha S_{\alpha,N})^{N/\alpha}-Cr^{(\tilde{\theta}+1)\alpha}}{(k_\alpha S_{\alpha,N})^{N/\alpha}-Cr^{\tilde{\theta}N}}\Big)^{\frac{N-\alpha}{2}}\\&\leq\frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha}-Cr^{(\tilde{\theta}+1)\alpha}\\&=\frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha}-Cm^{-(\tilde{\theta}+1)\alpha}.\end{aligned}\tag{3.41}$$

Similarly, for $\alpha = 1$, by (3.36)–(3.40), we obtain

$$
\begin{aligned}
&I(\tau\zeta_\eta(x-\tilde{x},t)\tilde{W}_r(x,t)) \\
&= \frac{\tau^2}{2}[k_1\int_{\mathcal{C}_\Omega}|\zeta_\eta(x-\tilde{x},t)\tilde{W}_r(x,t)|^2\,dx\,dt \\
&\quad -\lambda\int_\Omega|\zeta_\eta(x-\tilde{x},0)\tilde{w}_r(x)|^2dx] - \frac{\tau^{2_1^*}}{2_1^*}\int_\Omega|\tilde{w}_r(x)|^{2_1^*}dx \\
&\le \max_{\tau\ge 0}\frac{\tau^2}{2}((k_1S_{1,N})^N + Cr^{\tilde{\theta}(N-1)}|\log r| - Cr^{(\tilde{\theta}+1)\alpha}) \\
&\quad - \frac{\tau^{2_1^*}}{2_1^*}((k_1S_{1,N})^N - Cr^{\tilde{\theta}N}) \\
&\le \frac{1}{2N}((k_1S_{1,N})^N - Cr^{\tilde{\theta}+1})(\frac{(k_1S_{1,N})^N - Cr^{\tilde{\theta}+1}}{(k_1S_{1,N})^N - Cr^{\tilde{\theta}N}})^{\frac{N-1}{2}} \\
&\le \frac{1}{2N}(k_1S_{1,N})^N - Cr^{\tilde{\theta}+1} \\
&= \frac{1}{2N}(k_1S_{1,N})^N - Cm^{-(\tilde{\theta}+1)}.
\end{aligned}
\tag{3.42}
$$

Therefore, by (3.41) and (3.42), there exist $C_3 > 0$ and $m_1 > m_0$ such that for any $\alpha \in (0,2)$ and $m \ge m_1$,

$$
\sup_{t\ge 0} I(t\zeta_\eta(x-\tilde{x},y)\tilde{W}_r(x,t)) \le \frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha} - C_3m^{-(\tilde{\theta}+1)\alpha} =: S_m
$$

and $C_3m^{-(\tilde{\theta}+1)\alpha} > C_2m^{\frac{N(\alpha-N)}{\alpha}}$ due to $0 < (\tilde{\theta}+1)\alpha < \frac{N(N-\alpha)}{\alpha}$. The lemma follows immediately. $\square$

4. Proof of Theorem 1.1

In this section, we have all the tools to prove our main result. Now, we fix $m \ge m_1$.Note that $r_1 = \frac{1}{6m}$ and $\epsilon_r = r^{\tilde{\theta}+1}$.Let $\eta_r = r^{2\tilde{\theta}+1}$ and $\tilde{x} \in \Omega$.Then for any $0 < r \le r_1$, we have

$$
\tilde{W}_r(x+\tilde{x},t) \in X_0^\alpha\big(\overline{\mathbb{B}_r^+(-\tilde{x})}\big), \tag{4.1}
$$

$$
\zeta_{\eta_r}(x,t)\tilde{W}_r(x+\tilde{x},t) \in X_0^\alpha\Big(\overline{\mathbb{B}_r^+(-\tilde{x})\backslash\mathbb{B}_{\frac{\eta_r}{2}}^+(0)}\Big). \tag{4.2}
$$

We write $B^j = \{x \in \mathbb{R}^j : |x| \le 1\}$ and $S^j = \{x \in \mathbb{R}^{j+1} : |x| = 1\}$ for any integer $j \ge 1$. Denote $u^\pm := \max\{\pm u, 0\}$. We have the following lemma.

Lemma 4.1. *For any integer $k \ge 0$, there exists an odd continuous map $\bar{h}$: $\mathbb{R}^{k+N+2} \to X_0^\alpha(\mathcal{C}_\Omega)$ such that $\lim_{|x|\to+\infty} I(\bar{h}(x)) = -\infty$ and $\sup_{U\in\bar{h}(\mathbb{R}^{k+N+2})} I(U) < \frac{\alpha}{N}(k_\alpha S_{\alpha,N})^{N/\alpha}$.*

Proof. The proof follows the same idea as in [9], so we only sketch the proof.

