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GLOBAL WELL-POSEDNESS OF THE 2D MICROPOLAR FLUID FLOWS WITH MIXED DISSIPATION

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Abstract. This article concerns the global well-posedness of the 2D micropolar fluid flows with mixed dissipation:

$$-\partial_{yy}(-\Delta)^{\alpha}u_1, \quad -\partial_{xx}(-\Delta)^{\alpha}u_2, \quad (-\Delta)^{\beta}w.$$

We prove the existence and uniqueness of global smooth solution of 2D micropolar fluid flows when $\alpha+\beta \geq 1/2$.

1. Introduction

Micropolar fluid flows derived by Eringe [10] are an important mathematical model in some polymeric fluids and fluids which contain certain additives in narrow films([15, 16]). They are non-Newtonian fluids with nonsymmetric stress tensor which are coupled with the kinematic viscous effect, microrotational effects as well as microrotational inertia. The two-dimensional (2D) incompressible micropolar fluid flows is governed by

$$\begin{gathered}
\partial_t u-(\nu+\kappa)\Delta u-2\kappa\nabla\times w+u\cdot\nabla u+\nabla\pi=0,\\
\nabla\cdot u=0,\\
\partial_t w-\gamma\Delta w+4\kappa w-2\kappa\nabla\times u+u\cdot\nabla w=0.
\end{gathered}\tag{1.1}$$

Where $u(x,y,t)=(u_1(x,y,t),u_2(x,y,t))$ is the unknown velocity vector field, $\pi(x,y,t)$ is the unknown scalar pressure field and $w(x,y,t)$ is the unknown scalar micro-rotation angular velocity of the rotation of the particles of the fluid. $\nu,\kappa,\gamma\geq 0$ are viscous coefficients. u_0 and w_0 represent the prescribed initial data for the velocity and micro-rotation fields. Here and in what follows,

$$\nabla\times u=\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}, \quad \nabla\times w=\Big(\frac{\partial w}{\partial y},-\frac{\partial w}{\partial x}\Big).$$

Because of its importance in mathematics, there is much attention on the well-posedness of the micropolar fluid flows [6, 11, 14]. In particular, when there is full dissipation, Lukaszewicz [14] examined the global well-posedness of smooth solution to the 2D micropolar fluid flows (1.1). A more explicit existence and uniqueness result which is based on the decay estimates of the linearized equations is recently investigated in Dong and Chen [7]. When there is partial dissipation, the issue on global regularity becomes more difficult. Due to some new observation, Dong and

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Zhang [8] recently obtained the global existence and uniqueness of classic solution of micropolar fluid flows (1.1) with only velocity dissipation Δu. Xue[18] also examined the well-posedness of the micropolar fluid flows (1.1) with only velocity dissipation Δu under the Besov space framework. Yamaziki[19] futher extended the results of above the results to the magneto-micropolar equations. Chen[5] examined the existence and uniqueness of smooth solution of micropolar fluid flows (1.1) with partial dissipation $(\partial_{yy}u, \partial_{xx}w)$. Very recently, Dong, Li and Wu [9] proved the global well-posedness of 2D micropolar fluid flows (1.1) with only angular viscosity dissipation Δw. One may also refer to some important and interesting results of the 2D fluid dynamical models with partial dissipation such as the 2D Boussinesq equations [1, 4, 12, 17] and the 2D magnetohydrodynamic (MHD) equations [3, 2].

Motivated by the above results of the 2D fluid dyanmical models with partial dissipation, the purpose of this study is investigate the global existence and uniqueness of the smooth solution of 2D micropolar fluid flows with mixed dissipation $(\partial_{yy}(-\Delta)^{\alpha}u_1, \partial_{xx}(-\Delta)^{\alpha}u_2, -(-\Delta)^{\beta}w)$. More precisely, we will examine the global regularity issue of the following 2D micropolar fluid flows with unit viscosity

$$\begin{gathered}
\partial_t u_1 + (u\cdot\nabla)u_1 - \partial_y w + \partial_x \pi = \partial_{yy}(-\Delta)^{\alpha}u_1,\\
\partial_t u_2 + (u\cdot\nabla)u_2 + \partial_x w + \partial_y \pi = \partial_{xx}(-\Delta)^{\alpha}u_2,\\
\partial_x u_1 + \partial_y u_2 = 0,\\
\partial_t w + 2w - \nabla\times u + u\cdot\nabla w = -(-\Delta)^{\beta}w.
\end{gathered} \tag{1.2}$$

where $0 < \alpha, \beta < 1$. We are able to prove the following existence and uniqueness of smooth global solutions for (1.2).

