*Electronic Journal of Differential Equations*, Vol. 2016 (2016), No. 157, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# EXISTENCE OF SOLUTIONS TO BURGERS EQUATIONS IN DOMAINS THAT CAN BE TRANSFORMED INTO RECTANGLES

YASSINE BENIA, BOUBAKER-KHALED SADALLAH

Abstract. This work is concerned with Burgers equation $\partial_t u + u\partial_x u - \partial_x^2 u = f$ (with Dirichlet boundary conditions) in the non rectangular domain $\Omega = \{(t,x) \in R^2;\ 0 < t < T,\ \varphi_1(t) < x < \varphi_2(t)\}$ (where $\varphi_1(t) < \varphi_2(t)$ for all $t \in [0;T]$). This domain will be transformed into a rectangle by a regular change of variables. The right-hand side lies in the Lebesgue space $L^2(\Omega)$, and the initial condition is in the usual Sobolev space $H_0^1$. Our goal is to establish the existence, uniqueness and the optimal regularity of the solution in the anisotropic Sobolev space.

## 1. Introduction

One of the most important partial differential equations of the theory of nonlinear conservation laws, is the semilinear diffusion equation, called Burgers equation:

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = f, \tag{1.1}$$

where $u$ stands, generally, for a velocity, $t$ the time variable, $x$ the space variable and $\nu$ the constant of viscosity (or the diffusion coefficient). Homogeneous Burgers equation (equation (1.1) with $f = 0$), is one of the simplest models of nonlinear equations which have been studied.

The mathematical structure of this equation includes a nonlinear convection term $u\partial_x u$ which makes the equation more interesting, and a viscosity term of higher order $\partial_x^2 u$ which regularizes the equation and produces a dissipation effect of the solution near a shock. When the viscosity coefficient vanishes, $\nu = 0$, the Burgers equation reduced to the transport equation, which represents the inviscid Burgers equation $\partial_t u + u\partial_x u = f$.

The study of the equation (1.1) has a long history: In 1906, Forsyth, treated an equation which converts by some variable changes to the Burgers equation. In 1915, Bateman [2] introduced the equation (1.1): He was interested in the case when $\nu \to 0$, and in studying the movement behavior of a viscous fluid when the viscosity tends to zero. Burgers (1948) has published a study on the equation (1.1) (which it owes his name), in his document [6] about modeling the turbulence phenomena. Using the transformation discovered later by [8] in 1951, about the

---

2010 *Mathematics Subject Classification.* 35K58, 35Q35.
*Key words and phrases.* Semilinear parabolic problem; Burgers equation; existence; uniqueness; anisotropic Sobolev space.

Submitted April 15, 2016. Published June 21, 2016.

same time and independently by Hopf [10], (called the Hopf-Cole transformation), Burgers continued his study of what he called "nonlinear diffusion equation". This study treated mainly the static aspects of the equation. The results of these works can be found in the book [5].

The objective of Burgers was to consider a simplified version of the incompressible Navier Stokes equation $\partial_t u + (u\cdot\nabla)u = \nu\Delta u - \nabla p$ by neglecting the pressure term.

Among the most interesting applications of the one-dimensional Burgers equation, we mention traffic flow, growth of interfaces, and financial mathematics (see for example [11, 15]).

The nonlinear Burgers equation (1.1), with $f = 0$, can be converted to the linear heat equation and then explicitly solved by the Hopf-Cole transformation. We usually look for explicit solutions for the forced Burgers equation $\partial_t u + u\partial_x u - \nu\partial_x^2 u = f$, where $f(x,t)$ is the forcing term in a rectangular domain. In this work we are interested in proving a result of existence, uniqueness and regularity for the inhomogeneous Burgers problem.

For $f(x,t) = -\lambda\partial_x\eta(x,t)$, Burgers equation becomes

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = -\lambda\partial_x\eta(x,t), \tag{1.2}$$

which is Burgers stochastic equation, where $\eta(x,t)$ stands for the white noise. Using the transformation $u(x,t) = -\lambda\partial_x h(x,t)$, we find that (1.2) is equivalent to the equation of KPZ

$$\partial_t h(x,t) - \frac{\lambda}{2}(\partial_x h(x,t))^2 - \nu\partial_x^2 h(x,t) = \eta(x,t).$$

This equation has been introduced by Kardar, Parisi and Zhang in 1986, and quickly became the default model for random interface growth in physics.

In a paper by Morandi Cecchi *et al.* [12], the main result was the existence and uniqueness of a solution to the Burgers problem (with constant coefficients) in the anistropic Sobolev space

$$H^{1,2}(R) = \big\{u \in L^2(R) : \partial_t u \in L^2(R),\ \partial_x u \in L^2(R),\ \partial_x^2 u \in L^2(R)\big\}$$

where $R$ is a rectangle. The authors used a wrong inequality (namely $\int_\Omega M(u-M)^+(t)dx \le M\|(u-M)^+(t)\|^2$) at the end of the proof of Theorem 2 (maximum principle); the inequality appears in the line 14, page 165 (and line 15 page 167). To rectify this part of the proof it suffices to show that $u \in L^\infty(Q)$. The proof given by the authors remains true only when $f = 0$ (but this was not the objective of their paper), this case being treated by Bressan in [3]. However, in our work, using another method, we prove a more general result concerning the existence, uniqueness and regularity of a solution to the Burgers problem with variable coefficients in a rectangle. Then, the existence, uniqueness and regularity of a solution to the Burgers problem in a domain that can be transformed into a rectangle.

**Setting of the problem.** Recall that $L^p(0,a)$ and $H^m(0,a)$ are the usual spaces of Lebesgue and Sobolev, respectively, for $1 \le p \le \infty$ and $m \in \mathbb{Z}$. For any Banach space $X$, we define $L^p(0,T;X)$ to be the space of measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u\|_X^p \,\mathrm{d}t\Big)^{1/p} < \infty$$

for $1 \leq p < \infty$ and $\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0<t<T} \|u\|_X < \infty$ if $p = \infty$. $L^p(0,T;X)$ is a Banach space. Of course, we have $L^p(R) = L^p(0,T;L^p(0,a))$.