Step 1: First, we construct an odd continuous map $h_1 : B^N \to X_0^\alpha(\mathbb{B}_{\frac{1}{2m}}(0))$ such that

$$
\operatorname{supp} h_1(y)^+ \cap \operatorname{supp} h_1(y)^- = \emptyset \text{ and } \sup_{\tau\ge 0} I(\tau h_1(y)) < S_m + \frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha}, \tag{4.3}
$$

for all $y \in B^N$. For any $y \in B^N$, set $s = |y|, \theta = \frac{y}{|y|}$ and define $h_1 : B^N \to X_0^\alpha(\mathbb{B}_{\frac{1}{2m}}(0))$ by

$$h_1(y)(x,t) = \begin{cases} \tilde{W}_{\frac{\eta_{r_1}}{2}}(x,t) - \xi_{\eta_{r_1}}(x,t)\tilde{W}_{r_1}(x+4sr_1\theta,t) \\ \quad \text{if } 0 \le s \le \frac{1}{2}, \\ \tilde{W}_{s(2r_1-\eta_{r_1})-r_1+\eta_{r_1}}(x-2r_1(2s\theta-\theta),t) - \xi_{\eta_{r_1}}(x,t)\tilde{W}_{r_1}(x+2r_1\theta,t) \\ \quad \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

Clearly, (4.3) follows from (4.1),(4.2) and Proposition 3.7.

Step 2: The map h_1 induces an odd continuous mapping $h_2 : \mathbb{S}^N \to X_0^\alpha\big(\mathbb{B}_{\frac{1}{2m}}(0)\big)$ by

$$h_2(x_1,\dots,x_{N+1}) = \begin{cases} h_1(x_1,\dots,x_N) & \text{if } x_{N+1} \ge 0, \\ -h_1(-x_1,\dots,-x_N) & \text{if } x_{N+1} \le 0. \end{cases}$$

Since h_1 is odd on $\mathbb{S}^{N-1}$, we have

$$\operatorname{supp} h_2(\theta)^+ \cap \operatorname{supp} h_2(\theta)^- = \emptyset \text{ and } \sup_{\tau\ge 0} I(\tau h_2(\theta)) < S_m + \frac{\alpha}{2N}S^{N/\alpha},\ \forall \theta \in \mathbb{S}^N. \tag{4.4}$$

Step 3: There exists an odd continuous map $h_3 : \mathbb{R}^{N+2} \to X_0^\alpha(\mathbb{B}_{\frac{1}{m}}(0))$ such that

$$\sup_{U\in h_3(\mathbb{R}^{N+2})} I(U) < S_m + \frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha}. \tag{4.5}$$

Indeed, define a cylindric surface in $\mathbb{R}^{N+2}$ by

$$Z := (\mathbb{S}^N \times [-1,1]) \cup (B^{N+1} \times \{-1,1\}) \subset \mathbb{R}^{N+2},$$

and choose a positive function $v_0 := \xi_{\eta_{r_1}}(x,t)\tilde{W}_{\frac{1}{6m}}(x+y_0,t) \in X_0^\alpha\big(\mathbb{B}_{\frac{1}{m}}(0)\backslash\mathbb{B}_{\frac{1}{2m}}(0)\big)$ with $y_0 \in \Omega$ and $|y_0| = \frac{3}{4m}$. Then, it follows from Proposition 3.7 that

$$\sup_{t\ge 0} I(tv_0) \le S_m. \tag{4.6}$$

For $\theta \in \mathbb{S}^N, s_1 \in [0,1], s_2 \in [-1,1]$, set

$$\tilde{h}_2(s_1\theta,s_2) := \begin{cases} (1-s_2)h_2(\theta)^- + (1+s_2)h_2(\theta)^+ & \text{if } s_1 = 1, \\ 2s_1h_2(\theta)^+ + (1-s_1)v_0 & \text{if } s_2 = 1, \\ 2s_1h_2(\theta)^- + (1-s_1)v_0 & \text{if } s_2 = -1. \end{cases}$$

It is easy to check that $\operatorname{supp}\tilde{h}_2(s_1\theta,s_2)^+ \cap \operatorname{supp}\tilde{h}_2(s_1\theta,s_2)^- = \emptyset$. Now, we extend $\tilde{h}_2$ to a map $h_3 : \mathbb{R}^{N+2} \to X_0^\alpha\Big(\mathbb{B}_{\frac{1}{m}}(0)\Big)$ by

$$h_3(\tilde{t}z) := \tilde{t}\tilde{h}_2(z) \quad \text{for } z \in Z,\ \tilde{t} \ge 0.$$

Thus, (4.5) follows from (4.1), (4.2), (4.6) and Proposition 3.7 immediately.