Theorem 1.1. *Assume* $(u_0, w_0) \in H^s(\mathbb{R}^2)$, $s > 2$ *and* $\nabla\cdot u_0 = 0$. *There exists a unique global smooth solution* (u, w) *for the 2D micropolar fluid flows* (1.2) *with* $\alpha+\beta \geq \frac{1}{2}$

$$\begin{gathered}
u \in C([0,\infty); H^s(\mathbb{R}^2)), \quad u \in L^2(0,T; H^{s+1+\alpha}(\mathbb{R}^2)),\\
w \in C([0,\infty); H^s(\mathbb{R}^2)), \quad w \in L^2(0,T; H^{s+\beta}(\mathbb{R}^2)), \quad \forall T>0.
\end{gathered}$$

Clearly, Theorem 1.1 generalizes the global well-posedness results of Lukaszewicz [14], Dong and Chen [7] where there is full dissipation. Moreover, Theorem 1.1 has no inclusion relation between previous results of partial dissipative micropolar fluid flows [5, 8, 18, 19]. Especially, the main trick based on a new quantity $\nabla\times u - w$ in [8, 18, 19] is not available here any more.

We briefly summarize the main challenge and explain what we have done to achieve the global regularity. In order to prove Theorem 1.1, we need global *a priori* bounds of (u, w) in sufficiently smooth functional spaces. More precisely, if we can prove the Beale-Kato-Majda criterion of solutions for the 2D micropolar fluid flows (1.2)

$$\int_0^T \|(\nabla u(t), \nabla w(t))\|_{L^\infty}\, dt < \infty, \tag{1.3}$$

then the proof of Theorem 1.1 is a more or less standard procedure. The next natural step is to examine a global H^1-bound for (u, w) of (1.2) after the based L^2 energy estimates of (u, w). However, unlike the previous results where the H^1-bound can be derived from vorticity equation, the vorticity structure here is destroyed due to the mixed dissipation $(\partial_{yy}(-\Delta)^{\alpha}u_1, \partial_{xx}(-\Delta)^{\alpha}u_2)$. If we directly take the L^2 inner product (1.2) with $(\Delta u, \Delta w)$, then the H^1-bound for (u, w) of

(1.2) at least requires higher dissipation $\alpha+\beta \geq 1$. To overcome the difficulty, we first derive the optimal fractional-order derivative estimates of u,

$$\|\Lambda^{\alpha+\beta} u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{1+2\alpha+\beta} u(s)\|_{L^2}^2 ds \leq C(t, \|(u_0, w_0)\|_{H^s}),$$

and then the optimal fractional-order derivative estimates of w

$$\|\Lambda^{2(\alpha+\beta)} w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{2\alpha+3\beta} w(s)\|_{L^2}^2 ds \leq C(t, \|(u_0, w_0)\|_{H^s}).$$

The Beale-Kato-Majda criterion of solutions then can be derived by an iterative procedure.

2. Beale-Kato-Majda criterion

To prove the existence of global smooth solution for the 2D micropolar fluid flows (1.2), we will examine the Beale-Kato-Majda criterion in this section. For simplicity, we only need to prove the case $\alpha+\beta = 1/2$, the case $\alpha+\beta > 1/2$ can be done in a slightly modification. To do so, we first examine the following lemma which is mainly based on the divergence free of velocity.

Lemma 2.1. *Suppose $u = (u_1, u_2)$ is divergence free and for any $s \in \mathbb{R}$, we have*

$$\int_{\mathbb{R}^2} \left(-\partial_{yy} u_1 \Lambda^{2s} u_1 - \partial_{xx} u_2 \Lambda^{2s} u_2\right) dx\, dy \geq \frac{1}{2} \|\Lambda^{1+s} u\|_{L^2}^2 \tag{2.1}$$

where $\Lambda = (-\Delta)^{1/2}$ denotes the Zygmund operator, defined via the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