This article is concerned with two questions regarding the Burgers equation. The first one is to study the existence, uniqueness and regularity of the solution of the semilinear parabolic problem:

$$
\begin{gathered}
\partial_t u(t,x) + \alpha(t)u(t,x)\partial_x u(t,x) - \beta(t)\partial_x^2 u(t,x) + \gamma(t,x)\partial_x u(t,x) \\
= f(t,x) \quad (t,x) \in R, \\
u(0,x) = u_0(x) \quad x \in I, \\
u(t,0) = u(t,a) = 0 \quad t \in (0,T),
\end{gathered} \tag{1.3}
$$

in the rectangle $R = I \times (0,T)$ where $I = (0,a)$, $a \in R^+$ ($T$ is finite); $f \in L^2(R)$ and $u_0 \in H_0^1(I)$ are given functions. We assume that the functions $\alpha$, $\beta$ depend only on $t$ and the function $\gamma$ depends on $t$ and $x$. We also suppose that there exist positive constants $(\alpha_i)_{i=1,2}$, $(\beta_i)_{i=1,2}$ and $\gamma_1$, such that

$$
\begin{gathered}
\alpha_1 \leq \alpha(t) \leq \alpha_2, \quad \beta_1 \leq \beta(t) \leq \beta_2, \quad \forall t \in [0,T] \\
\text{and } |\partial_x \gamma(t,x)| \leq \gamma_1 \text{ or } |\gamma(t,x)| \leq \gamma_1 \quad \forall (t,x) \in R.
\end{gathered} \tag{1.4}
$$

The second question concerns the semilinear parabolic Burgers problem:

$$
\begin{gathered}
\partial_t u(t,x) + u(t,x)\partial_x u(t,x) - \nu \partial_x^2 u(t,x) = f(t,x) \quad \text{in } \Omega, \\
u|_{t=0} = u_0(x) \quad x \in J, \\
u|_{x=\varphi_i(t)} = 0 \quad i = 1,2
\end{gathered} \tag{1.5}
$$

in $\Omega \subset \mathbb{R}^2$, such as

$$
\Omega = \{(t,x) \in R^2; 0 \leq t \leq T,\, \varphi_1(t) < x < \varphi_2(t)\}
$$

where $J = [\varphi_1(0), \varphi_2(0)]$ and $\nu$ is a positive constant, $\varphi_1$, $\varphi_2$ are functions defined on $[0,T]$ belonging to $C^1(]0,T[)$. We assume that $\varphi_1(t) < \varphi_2(t)$ for $t \in [0,T]$.

Using the results obtained in the first part of this work, we look for conditions on the functions $(\varphi_i)_{i=1,2}$ which guarantee that problem (1.5) admits a unique solution $u \in H^{1,2}(\Omega)$. In order to solve problem (1.5), we will follow the method which was used, for example, in Sadallah[13] and Clark *et al.* [7]. This method consists in proving that this problem admits a unique solution when $\Omega$ is transformed into a rectangle, using a change of variables preserving the anisotropic Sobolev space $H^{1,2}(\Omega)$.

To establish the existence (and uniqueness) of the solution to (1.5), we impose the assumption

$$
|\varphi'(t)| \leq c \quad \text{for all } t \in [0,T] \tag{1.6}
$$

where $c$ is a positive constant, and $\varphi(t) = \varphi_2(t) - \varphi_1(t)$ for all $t \in [0,T]$.

The result related to the existence of the solution $u$ of (1.3) in a rectangle is obtained thanks to a personal (and detailed) communication of professor Luc Tartar about the Burgers equation with constant coefficients in a rectangle. The authors would like to thank him for his appreciate comments and hints. Our main result is as follows:

**Theorem 1.1.** *If $u_0 \in H_0^1(I)$, $f \in L^2(R)$ and $\alpha, \beta$ , $\gamma$ satisfy the assumption* (1.4), *then problem* (1.3) *admits a (unique) solution $u \in H^{1,2}(R)$.*

**Theorem 1.2.** *If $u_0 \in H_0^1(J)$, $f \in L^2(\Omega)$ and $\varphi_1$, $\varphi_2$ satisfy the assumption* (1.6), *then problem* (1.5) *admits a (unique) solution $u \in H^{1,2}(\Omega)$.*

The proof of Theorem 1.1 is based on the Faedo-Galerkin method. We introduce approximate solution by reduction to the finite dimension. By the Faedo-Galerkin method, we obtain the existence of an approximate solution using an existence theorem of solutions for a system of ordinary differential equations. We approximate the equation of problem (1.3) by a simple equation. Then we make the passage to the limit using a compactness argument. The proof of Theorem 1.2 needs an appropriate change of variables which allows us to use Theorem 1.1.

## 2. Proof of Theorem 1.1

Multiplying the equation of problem (1.3) by a test function $w \in H_0^1(I)$, and integrating by parts from 0 to $a$, we obtain

$$\begin{aligned} &\int_0^a \partial_t u w \,\mathrm{d}x + \alpha(t) \int_0^a u \partial_x u w \,\mathrm{d}x \\ &+ \beta(t) \int_0^a \partial_x u \partial_x w \,\mathrm{d}x + \int_0^a \gamma(t,x) \partial_x u w \,\mathrm{d}x \\ &= \int_0^a f w \,\mathrm{d}x, \quad \forall w \in H_0^1(I),\ t \in (0,T), \end{aligned} \tag{2.1}$$

This is the weak formulation of problem (1.3). The solution of (2.1) satisfying the conditions of problem (1.3) is called *weak solution*.