Step 4: For $k \ge 1$, define an odd continuous map $\bar{h} : \mathbb{R}^{k+N+2} \to X_0^\alpha(\mathcal{C}_\Omega)$ by

$$\bar{h}(y,z) = \tilde{h}_3(y) + h_3(z) \quad \text{for all } y \in \mathbb{R}^k, z \in \mathbb{R}^{N+2},$$

where $\tilde{h}_3 : \mathbb{R}^k \to X_0^\alpha\Big(\Omega\backslash\mathbb{B}_{\frac{1}{m}}(0)\Big)$ is an odd map defined by $\tilde{h}_3(y_1,\dots,y_k) := \sum_{i=1}^k y_iE_i^m$. It is easy to see that $\lim_{|(y,z)|\to+\infty} I(\bar{h}(y,z)) = -\infty$. Note that

$\operatorname{supp}\tilde{h}_3(y)\cap\operatorname{supp}h_3(z)=\emptyset$ for all $y\in\mathbb{R}^k, z\in\mathbb{R}^{N+2}$, then by Lemma 3.4 and Proposition 3.7, we have

$$\begin{aligned}\sup_{(y,z)\in\mathbb{R}^{k+N+2}} I(\bar{h}(y,z)) &\le \sup_{y\in\mathbb{R}^k} I(\tilde{h}_3(y)) + \sup_{z\in\mathbb{R}^{N+2}} I(h_3(z))\\ &< C_2 m^{\frac{N(\alpha-N)}{\alpha}} + S_m + \frac{\alpha}{2N}(k_\alpha S_{\alpha,N})^{N/\alpha}\\ &< \frac{\alpha}{N}(k_\alpha S_{\alpha,N})^{N/\alpha}.\end{aligned}$$

Step 5: For $k=0$, we define $\bar{h}:\mathbb{R}^{N+2}\to X_0^\alpha(\mathcal{C}_\Omega)$ by $\bar{h}=h_3$. Clearly, it follows from Proposition 3.7 that

$$\sup_{z\in\mathbb{R}^{N+2}} I(\bar{h}(z)) < \frac{\alpha}{N}(k_\alpha S_{\alpha,N})^{N/\alpha}.$$

Therefore, Step 4 and Step 5 yield our conclusion. □

Note that $\lambda_0=0$ and $\lambda_k^{\alpha/2}\le\lambda<\lambda_{k+1}^{\alpha/2}$ for some $k\ge 0$. Then, we have the following lemma.

Lemma 4.2. $0<\beta_{k+1}\le\cdots\le\beta_{k+N+2}<2^{\alpha/N}k_\alpha S_{\alpha,N}$.

Proof. According to the definition of β_j, we obtain $\beta_{k+1}\le\cdots\le\beta_{k+N+2}$. Moreover, by (2.8), we have $\beta_{k+1}>0$. So we only need to verify $\beta_{k+N+2}<2^{\alpha/N}k_\alpha S_{\alpha,N}$. By using the same idea as in [9], we conclude that $\gamma(\mathfrak{A})\ge k+N+2$, where $\mathfrak{A}:=\{U\in\bar{h}(\mathbb{R}^{k+N+2}):\|u\|_{L^{2^*_\alpha}(\Omega)}=1\}$. Then, it follows from Lemma 4.1 that for any function $U\in\mathfrak{A}$,

$$\frac{\alpha}{N}(k_\alpha S_{\alpha,N})^{N/\alpha} > \sup_{\tau\ge 0} I(\tau U) \ge \frac{\alpha}{2N}\Big(\frac{\|U\|^2_{X_0^\alpha(\mathcal{C}_\Omega)}-\lambda\|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^{2^*_\alpha}(\Omega)}}\Big)^{N/\alpha} = \frac{\alpha}{2N}J(U)^{N/\alpha}.$$

This implies that $\sup_{U\in\mathfrak{A}}J(U)<2^{\alpha/N}k_\alpha S_{\alpha,N}$. Therefore, by the definition of β_{k+N+2}, we conclude that $\beta_{k+N+2}<2^{\alpha/N}k_\alpha S_{\alpha,N}$. We completed the proof. □

Proof of Theorem 1.1. There are two cases to complete our proof. If K^β is infinite for some $\beta\in(0,2^{\alpha/N}k_\alpha S_{\alpha,N})$, then by Lemma 2.2, J has infinitely many critical points and hence we complete our proof. If K^β is finite for all $\beta\in(0,2^{\alpha/N}k_\alpha S_{\alpha,N})$, then according to Lemmas 2.2 and 4.2, we may assume $0<\beta_{k+1}<\cdots<\beta_{k+N+2}<2^{\alpha/N}k_\alpha S_{\alpha,N}$. Let $j_0\ge 1$ be an integer such that $\beta_{k+j_0}\ge k_\alpha S_{\alpha,N}$, Then Lemma 2.1 implies that J has at least $\max\{j_0-1,N+2-j_0\}\ge[\frac{N+1}{2}]$ pairs of nontrivial critical points, and so do I. The proof is complete. □

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Hui Guo
College of Mathematics and Econometrics, Hunan University, Changsha 410082, China
E-mail address: huiguo_math@163.com