Proof. Integrating by parts and applying the divergence free property of velocity, it is easy to check that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \big(- \partial_{yy} u_1 \Lambda^{2s} u_1 - \partial_{xx} u_2 \Lambda^{2s} u_2\big)\, dx\, dy \\
&= \int_{\mathbb{R}^2} \big(\partial_{yy} u_1 \ \Delta \Lambda^{2s-2} u_1 + \partial_{xx} u^2 \ \Delta \Lambda^{2s-2} u_2\big)\, dx\, dy \\
&= \int_{\mathbb{R}^2} \Big(\nabla \partial_y \Lambda^{s-1} u_1 \nabla \partial_y \Lambda^{s-1} u_1 + \nabla \partial_x \Lambda^{s-1} u_2 \nabla \partial_x \Lambda^{s-1} u_2\Big)\, dx\, dy \\
&= \int_{\mathbb{R}^2} \Big(|\partial_{yy} \Lambda^{s-1} u_1|^2 + |\partial_{xy} \Lambda^{s-1} u_1|^2 + |\partial_{xx} \Lambda^{s-1} u_2|^2 + |\partial_{xy} \Lambda^{s-1} u_2|^2\Big)\, dx\, dy \\
&= \int_{\mathbb{R}^2} \Big(|\partial_{yy} \Lambda^{s-1} u_1|^2 + |\partial_{yy} \Lambda^{s-1} u_2|^2 + |\partial_{xx} \Lambda^{s-1} u_2|^2 + |\partial_{xx} \Lambda^{s-1} u_1|^2\Big)\, dx\, dy \\
&\geq \frac{1}{2} \int_{\mathbb{R}^2} \Big(|\Delta \Lambda^{s-1} u_1|^2 + |\Delta \Lambda^{s-1} u_2|^2\Big)\, dx\, dy = \frac{1}{2} \|\Lambda^{1+s} u\|_{L^2}^2.
\end{aligned}$$

□

To prove *a priori* estimates of solutions, we recall the following classical commutator estimate [13].

Lemma 2.2. *Let $s > 0$. Let $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ with $q_1, p_2 \in (1, \infty)$ and $p_1, q_2 \in [1, \infty]$. Then,*

$$\|[\Lambda^s, f] g\|_{L^r} \leq C \left(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}\right),$$

where C is a constant depending on the indices s, r, p_1, q_1, p_2 and q_2.

Proposition 2.3. *Under the conditions of Theorem 1.1 and let (u, w) be the corresponding solution of (1.2). Then (u, w) obeys the following global bounds, for any $0 < t < \infty$,*

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^{1+\alpha} u(s)\|_{L^2}^2 + \|\Lambda^\beta w(s)\|_{L^2}^2) ds \le C, \tag{2.2}$$

$$\|\Lambda^{\alpha+\beta} u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{1+2\alpha+\beta} u(s)\|_{L^2}^2 ds \le C, \tag{2.3}$$

$$\|\Lambda^{2\alpha+2\beta} w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{2\alpha+3\beta} w(s)\|_{L^2}^2 ds \le C, \tag{2.4}$$

where the positive constants C depend on t and $\|(u_0, w_0)\|_{H^s}$ only.

Proof. Taking the L^2 inner product of equations (1.2) with (u_1, u_2, w), it is easy to verify after applying Lemma 2.1, the Hölder inequality and the Young inequality

$$\begin{aligned}
&\frac{d}{dt}\big(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\big) + \|\Lambda^{1+\alpha} u(t)\|_{L^2}^2 + 2\|\Lambda^\beta w(t)\|_{L^2}^2 + 4\|w(t)\|_{L^2}^2 \\
&\le 2\int_{\mathbb{R}^2} \{(\nabla \times w)\cdot u + (\nabla \times u) w\}\, dx\, dy \\
&\le 4\|\Lambda^{1-\beta} u\|_{L^2} \|\Lambda^\beta w\|_{L^2} \le C\|u(t)\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{1+\alpha} u(t)\|_{L^2}^2 + \|\Lambda^\beta w(t)\|_{L^2}^2,
\end{aligned}$$

where we have used the following fact due to the divergence free of velocity u,

$$\int_{\mathbb{R}^2} (u\cdot\nabla u)\cdot u\, dx\, dy = 0, \quad \int_{\mathbb{R}^2} (u\cdot\nabla w) w\, dx\, dy = 0.$$

Integrating in time for $0 < t < \infty$,

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^{1+\alpha} u(s)\|_{L^2}^2 + \|\Lambda^\beta w(s)\|_{L^2}^2) ds \le e^{ct}(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2)$$

which implies (2.2).