To prove the existence of a weak solution to (1.3), we choose the basis $(e_j)_{j\in\mathbb{N}^\star}$ of $L^2(I)$ defined as a subset of the eigenfunctions of $-\partial_x^2$ for the Dirichlet problem

$$\begin{gathered} -\partial_x^2 e_j = \lambda_j e_j, \quad j \in \mathbb{N}^*, \\ e_j = 0 \quad \text{on } \Gamma = \{0, a\}. \end{gathered}$$

More precisely,

$$e_j(x) = \frac{\sqrt{2}}{\sqrt{a}} \sin \frac{j\pi x}{a}, \quad \lambda_j = (\frac{j\pi}{a})^2, \quad \text{for } j \in \mathbb{N}^*.$$

As the family $(e_j)_{j\in\mathbb{N}^\star}$ is an orthonormal basis of $L^2(I)$, then it is an orthogonal basis of $H_0^1(I)$. In particular, for $v \in L^2(R)$, we can write

$$v = \sum_{k=1}^{\infty} b_k(t) e_k,$$

where $b_k = (v, e_k)_{L^2(I)}$ and the series converges in $L^2(I)$. Then, we introduce the approximate solution $u_n$ by

$$\begin{gathered} u_n(t) = \sum_{j=1}^{n} c_j(t) e_j, \\ u_n(0) = u_{0n} = \sum_{j=1}^{n} c_j(0) e_j, \end{gathered}$$

which has to satisfy the approximate problem

$$\begin{gathered}
\begin{aligned}
&\int_0^a \partial_t u_n e_j \,\mathrm{d}x + \alpha(t) \int_0^a u_n \partial_x u_n e_j \,\mathrm{d}x \\
&+ \beta(t) \int_0^a \partial_x u_n \partial_x e_j \,\mathrm{d}x + \int_0^a \gamma(t,x) \partial_x u_n e_j \,\mathrm{d}x \\
&= \int_0^a f e_j \,\mathrm{d}x,
\end{aligned} \\
u_n(0) = u_{0n}.
\end{gathered} \tag{2.2}$$

for all $j = 1, \dots, n$, and$0 \leq t \leq T$.

**Remark 2.1.** The coefficients $c_j(0)$ (which depend on $j$ and $n$) will be chosen such that the sequence $(u_{0n})$ converges in $H_0^1(I)$ to $u_0$. Then we suppose in the sequel that $\lim u_{0n} = u_0$ in $H_0^1(I)$.

2.1. **Solution of the approximate problem.**

**Lemma 2.2.** *For all $j$, there exists a unique solution $u_n$ of Problem* (2.2).

*Proof.* As $e_1, \cdots, e_n$ are orthonormal in $L^2(I)$, then

$$\int_0^a \partial_t u_n e_j \,\mathrm{d}x = \sum_{i=1}^n c_i'(t) \int_0^a e_i e_j \,\mathrm{d}x = c_j'(t).$$

On the other hand, $-\partial_x^2 e_i = \lambda_i e_i$, then $\partial_x^2 u_n(t) = -\sum_{i=1}^n c_i(t) \lambda_i e_i$. Therefore, for all $t \in [0, T]$,

$$-\beta(t) \int_0^a \partial_x^2 u_n e_j \,\mathrm{d}x = \beta(t) \sum_{i=1}^n c_i(t) \lambda_i \int_0^a e_i e_j \,\mathrm{d}x = \beta(t) \lambda_j c_j(t).$$

Now, if we introduce

$$f_j(t) = \int_0^a f e_j \,\mathrm{d}x, \quad k_j(t) = -\alpha(t) \int_0^a u_n \partial_x u_n e_j \,\mathrm{d}x,$$
$$h_j(t) = -\int_0^a \gamma(t,x) \partial_x u_n e_j \,\mathrm{d}x,$$

for $j \in \{1, \dots, n\}$, then (2.2) is equivalent to the following system of $n$ uncoupled linear ordinary differential equations:

$$c_j'(t) = -\beta(t) \lambda_j c_j(t) + k_j(t) + h_j(t) + f_j(t), \quad j = 1, \dots, n. \tag{2.3}$$

Observe that the terms $k_j(t), h_j(t)$ are well defined (because $e_j$ and $\gamma(t,x)$ are regular) and $f_j$ is integrable (because $f \in L^2(R)$). Taking into account the initial condition $c_j(0)$, for each fixed $j \in \{1, \dots, n\}$, (2.3) has a unique regular solution $c_j$ in some interval $(0, T')$ with $T' \leq T$. In fact, we can prove here that $T' = T$. $\square$

2.2. **A priori estimate.**

**Lemma 2.3.** *There exists a positive constant $K_1$ independent of $n$, such that for all $t \in [0, T]$*

$$\|u_n\|_{L^2(I)}^2 + \beta_1 \int_0^t \|\partial_x u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s \leq K_1.$$

*Proof.* Multiplying (2.2) by $c_j$ and summing for $j = 1, \dots, n$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^a u_n^2 \,\mathrm{d}x + \beta(t)\int_0^a (\partial_x u_n)^2 \,\mathrm{d}x - \frac{1}{2}\int_0^a \partial_x\gamma(t,x)u_n^2 \,\mathrm{d}x = \int_0^a f u_n \,\mathrm{d}x.$$

Indeed, because of the boundary conditions, we have

$$\alpha(t)\int_0^a u_n^2\partial_x u_n \,\mathrm{d}x = \frac{\alpha(t)}{3}\int_0^a \partial_x(u_n)^3 \,\mathrm{d}x = 0,$$

and an integration by parts gives

$$-\frac{1}{2}\int_0^a \partial_x\gamma(t,x)u_n^2 \,\mathrm{d}x = \int_0^a \gamma(t,x)u_n\partial_x u_n \,\mathrm{d}x.$$

Then, by integrating with respect to $t$ ($t \in (0,T)$), and according to (1.4), we find that

$$\begin{aligned}
&\frac{1}{2}\|u_n\|_{L^2(I)}^2 + \beta_1\int_0^t \|\partial_x u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s \\
&\le \frac{1}{2}\|u_{0n}\|_{L^2(I)}^2 + \int_0^t \|f(s)\|_{L^2(I)}\|u_n(s)\|_{L^2(I)} \,\mathrm{d}s + \frac{\gamma_1}{2}\int_0^t \|u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s.
\end{aligned}$$

Using Poincaré's inequality

$$\|u_n\|_{L^2(I)}^2 \le \frac{a^2}{2}\|\partial_x u_n\|_{L^2(I)}^2,$$

both with the elementary inequality

$$|rs| \le \frac{\varepsilon}{2}r^2 + \frac{s^2}{2\varepsilon}, \quad \forall r,s \in R,\ \forall\varepsilon > 0, \tag{2.4}$$

with $\varepsilon = \frac{2\beta_1}{a^2}$, we obtain

$$\begin{aligned}
&\|u_n\|_{L^2(I)}^2 + \beta_1\int_0^t \|\partial_x u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s \\
&\le \|u_{0n}\|_{L^2(I)}^2 + \frac{a^2}{2\beta_1}\int_0^t \|f(s)\|_{L^2(I)}^2 \,\mathrm{d}s + \gamma_1\int_0^t \|u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s,
\end{aligned}$$

so

$$\begin{aligned}
&\|u_n\|_{L^2(I)}^2 + \beta_1\int_0^t \|\partial_x u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s \\
&\le \|u_{0n}\|_{L^2(I)}^2 + \frac{a^2}{2\beta_1}\int_0^t \|f(s)\|_{L^2(I)}^2 \,\mathrm{d}s \\
&\quad + \gamma_1\int_0^t \Big(\|u_n(s)\|_{L^2(I)}^2 + \beta_1\int_0^s \|\partial_x u_n(\tau)\|_{L^2(I)}^2 \,\mathrm{d}\tau\Big) \,\mathrm{d}s.
\end{aligned}$$

As the sequence $(u_{0n})$ converges in $H_0^1(I)$ to $u_0$ (see Remark 2.1) and $f \in L^2(R)$, there exists a positive constant $C_1$ independent of $n$ such that

$$\|u_{0n}\|_{L^2(I)}^2 + \frac{a^2}{2\beta_1}\|f\|_{L^2(R)}^2 \le C_1$$

and

$$\|u_n\|_{L^2(I)}^2 + \beta_1\int_0^t \|\partial_x u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s$$

$$\leq C_1 + \gamma_1 \int_0^t \Big( \|u_n(s)\|^2_{L^2(I)} + \beta_1 \int_0^s \|\partial_x u_n(\tau)\|^2_{L^2(I)}\, \mathrm{d}\tau \Big)\, \mathrm{d}s,$$

then by Gronwall's inequality,

$$\|u_n\|^2_{L^2(I)} + \beta_1 \int_0^t \|\partial_x u_n(s)\|^2_{L^2(I)}\, \mathrm{d}s \leq C_1 \exp(\gamma_1 t).$$

Taking $K_1 = C_1 \exp(\gamma_1 T)$, we obtain

$$\|u_n\|^2_{L^2(I)} + \beta_1 \int_0^t \|\partial_x v_n(s)\|^2_{L^2(I)}\, \mathrm{d}s \leq K_1.$$

□

**Lemma 2.4.** *There exists a positive constant $K_2$ independent of $n$, such that for all $t \in [0,T]$*

$$\|\partial_x u_n\|^2_{L^2(I)} + \beta_1 \int_0^t \|\partial_x^2 u_n(s)\|^2_{L^2(I)}\, \mathrm{d}s \leq K_2.$$

*Proof.* As $-\partial_x^2 e_j = \lambda_j e_j$, we deduce that

$$\sum_{j=1}^n c_j(t)\lambda_j e_j = -\sum_{j=1}^n c_j(t)\partial_x^2 e_j = -\partial_x^2 u_n(t),$$

then, multiplying both sides of (2.2) by $c_j\lambda_j$ and summing for $j = 1, \dots, n$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^a (\partial_x u_n)^2 \mathrm{d}x + \beta(t)\int_0^a (\partial_x^2 u_n)^2\, \mathrm{d}x \\ &= -\int_0^a f\partial_x^2 u_n\, \mathrm{d}x + \int_0^a \gamma(t,x)\partial_x u_n \partial_x^2 u_n\, \mathrm{d}x + \alpha(t)\int_0^a u_n\partial_x u_n \partial_x^2 u_n\, \mathrm{d}x. \end{aligned} \tag{2.5}$$

Using Cauchy-Schwartz inequality, (2.4) with $\varepsilon = \beta_1/2$ leads to

$$\begin{aligned} |\int_0^a f\partial_x^2 u_n\, \mathrm{d}x| &\leq \Big(\int_0^a |\partial_x^2 u_n|^2\, \mathrm{d}x\Big)^{1/2}\Big(\int_0^a |f|^2\, \mathrm{d}x\Big)^{1/2} \\ &\leq \frac{\beta_1}{4}\int_0^a |\partial_x^2 u_n|^2\, \mathrm{d}x + \frac{1}{\beta_1}\int_0^a |f|^2\, \mathrm{d}x, \end{aligned} \tag{2.6}$$

and

$$|\int_0^a \gamma(t,x)\partial_x u_n \partial_x^2 u_n\, \mathrm{d}x| = \frac{1}{2}\Big|\int_0^a \partial_x\gamma(t,x)\partial_x u_n^2\, \mathrm{d}x\Big| \leq \frac{\gamma_1}{2}\int_0^a |\partial_x u_n|^2\, \mathrm{d}x. \tag{2.7}$$

Now, we have to estimate the last term of (2.5). An integration by parts gives

$$\int_0^a u_n\partial_x u_n \partial_x^2 u_n\, \mathrm{d}x = \int_0^a u_n \partial_x(\frac{1}{2}(\partial_x u_n)^2)\, \mathrm{d}x = -\frac{1}{2}\int_0^a (\partial_x u_n)^3\, \mathrm{d}x.$$