To examine (2.3), we take the L^2 inner product of the first two equations of (1.2) with $(\Lambda^{2\alpha+2\beta} u_1, \Lambda^{2\alpha+2\beta} u_2)$ and use Lemma 2.1 to obtain

$$\begin{aligned}
&\frac{d}{dt}\|\Lambda^{\alpha+\beta} u(t)\|_{L^2}^2 + \|\Lambda^{1+2\alpha+\beta} u(t)\|_{L^2}^2 \\
&= 2\int_{\mathbb{R}^2} (\nabla\times w)\cdot \Lambda^{2\alpha+2\beta} u\, dx\, dy + 2\int_{\mathbb{R}^2} \Lambda^{\alpha+\beta}[u\cdot\nabla u]\Lambda^{\alpha+\beta} u\, dx\, dy \\
&= I_1 + I_2
\end{aligned} \tag{2.5}$$

For I_1, it is easy to check that

$$\begin{aligned}
2\int_{\mathbb{R}^2} (\nabla\times w)\cdot\Lambda^{2\alpha+2\beta} u\, dx\, dy &\le C\|\Lambda^{1+2\alpha+\beta} u(t)\|_{L^2}\|\Lambda^\beta w(t)\|_{L^2} \\
&\le \frac{1}{4}\|\Lambda^{1+2\alpha+\beta} u(t)\|_{L^2}^2 + C\|\Lambda^\beta w(t)\|_{L^2}^2
\end{aligned}$$

For I_2, using Lemma 2.2, the Hölder inequality, Young inequality, and interpolation inequality gives

$$I_2 = 2\int_{\mathbb{R}^2} \Lambda^{\alpha+\beta}[u\cdot\nabla u]\Lambda^{\alpha+\beta} u\, dx\, dy$$

$$
\begin{aligned}
&\le C\|\nabla u\|_{L^{\frac{2}{1-\beta}}}\|\Lambda^{\alpha+\beta}u\|_{L^{\frac{2}{\beta}}}\|\Lambda^{\alpha+\beta}u\|_{L^2}\\
&\le C\|\Lambda^{1+\beta}u\|_{L^2}\|\Lambda^{1+\alpha}u\|_{L^2}\|\Lambda^{\alpha+\beta}u\|_{L^2}\\
&\le \frac{1}{2}\|\Lambda^{1+\beta}u\|_{L^2}^2 + C\|\Lambda^{1+\alpha}u\|_{L^2}^2\|\Lambda^{\alpha+\beta}u\|_{L^2}^2\\
&\le \frac{1}{4}\|u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}^2 + C\|\Lambda^{1+\alpha}u\|_{L^2}^2\|\Lambda^{\alpha+\beta}u\|_{L^2}^2.
\end{aligned}
$$

Using the estimates of I_1, I_2 into (2.5) gives

$$
\begin{aligned}
&\frac{d}{dt}\|\Lambda^{\alpha+\beta}u(t)\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{1+2\alpha+\beta}u(t)\|_{L^2}^2\\
&\le C\|\Lambda^{\beta}w(t)\|_{L^2}^2 + C\|\mathbf{u}(t)\|_{L^2}^2 + C\|\Lambda^{1+\alpha}u\|_{L^2}^2\|\Lambda^{\alpha+\beta}u\|_{L^2}^2
\end{aligned}
$$

Taking the Gronwall inequality into consideration together with (2.2) implies

$$
\begin{aligned}
&\|\Lambda^{\alpha+\beta}u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{1+2\alpha+\beta}u(s)\|_{L^2}^2 ds\\
&\le \exp\Big(C\int_0^t \|\Lambda^{1+\alpha}u\|_{L^2}^2 d\tau\Big)\big(\|\Lambda^{\alpha+\beta}u_0\|_{L^2}^2 + Ct\big) \le C, \quad 0<t<\infty
\end{aligned}
$$

which is (2.3).