Since $\partial_x u_n$ satisfies $\int_0^a \partial_x u_n\, \mathrm{d}x = 0$ we deduce that the continuous function $\partial_x u_n$ is zero at some point $y_{0n} \in (0,a)$, and by integrating $2\partial_x u_n \partial_x^2 u_n$ between $y_{0n}$ and $y$, we obtain

$$|\partial_x u_n|^2 = |\int_{y_{0n}}^y \partial_x(\partial_x u_n)^2\, \mathrm{d}x| = 2|\int_{y_{0n}}^y u_n\partial_x^2 u_n\, \mathrm{d}x|,$$

the Cauchy-Schwartz inequality gives

$$\|\partial_x u_n\|^2_{L^\infty(I)} \leq 2\|\partial_x u_n\|_{L^2(I)}\|\partial_x^2 u_n\|_{L^2(I)}.$$

But
$$\|\partial_x u_n\|_{L^3(I)}^3 \le \|\partial_x u_n\|_{L^2(I)}^2 \|\partial_x u_n\|_{L^\infty(I)}.$$
So, (1.4) yields
$$|\int_0^a \alpha(t) u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x| \le \Big(\int_0^a |\partial_x^2 u_n|^2 \,\mathrm{d}x\Big)^{1/4} \Big(\alpha_2^{4/5} \int_0^a |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{5/4}.$$

Finally, thanks to Young's inequality $|AB| \le \frac{|A|^p}{p} + \frac{|B|^{p'}}{p'}$, with $1 < p < \infty$ and $p' = \frac{p}{p-1}$, we have
$$|AB| = |(\beta_1^{1/p} A)(\beta_1^{1/p'} \frac{B}{\beta_1})| \le \frac{\beta_1}{p} |A|^p + \frac{\beta_1}{p' \beta_1^{p'}} |B|^{p'}.$$
Choosing $p = 4$ (then $p' = \frac{4}{3}$) in the previous formula,
$$A = (\int_0^a |\partial_x^2 u_n|^2 \,\mathrm{d}x)^{1/4}, \quad B = \Big(\alpha_2^{4/5} \int_0^a |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{5/4},$$
the estimate of the last term of (2.5) becomes
$$|\int_0^a \alpha(t) u_n \partial_x u_n \partial_x^2 u_n \,\mathrm{d}x| \le \frac{\beta_1}{4} \int_0^a |\partial_x^2 u_n|^2 \,\mathrm{d}x + \frac{3}{4} \frac{\alpha_2^{4/3}}{\beta_1^{1/3}} \Big(\int_0^a |\partial_x u_n|^2 \,\mathrm{d}x\Big)^{5/3}. \quad (2.8)$$
Let us return to inequality (2.5): By integrating between 0 and $t$, from the estimates (2.6), (2.7), and (2.8) we obtain
$$\begin{aligned}
&\|\partial_x u_n\|_{L^2(I)}^2 + \beta_1 \int_0^t \|\partial_x^2 u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s \\
&\le \|\partial_x u_{0n}\|_{L^2(I)}^2 + \frac{2}{\beta_1} \int_0^t \|f(s)\|_{L^2(I)}^2 \,\mathrm{d}s \\
&\quad + C_2 \int_0^t \Big(\|\partial_x u_n(s)\|_{L^2(I)}^2\Big)^{5/3} \,\mathrm{d}s + \gamma_1 \int_0^t \|\partial_x u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s,
\end{aligned}$$
where $C_2 = \frac{3}{2} \frac{\alpha_2^{4/3}}{\beta_1^{1/3}}$. Observe that $f \in L^2(R))$, and $\|\partial_x u_{0n}\|_{L^2(I)}^2$ is bounded (see Remark 2.1). Then, there exists a constant $C_3$ such that
$$\begin{aligned}
&\|\partial_x u_n\|_{L^2(I)}^2 + \beta_1 \int_0^t \|\partial_x^2 u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s \\
&\le C_3 + C_2 \int_0^t \Big(\|\partial_x u_n(s)\|_{L^2(I)}^2\Big)^{2/3} \|\partial_x u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s + \gamma_1 \int_0^t \|\partial_x u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s.
\end{aligned}$$
Consequently, the function
$$\varphi(t) = \|\partial_x u_n\|_{L^2(I)}^2 + \beta_1 \int_0^t \|\partial_x^2 u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s$$
satisfies the inequality
$$\varphi(t) \le C_3 + \int_0^t (C_2 \|\partial_x u_n(s)\|_{L^2(I)}^{4/3} + \gamma_1) \varphi(s) \mathrm{d}s.$$
Gronwall's inequality shows that
$$\varphi(t) \le C_3 \exp\Big(\int_0^t (C_2 \|\partial_x u_n(s)\|_{L^2(I)}^{4/3} + \gamma_1) \mathrm{d}s\Big).$$

According to Lemma 2.3 the integral $\int_0^t \|\partial_x u_n\|_{L^2(I)}^{4/3} \mathrm{d}s$ is bounded by a constant independent of $n$ (and $t$). So there exists a positive constant $K_2$ such that

$$\|\partial_x u_n\|_{L^2(I)}^2 + \beta_1 \int_0^t \|\partial_x^2 u_n(s)\|_{L^2(I)}^2 \,\mathrm{d}s \le K_2.$$

□

**Lemma 2.5.** *There exists a positive constant $K_3$ independent of $n$, such that for all $t \in [0,T]$*

$$\|\partial_t u_n\|_{L^2(R)}^2 \le K_3.$$

*Proof.* Let

$$g_n = f - \alpha(t) u_n \partial_x u_n + \beta(t)\partial_x^2 u_n - \gamma(t,x)\partial_x u_n.$$

To show that $\partial_t u_n$ is bounded in $L^2(R)$, we will first show that $g_n$ is bounded in $L^2(R)$. According to Lemmas 2.3 and 2.4, the terms $\gamma(t,x)\partial_x u_n$ and $\beta(t)\partial_x^2 u_n$ are bounded in $L^2(R)$. On the other hand, by the hypothesis $f \in L^2(R)$. It remains only to show that $\alpha(t)u_n\partial_x u_n \in L^2(R)$.