For (2.4), we take the inner product of fourth equation of (1.2) with $\Lambda^{4(\alpha+\beta)}w$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{2\alpha+2\beta}w\|_{L^2}^2 + \|\Lambda^{2\alpha+3\beta}w\|_{L^2}^2 + 2\|\Lambda^{2\alpha+2\beta}w\|_{L^2}^2\\
&= 2\int_{\mathbb{R}^2}(\nabla\times u)\Lambda^{4(\alpha+\beta)}w\,dx\,dy - \int_{\mathbb{R}^2}[\Lambda^{2\alpha+2\beta}, u\cdot\nabla]w\Lambda^{2\alpha+2\beta}w\,dx\,dy\\
&:= J_1 + J_2.
\end{aligned}
\tag{2.6}
$$

Expression J_1 is bounded by applying the Hölder inequality and the Young inequality,

$$
J_1 \le C\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}\|\Lambda^{2\alpha+3\beta}w\|_{L^2} \le \frac{1}{4}\|\Lambda^{2\alpha+3\beta}w\|_{L^2}^2 + C\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}^2.
$$

As in the estimates for I_2, employing Lemma 2.2, the Sobolev's imbedding inequality and the Young inequality, it follows that

$$
\begin{aligned}
J_2 &= -\int_{\mathbb{R}^2}[\Lambda^{2\alpha+2\beta}, u\cdot\nabla]w\Lambda^{2\alpha+2\beta}w\,dx\,dy\\
&\le \Big(\|\nabla u\|_{L^{\frac{2}{\beta}}}\|\Lambda^{2\alpha+2\beta}w\|_{L^{\frac{2}{1-\beta}}} + \|\Lambda^{2\alpha+2\beta}u\|_{L^{\frac{2}{\beta}}}\|\nabla w\|_{L^{\frac{2}{1-\beta}}}\Big)\|\Lambda^{2\alpha+2\beta}w\|_{L^2}\\
&\le C\left(\|\Lambda^{2-\beta}u\|_{L^2}\|\Lambda^{1+\beta}w\|_{L^2} + \|\Lambda^{1+2\alpha+\beta}u\|_{L^2}\|\Lambda^{2\alpha+3\beta}w\|_{L^2}\right)\|\Lambda^{2\alpha+2\beta}w\|_{L^2}\\
&\le C\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}\|\Lambda^{2\alpha+3\beta}w\|_{L^2}\|\Lambda^{2\alpha+2\beta}w\|_{L^2}\\
&\le \frac{1}{4}\|\Lambda^{2\alpha+3\beta}w\|_{L^2}^2 + C\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}^2\|\Lambda^{2\alpha+2\beta}w\|_{L^2}^2
\end{aligned}
$$

Inserting the estimates for J_1 and J_2 into (2.6) gives

$$
\frac{d}{dt}\|\Lambda^{2\alpha+2\beta}w\|_{L^2}^2 + \|\Lambda^{2\alpha+3\beta}w\|_{L^2}^2 \le C\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}^2\left(\|\Lambda^{2\alpha+2\beta}w\|_{L^2}^2 + 1\right)
$$

and applying Gronwall's inequality, we have

$$
\|\Lambda^{2\alpha+2\beta}w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{2\alpha+3\beta}w(s)\|_{L^2}^2 ds \le C(t, u_0, w_0)
$$

which gives (2.4). This completes the proof of Proposition 2.3.

To obtain the Beale-Kato-Majda criterion for the solutions. we further need to prove the following *a priori* bounds for higher-order derivatives of the solutions.

Proposition 2.4. *Suppose* (u, w) *is the corresponding solution in Theorem 1.1. Then* (u, w) *obeys the following global bounds, for any* $0 < t < \infty$,

$$\|\Lambda^{3\alpha+3\beta}u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{1+4\alpha+3\beta}u(s)\|_{L^2}^2 ds \leq C, \tag{2.7}$$

$$\|\Lambda^{4\alpha+4\beta}w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{4\alpha+5\beta}w(s)\|_{L^2}^2 ds \leq C. \tag{2.8}$$

In particular, we have the Beale-Kato-Majda criterion of solutions for the 2D micropolar fluid flows (1.2),

$$\int_0^T \|(\nabla u(t), \nabla w(t))\|_{L^\infty}\, dt < \infty. \tag{2.9}$$

Proof. Similar to the estimation of (2.2), multiplying the both sides of the first two equations of (1.2) with $(\Lambda^{6\alpha+6\beta}u_1, \Lambda^{6\alpha+6\beta}u_2)$ and applying Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\frac{d}{dt}\|\Lambda^{3\alpha+3\beta}u\|_{L^2}^2 + \|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}^2 \\
&= 2\int_{\mathbb{R}^2}(\nabla\times w)\cdot\Lambda^{6\alpha+6\beta}u\,dx\,dy + 2\int_{\mathbb{R}^2}\Lambda^{3\alpha+3\beta}[u\cdot\nabla u]\Lambda^{3\alpha+3\beta}u\,dx\,dy \\
&\leq C\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}\|\Lambda^{2\alpha+3\beta}w\|_{L^2} \\
&\quad + C\|\nabla u\|_{L^{\frac{2}{1-(2\alpha+\beta)}}}\|\Lambda^{3\alpha+3\beta}u\|_{L^{\frac{2}{2\alpha+\beta}}}\|\Lambda^{3\alpha+3\beta}u\|_{L^2} \\
&\leq C\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}\|\Lambda^{2\alpha+3\beta}w\|_{L^2} \\
&\quad + C\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}\|\Lambda^{1+\alpha+2\beta}u\|_{L^2}\|\Lambda^{3\alpha+3\beta}u\|_{L^2} \\
&\leq C\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}\|\Lambda^{2\alpha+3\beta}w\|_{L^2} \\
&\quad + C\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}(\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2} + \|u\|_{L^2})\|\Lambda^{3\alpha+3\beta}u\|_{L^2} \\
&\leq \frac{1}{2}\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}^2 + C\|\Lambda^{2\alpha+3\beta}w\|_{L^2}^2 \\
&\quad + C\|\Lambda^{1+2\alpha+\beta}u\|_{L^2}^2\|\Lambda^{3\alpha+3\beta}u\|_{L^2}^2 + C.
\end{aligned} \tag{2.10}$$

Employing the Gronwall inequality and Proposition 2.3 gives

$$\|\Lambda^{3\alpha+3\beta}u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{1+4\alpha+3\beta}u(s)\|_{L^2}^2 ds \leq C(t, u_0, w_0), \quad 0 < t < \infty$$

which gives (2.7).

To prove (2.8), similarly, multiplying both sides of the fourth equation in (1.2) by $\Lambda^{8(\alpha+\beta)}w$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{4\alpha+4\beta}w\|_{L^2}^2 + \|\Lambda^{4\alpha+5\beta}w\|_{L^2}^2 + 2\|\Lambda^{4\alpha+4\beta}w\|_{L^2}^2 \\
&= 2\int_{\mathbb{R}^2}(\nabla\times u)\Lambda^{8(\alpha+\beta)}w\,dx\,dy - \int_{\mathbb{R}^2}[\Lambda^{4\alpha+4\beta}, u\cdot\nabla]w\Lambda^{4\alpha+4\beta}w\,dx\,dy \\
&\le C\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}\|\Lambda^{4\alpha+5\beta}w\|_{L^2} + C\Big(\|\nabla u\|_{L^{\frac{2}{\beta}}}\|\Lambda^{4\alpha+4\beta}w\|_{L^{\frac{2}{1-\beta}}} \\
&\quad + \|\Lambda^{4\alpha+4\beta}u\|_{L^{\frac{2}{\beta}}}\|\nabla w\|_{L^{\frac{2}{1-\beta}}}\Big)\|\Lambda^{4\alpha+4\beta}w\|_{L^2} \\
&\le C\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}\|\Lambda^{4\alpha+5\beta}w\|_{L^2} + C\Big(\|\Lambda^{2-\beta}u\|_{L^2}\|\Lambda^{4\alpha+5\beta}w\|_{L^2} \\
&\quad + \|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}\|\Lambda^{2-\beta}w\|_{L^2}\Big)\|\Lambda^{2\alpha+2\beta}w\|_{L^2} \\
&\le C\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}\|\Lambda^{4\alpha+5\beta}w\|_{L^2} + C(\|w\|_{L^2} + \|\Lambda^{4\alpha+5\beta}w\|_{L^2}) \\
&\quad \times\big(\|u\|_{L^2} + \|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}\|\Lambda^{2-\beta}w\|_{L^2}\big)\|\Lambda^{2\alpha+2\beta}w\|_{L^2} \\
&\le \frac{1}{2}\|\Lambda^{4\alpha+5\beta}w\|_{L^2}^2 + C(1+\|\Lambda^{1+4\alpha+3\beta}u\|_{L^2}^2\|\Lambda^{4\alpha+4\beta}w\|_{L^2}^2 + C.
\end{aligned}
\tag{2.11}
$$

and then Gronwall's inequality gives

$$
\|\Lambda^{4\alpha+4\beta}w(t)\|_{L^2}^2 + \int_0^t\|\Lambda^{4\alpha+5\beta}w(s)\|_{L^2}^2ds \le C(t,u_0,w_0)
$$

which is (2.8).