Lemma 2.3 proves that $\|u_n\|_{L^\infty(0,T;H_0^1(I))}^2$ is bounded. Then, using the injection of $H_0^1(I)$ in $L^\infty(I)$, we obtain

$$\begin{aligned}
|\int_0^T \int_0^a (\alpha(t) u_n \partial_x u_n)^2 \,\mathrm{d}x\,\mathrm{d}t| &\le \alpha_2^2 \int_0^T \Big(\|u_n\|_{L^\infty(I)}^2 \int_0^a |\partial_x u_n|^2 \,\mathrm{d}x\Big)\,\mathrm{d}t \\
&\le \alpha_2^2 C_I \int_0^T \|u_n\|_{H_0^1(I)}^2 \|\partial_x u_n\|_{L^2(I)}^2 \,\mathrm{d}t \\
&\le \alpha_2^2 C_I \|u_n\|_{L^\infty(0,T;H_0^1(I))}^2 \|\partial_x u_n\|_{L^2(R)}^2,
\end{aligned}$$

where $C_I$ is a constant independent of $n$. Hence $g_n$ is bounded in $L^2(R)$. So, $\partial_t u_n$ is also bounded in $L^2(R)$. Indeed, from (2.2) for $j = 1,\dots,n$, we have

$$\begin{aligned}
\int_0^a \partial_t u_n e_j \,\mathrm{d}x &= \int_0^a (f - \alpha(t)u_n\partial_x u_n + \beta(t)\partial_y^2 u_n - \gamma(t,x)\partial_x u_n) e_j \,\mathrm{d}x, \\
&= \int_0^a g_n e_j \,\mathrm{d}x,
\end{aligned}$$

multiplying both sides by $c_j'$ and summing for $j = 1,\dots,n$,

$$\|\partial_t u_n\|_{L^2(I)}^2 = \int_0^a g_n \partial_t u_n \,\mathrm{d}x,$$

we deduce that $\|\partial_t u_n\|_{L^2(R)} \le \|g_n\|_{L^2(R)}$. □

2.3. **Existence and uniqueness.** Lemmas 2.3, 2.4 and 2.5 show that the Galerkin approximation $u_n$ is bounded in $L^\infty(0,T,L^2(I))$, and in $L^2(0,T,H^2(I))$, and $\partial_t u_n$ is bounded in $L^2(R)$. So, it is possible to extract a subsequence from $u_n$ (that we continue to denote $u_n$) such that

$$u_n \to u \quad \text{weakly in } L^2(0,T,H_0^1(I)), \tag{2.9}$$

$$u_n \to u \quad \text{strongly in } L^2(0,T,L^2(I)) \text{ and a.e. in } R, \tag{2.10}$$

$$\partial_t u_n \to \partial_t u \quad \text{strongly in } L^2(R). \tag{2.11}$$

**Lemma 2.6.** *Under the assumptions of Theorem 1.1, problem* (1.3) *admits a weak solution $u \in H^{1,2}(R)$.*

*Proof.* Note that (2.11) implies

$$\int_0^T \int_0^a \partial_t u_n w \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_0^a \partial_t u w \,\mathrm{d}x\,\mathrm{d}t, \quad \forall w \in L^2(R).$$

From (2.9) and (2.10),

$$u_n \partial_x u_n \to u\partial_x u \quad \text{weakly in} \quad L^2(R)\ ,$$

then

$$\int_0^T \int_0^a \alpha(t) u_n \partial_x u_n w \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_0^a \alpha(t) u \partial_x u w \,\mathrm{d}x\,\mathrm{d}t, \quad \forall w \in L^2(R),$$

and

$$\int_0^T \int_0^a \gamma(t,x) \partial_x u_n w \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_0^a \gamma(t,x) \partial_x u w \,\mathrm{d}x\,\mathrm{d}t, \quad \forall w \in L^2(R).$$

Our goal is to use these properties to pass to the limit. In problem (2.2), when $n \to +\infty$, for each fixed index $j$, we have

$$\begin{aligned} &\int_0^a \big(\partial_t u + \alpha(t) u \partial_x u\big) e_j \,\mathrm{d}x + \beta(t) \int_0^a \partial_x u \partial_x e_j \,\mathrm{d}x + \int_0^a \gamma(t,x) \partial_x u e_j \,\mathrm{d}x \\ &= \int_0^a f e_j \,\mathrm{d}x, \end{aligned} \tag{2.12}$$

Since $(e_j)_{j\in\mathbb{N}}$ is a base of $H_0^1(I)$, for all $w \in H_0^1(I)$, we can write

$$w(t) = \sum_{k=1}^{\infty} b_k(t) e_k,$$

that is to say $w_N(t) = \sum_{k=1}^N b_k(t) e_k \to w(t)$ in $H_0^1(I)$ when $N \to +\infty$.

Multiplying (2.12) by $b_k$ and summing for $k = 1, \dots, N$, then

$$\begin{aligned} &\int_0^a \big(\partial_t u + \alpha(t) u \partial_x u\big) w_N \,\mathrm{d}x + \beta(t) \int_0^a \partial_x u \partial_x w_N \,\mathrm{d}x + \int_0^a \gamma(t,x) \partial_x u w_N \,\mathrm{d}x \\ &= \int_0^a f w_N \,\mathrm{d}x. \end{aligned}$$

Letting $N \to +\infty$, we deduce that

$$\int_0^a \big(\partial_t u + \alpha(t) u \partial_x u\big) w \,\mathrm{d}x + \beta(t) \int_0^a \partial_x u \partial_x w \,\mathrm{d}x + \int_0^a \gamma(t,x) \partial_x u w \,\mathrm{d}x = \int_0^a f w \,\mathrm{d}x,$$

so, $u$ satisfies the weak formulation (2.1) for all $w \in H_0^1(I)$ and $t \in [0; T]$.