Moreover, since

$$
1+4\alpha+3\beta,\quad\text{and}\quad 4\alpha+5\beta>2,
$$

a priori estimates (2.7) and (2.8) actually imply the following Beale-Kato-Majda criterion of solutions u and w,

$$
\int_0^T\|\nabla u(t)\|_{L^\infty}\,dt<\infty,\quad \int_0^T\|\nabla w(t)\|_{L^\infty}\,dt<\infty
$$

by the Sobolev imbedding inequality. The proof is complete. $\square$

3. Proof of Theorem 1.1

Under the Beale-Kato-Majda criterion in Section 2, the proof of Theorem 1.1 is more or less standard.

Existence. We now borrow the classic Friedrichs method to prove the existence of global smooth solutions. Firstly we consider the following approximate equations of (1.2)

$$
\begin{gathered}
\partial_t u_{n1} + J_nP(J_nu_n\cdot\nabla J_nu_{n1}) - P\partial_y w_n + \partial_x\pi_n = J_n\partial_{yy}(-\Delta)^\alpha u_{n1}, \\
\partial_t u_{n2} + J_nP(J_nu_n\cdot\nabla J_nu_{n2}) + P\partial_x w_n + \partial_y\pi_n = J_n\partial_{xx}(-\Delta)^\alpha u_{n2}, \\
\partial_x u_{n1} + \partial_y u_n = 0, \\
\partial_t w_n + 2w_n - \nabla\times u + J_nu_n\cdot\nabla J_nw_n = -J_n(-\Delta)^\beta w_n, \\
u_n(x,y,0) = J_nu_0,\quad w_n(x,y,0) = J_nw_0,
\end{gathered}
\tag{3.1}
$$

where J_n is defined as $J_n\varphi = \mathcal{F}^{-1}(\chi_{B(0,n)}(\xi)\mathcal{F}(\varphi)(\xi))$, and $\chi_{B(0,n)}$ denotes the characteristic function on the ball $B(0,n)$. P denotes the standard projection onto divergence-free vector fields. The standard Picard type theorem ensures that, for some $T_n > 0$, there exists a unique local smooth solution $(u_n, w_n) \in C([0, T_n); L^2)$. Additionally, it is easy to see that $(J_n u_n, J_n w_n)$ is also a local smooth solution of (3.1). Thus (u_n, w_n) also satisfies

$$\begin{gathered}
\partial_t u_{n1} + J_n(u_n \cdot \nabla u_{n1}) - \partial_y w_n + \partial_x \pi_n = \partial_{yy}(-\Delta)^\alpha u_{n1}, \\
\partial_t u_{n2} + J_n(u_n \cdot \nabla u_{n2}) + \partial_x w_n + \partial_y \pi_n = \partial_{xx}(-\Delta)^\alpha u_{n2}, \\
\partial_x u_{n1} + \partial_y u_n = 0, \\
\partial_t w_n + 2w_n - \nabla \times u + u_n \cdot \nabla w_n = -(-\Delta)^\beta w_n, \\
u_n(x, y, 0) = J_n u_0, \quad w_n(x, y, 0) = J_n w_0,
\end{gathered} \tag{3.2}$$

A basic L^2 energy estimate for (u_n, w_n) in (3.2) implies

$$\|u_n(t)\|_{L^2}^2 + \|w_n(t)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^{1+\alpha} u_n(\tau)\|_{L^2}^2 + \|\Lambda^\beta w_n(\tau)\|_{L^2}^2\right) \tau \le C(t, u_0, w_0),$$

where C is independent of n. Therefore, the local solution can be extended into a global one by the standard Picard Extension Theorem. Next we show that (u_n, w_n) admits a uniform global bound in $H^s(\mathbb{R}^2)$ with $(s > 2)$. By a standard energy estimate involving (3.2), it is easy to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_n\|_{H^s}^2 + \|w_n\|_{H^s}^2\right) + \|\Lambda^{1+\alpha} u_n\|_{H^s}^2 + \|\Lambda^\beta w_n\|_{H^s}^2 \\
&\le \int_{\mathbb{R}^2} \{(\nabla \times w_n) \cdot \Lambda^{2s} u_n + (\nabla \times u_n) \Lambda^{2s} w_n\}\, dxdy \\
&\quad + \int_{\mathbb{R}^2} [\Lambda^s, u_n \cdot \nabla] u_n \cdot \Lambda^s u_n \, dx\, dy + \int_{\mathbb{R}^2} [\Lambda^s, u_n \cdot \nabla] w_n \Lambda^s w_n \, dx\, dy \\
&\le \frac{1}{2}\left(\|\Lambda^{1+\alpha} u_n\|_{H^s}^2 + \|\Lambda^\beta w_n\|_{H^s}^2\right) \\
&\quad + C\left(\|\nabla u_n\|_{L^\infty} + \|\nabla w_n\|_{L^\infty} + 1\right)\left(\|u_n\|_{H^s}^2 + \|w_n\|_{H^s}^2\right).
\end{aligned}$$

Applying the Gronwall's inequality and the Beale-Kato-Majda criterion in Proposition 2.4, we have

$$\begin{aligned}
&\|u_n\|_{H^s}^2 + \|w_n\|_{H^s}^2 + \int_0^t (\|\Lambda^{1+\alpha} u_n\|_{H^s}^2 + \|\Lambda^\beta w\|_{H^s}^2) ds \\
&\le (\|u_0\|_{H^s}^2 + \|w_0\|_{H^s}^2) e^{C \int_0^t (\|\nabla u_n\|_{L^\infty} + \|\nabla w_n\|_{L^\infty} + 1) ds} \\
&\le C(t, \|(u_0, w_0)\|_{H^s}).
\end{aligned}$$

Hence the above uniform H^s estimates allows us to obtain the global existence of the desired solution (u, w) to (1.2) by a standard compactness argument.

3.1. **Uniqueness.** We show that any two solutions (u, w) and $(\bar{u}, \bar{w})$ to (1.2) must be the same. The difference (U, W) with $U = u - \bar{u}$ and $W = w - \bar{w}$ satisfies

$$\begin{gathered}
\partial_t U_1 + (U \cdot \nabla) u_1 + (\bar{u} \cdot \nabla) U_1 - \partial_y W + \partial_x \pi = \partial_{yy}(-\Delta)^\alpha U_1, \\
\partial_t U_2 + (U \cdot \nabla) u_2 + (\bar{u} \cdot \nabla) U_2 + \partial_x W + \partial_y \pi = \partial_{xx}(-\Delta)^\alpha U_2, \\
\partial_x U_1 + \partial_y U_2 = 0, \\
\partial_t W + 2W - \nabla \times U + (U \cdot \nabla) w + (\bar{u} \cdot \nabla) W = -(-\Delta)^\beta W.
\end{gathered} \tag{3.3}$$

Taking the L^2 inner product of (U,W) with (3.3), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|U\|_{L^2}^2+\|W\|_{L^2}^2\right)+\frac{1}{2}\|\Lambda^{1+\alpha}U\|_{L^2}^2+\|\Lambda^{\beta}W\|_{L^2}^2+2\|W\|_{L^2}^2\\
&=\int_{\mathbb{R}^2}(\nabla\times W\cdot U+\nabla\times UW)dx-2\int_{\mathbb{R}^2}(U\cdot\nabla u)\cdot Udx-\int_{\mathbb{R}^2}(U\cdot\nabla w)Wdx\\
&\le C\|\Lambda^{1-\beta}U\|_{L^2}\|\Lambda^{\beta}W\|_{L^2}+C\|\nabla u_1\|_{L^\infty}\|U\|_{L^2}^2+2\|U\|_{L^2}\|\nabla w_1\|_{L^\infty}\|W\|_{L^2}\\
&\le \frac{1}{2}\|\Lambda^{\beta}W\|_{L^2}^2+\frac{1}{4}\|\Lambda^{1+\alpha}U\|_{L^2}^2\\
&\quad +C\left(\|\nabla u\|_{L^\infty}+\|\nabla w\|_{L^\infty}+1\right)\left(\|U\|_{L^2}^2+\|W\|_{L^2}^2\right).
\end{aligned}
$$

Gronwall's inequality then implies

$$
\|U(t)\|_{L^2}^2+\|W(t)\|_{L^2}^2 \quad \le C(t,u_0,w_0)\left(\|U_0\|_{L^2}^2+\|W_0\|_{L^2}^2\right), \tag{3.4}
$$

which implies the uniqueness. $\square$

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