Finally, we recall that, by hypothesis, $\lim_{n\to+\infty} u_n(0) := u_0$. This completes the proof of the "existence" part of Theorem 1.1. □

**Lemma 2.7.** *Under the assumptions of Theorem 1.1, the solution of problem* (1.3) *is unique.*

*Proof.* Let us observe that any solution $u \in H^{1,2}(R)$ of problem (1.3) is in $u \in L^\infty(0, T, L^2(I))$. Indeed, it is not difficult to see that such a solution satisfies

$$\frac{1}{2} \frac{d}{dt} \int_0^a u^2 \,\mathrm{d}x + \beta(t) \int_0^a (\partial_x u)^2 \,\mathrm{d}x - \frac{1}{2} \int_0^a \partial_x \gamma(t,x) u^2 \,\mathrm{d}x = \int_0^a f u \,\mathrm{d}x,$$

because

$$\alpha(t) \int_0^a u^2 \partial_x u \,\mathrm{d}x = \frac{\alpha(t)}{3} \int_0^a \partial_x (u)^3 \,\mathrm{d}x = 0,$$

and

$$\int_0^a \gamma(t,x)\partial_x u u \,\mathrm{d}x = \int_0^a \gamma(t,x)\partial_x(\frac{u^2}{2}) \,\mathrm{d}x = -\frac{1}{2}\int_0^a \partial_x \gamma(t,x) u^2 \,\mathrm{d}x.$$

Consequently (see the proof of Lemma 2.3)

$$\begin{aligned}
&\|u\|^2_{L^2(I)} + \beta_1 \int_0^t \|\partial_x u(s)\|^2_{L^2(I)} \,\mathrm{d}s \\
&\le \|u_0\|^2_{L^2(I)} + \frac{a^2}{2\beta_1}\int_0^t \|f(s)\|^2_{L^2(I)} \,\mathrm{d}s + \gamma_1 \int_0^t \|u(s)\|^2_{L^2(I)} \,\mathrm{d}s,
\end{aligned}$$

so,

$$\begin{aligned}
&\|u\|^2_{L^2(I)} + \beta_1 \int_0^t \|\partial_x u(s)\|^2_{L^2(I)} \,\mathrm{d}s \\
&\le \|u_0\|^2_{L^2(I)} + \frac{a^2}{2\beta_1}\int_0^t \|f(s)\|^2_{L^2(I)} \,\mathrm{d}s \\
&\quad + \gamma_1 \int_0^t \Big(\|u(s)\|^2_{L^2(I)} + \beta_1 \int_0^s \|\partial_x u(\tau)\|^2_{L^2(I)} \,\mathrm{d}\tau\Big) \,\mathrm{d}s.
\end{aligned}$$

Then there exist a positive constant $C$ such that

$$\begin{aligned}
&\|u\|^2_{L^2(I)} + \beta_1 \int_0^t \|\partial_x u(s)\|^2_{L^2(I)} \,\mathrm{d}s \\
&\le C + \gamma_1 \int_0^t \Big(\|u(s)\|^2_{L^2(I)} + \beta_1 \int_0^s \|\partial_x u(\tau)\|^2_{L^2(I)} \,\mathrm{d}\tau\Big) \,\mathrm{d}s.
\end{aligned}$$

Hence, Gronwall's lemma gives

$$\|u\|^2_{L^2(I)} + \beta_1 \int_0^t \|\partial_x u(s)\|^2_{L^2(I)} \,\mathrm{d}s \le K,$$

where $K = C\exp(\gamma_1 T)$. This shows that $u \in L^\infty(0,T,L^2(I))$ for all $f \in L^2(I)$.

Now, let $u_1, u_2 \in H^{1,2}(R)$ be two solutions of (1.3). We put $u = u_1 - u_2$. It is clear that $u \in L^\infty(0,T,L^2(I))$. The equations satisfied by $u_1$ and $u_2$ lead to

$$\int_0^a [\partial_t u w + \alpha(t) u w \partial_x u_1 + \alpha(t) u_2 w \partial_x u + \beta(t)\partial_x u \partial_x w + \gamma(t,x) w \partial_x u] \,\mathrm{d}x = 0.$$

Taking, for $t \in [0,T]$, $w = u$ as a test function, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I)} + \beta(t)\|\partial_x u\|^2_{L^2(I)} \\
&= -\int_0^a \gamma(t,x) u \partial_x u \,\mathrm{d}x - \alpha(t)\int_0^a u^2 \partial_x u_1 \,\mathrm{d}x - \alpha(t)\int_0^a u_2 u \partial_x u \,\mathrm{d}x.
\end{aligned} \tag{2.13}$$

An integration by parts gives

$$\alpha(t)\int_0^a u^2 \partial_x u_1 \,\mathrm{d}x = -2\alpha(t)\int_0^a u \partial_x u u_1 \,\mathrm{d}x,$$

then (2.13) becomes

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I)} + \beta(t)\|\partial_x u\|^2_{L^2(I)} = \frac{1}{2}\int_0^a \partial_x \gamma(t,x) u^2 \,\mathrm{d}x + \int_0^a \alpha(t)(2u_1 - u_2) u \partial_x u \,\mathrm{d}x.$$

By (1.4) and inequality (2.4) with $\varepsilon = 2\beta_1$, we obtain

$$
\begin{aligned}
&|\int_0^a \alpha(t)(2u_1 - u_2)u\partial_x u \,\mathrm{d}x| \\
&\leq \frac{1}{4\beta_1}\alpha_2^2(2\|u_1\|_{L^\infty(0,T,L^2(I))} + \|u_2\|_{L^\infty(0,T,L^2(I))})^2\|u\|^2_{L^2(I)} + \beta_1\|\partial_x u\|^2_{L^2(I)}.
\end{aligned}
$$

Furthermore,

$$
\frac{1}{2}\int_0^a \partial_x\gamma(t,x)u^2 \,\mathrm{d}x \leq \frac{\gamma_1}{2}\|u\|^2_{L^2(I)}.
$$

So, we deduce that there exists a non-negative constant $D$, such as

$$
\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(I)} \leq D\|u\|^2_{L^2(I)},
$$

and Gronwall's lemma leads to $u = 0$. This completes the proof. $\square$

## 3. Proof of the theorem 1.2

Let

$$
\Omega = \{(t,x) \in R^2;\ 0 < t < T;\ \varphi_1(t) < x < \varphi_2(t)\},
$$

where $T$ is a positive finite number. The change of variables: $\Omega \to R$,

$$
(t,x) \mapsto (t,y) = (t, \frac{x - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)})
$$

transforms $\Omega$ into the rectangle $R = ]0,T[\times]0,1[$. Putting $u(t,x) = v(t,y)$ and $f(t,x) = g(t,y)$, then problem (1.5) becomes

$$
\begin{gathered}
\partial_t v(t,y) + \frac{1}{\varphi(t)}v(t,y)\partial_y v(t,y) \\
- \frac{\nu}{\varphi^2(t)}\partial_y^2 v(t,y) + \gamma(t,y)\partial_y v(t,y) = g(t,y) \quad \text{in } R, \\
v(0,y) = v_0(y) = u_0(\varphi_1(0) + \varphi(0)y), \quad y \in (0,1), \\
v(t,0) = v(t,1) = 0 \quad t \in (0,T),
\end{gathered}
\tag{3.1}
$$

where

$$
\begin{gathered}
\varphi(t) = \varphi_2(t) - \varphi_1(t), \\
\gamma(t,y) = -\frac{y\varphi'(t) + \varphi_1'(t)}{\varphi(t)}.
\end{gathered}
$$

Now, we take $I = (0,1)$, $\alpha(t) = \frac{1}{\varphi(t)}$, $\beta(t) = \frac{\nu}{\varphi^2(t)}$, then problem (3.1) can be written as

$$
\begin{gathered}
\partial_t v(t,y) + \alpha(t)v(t,y)\partial_y v(t,y) - \beta(t)\partial_y^2 v(t,y) + \gamma(t,y)\partial_y v(t,y) = g(t,y) \\
(t,y) \in R, \\
v(0,y) = v_0(y) \quad y \in I, \\
v(t,1) = v(t,0) = 0 \quad t \in (0,T),
\end{gathered}
$$

It is easy to see that this change of variables preserves the spaces $H_0^1$, $H^{1,2}$ and $L^2$. In other words

$$
\begin{gathered}
f \in L^2(\Omega) \ \Leftrightarrow\ g \in L^2(R) \\
u \in H^{1,2}(\Omega) \ \Leftrightarrow\ v \in H^{1,2}(R) \\
u_0 \in H_0^1(J) \ \Leftrightarrow\ v_0 \in H_0^1(I)
\end{gathered}
\tag{3.2}
$$

**Remark 3.1.** Observe that the hypotheses (1.4) are fulfilled. This means that the functions $\alpha, \beta$ and $\gamma$ satisfy the following conditions

$$\begin{gathered}
\alpha_1 < \alpha(t) < \alpha_2, \quad \forall t \in [0,T], \\
\beta_1 < \beta(t) < \beta_2, \quad \forall t \in [0,T], \\
|\partial_y \gamma(t,y)| \le \gamma_1, \quad \forall (t,y) \in R.
\end{gathered}$$

So, Burgers problem (1.5) is equivalent to problem (3.1), and by Theorem 1.1, there exists a unique solution $v \in H^{1,2}(R)$ of problem (3.1). Then (3.2) implies that the nonhomogeneous Burgers problem (1.5) in the domain $\Omega$ admits a unique solution $u \in H^{1,2}(R)$.

This work can be generalized to the case when $\varphi_1$, $\varphi_2$ are Lipshitz continuous functions on $[0,T]$ instead of $C^1(]0,T[)$. On the other hand, this is an interesting question: What happens if $\varphi_1(0) = \varphi_2(0)$? This is a singular case which needs some hypotheses on $\varphi_1$, $\varphi_2$ near $t = 0$. In a forthcoming work, we will answer this question.

## References


[1] R. A. Adams; *Sobolev spaces*, Academic Press, New York, 1975.

[2] H. Bateman; *Some recent researches on the motion of fluids*, Mon. Weather Rev. 43 (1915), 163-170.

[3] N. Bressan, A. Quarteroni; *An implicit/explicit spectral method for Burgers equation*, Calcolo, 23 (1987), 265-284.

[4] H. Brezis; *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011.

[5] J. M. Burgers; *The nonlinear diffusion equation*, Reidel, Dordrecht, 1974.

[6] J. M. Burgers; *A mathematical model illustrating the theory of turbulence*, Adv. Appl. Mech. 1 (1948) 171-199.

[7] H. R. Clark, M. A. Rincon, A. Silva; *Analysis and numerical simulation of viscous Burgers equation*, Numerical Functional Analysis and Optimization, 32:7 (2011) 695-716.

[8] J. D. Cole; *On a quasi-linear parabolic equation occurring in aerodynamics*, Quart. Appl. Math.9 (1951) 225-236.

[9] A. R. Forsyth; *Theory of differential equations*, Part IV - Partial Differential Equations, Campridg University press, Cambridge, 1906, Republished by Dover, New York, 1959

[10] E. Hopf; *The partial differential equation* $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 3 (1950) 201-230.

[11] J. Kevorkian; *Partial differential equations: Analytical solution techniques*, Brooks/Cole Pub. Pacific Grove, California, 1990.

[12] M. Morandi Cecchi, R. Nociforo, P. Patuzzo; *Burgers problems - Theoretical results*, Rivista di Matematica Pura ed Applicata (1997), 159-174.

[13] B.-K. Sadallah; *Étude d'un problème* $2m$ *-parabolique dans des domaines plan non rectangulaires*, Boll. U. M. I., (6), 2-B (1983), 51-112.

[14] R. Temam; *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1997.

[15] G. B. Whitham; *Lectures on wave propagation*, Narosa Pub. House, New Delhi, 1979.



Yassine Benia

Department of Mathematics, University of Tiaret, B.P. 78, 14000, Tiaret, Algeria

*E-mail address*: `benia.yacine@yahoo.fr`

Boubaker-Khaled Sadallah

Lab. PDE & Hist Maths; Dept of Mathematics, E.N.S., 16050, Kouba, Algiers, Algeria

*E-mail address*: `sadallah@ens-kouba.dz`