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FAMILY OF QUADRATIC DIFFERENTIAL SYSTEMS WITH INVARIANT HYPERBOLAS: A COMPLETE CLASSIFICATION IN THE SPACE $\mathbb{R}^{12}$

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Abstract. In this article we consider the class QS of all non-degenerate quadratic systems. A quadratic polynomial differential system can be identified with a single point of $\mathbb{R}^{12}$ through its coefficients. In this paper using the algebraic invariant theory we provided necessary and sufficient conditions for a system in QS to have at least one invariant hyperbola in terms of its coefficients. We also considered the number and multiplicity of such hyperbolas. We give here the global bifurcation diagram of the class QS of systems with invariant hyperbolas. The bifurcation diagram is done in the 12-dimensional space of parameters and it is expressed in terms of polynomial invariants. The results can therefore be applied for any family of quadratic systems in this class, given in any normal form.

1. Introduction and statement of main results

In this article, we consider differential systems of the form

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where $P, Q \in \mathbb{R}[x,y]$, i.e. P, Q are polynomials in x, y over $\mathbb{R}$ and their associated vector fields

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{1.2}$$

We call *degree* of a system (1.1) the integer $m = \max(\deg P, \deg Q)$. In particular we call *quadratic* a differential system (1.1) with $m = 2$. We denote here by QS the whole class of real *non-degenerate quadratic systems*, i.e. we assume that the polynomials P and Q are coprime.

Quadratic systems appear in the modeling of many natural phenomena described in different branches of science, in biological and physical applications and applications of these systems became a subject of interest for the mathematicians. Many papers have been published about quadratic systems, see for example [13] for a bibliographical survey.

Let V be an open and dense subset of $\mathbb{R}^2$, we say that a nonconstant differentiable function $H : V \to \mathbb{R}$ is a first integral of a system (1.1) on V if $H(x(t), y(t))$ is

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constant for all of the values of t for which $(x(t), y(t))$ is a solution of this system contained in V. Obviously H is a first integral of systems (1.1) if and only if

$$X(H) = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0, \tag{1.3}$$

for all $(x, y) \in V$. When a system (1.1) has a first integral we say that this system is integrable.

The knowledge of the first integrals is of particular interest in planar differential systems because they allow us to draw their phase portraits.

On the other hand given $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is an *invariant algebraic curve* of systems (1.1) if there exists $K \in \mathbb{C}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{1.4}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. When $K = 0$, f is a polynomial first integral.

Quadratic systems with an invariant algebraic curve have been studied by many authors, for example Schlomiuk and Vulpe [14, 16] have studied quadratic systems with invariant straight lines, Qin Yuan-xum [10] has investigated the quadratic systems having an ellipse as limit cycle, Druzhkova [7] has presented necessary and sufficient conditions for existence and uniqueness of an invariant algebraic curve of second degree in terms of the coefficients of quadratic systems, and Cairo and Llibre [3] have studied the quadratic systems having invariant algebraic conics in order to investigate the Darboux integrability of such systems.

The motivation for studying the systems in the quadratic class is not only because of their usefulness in many applications but also for theoretical reasons, as discussed by Schlomiuk and Vulpe in the introduction of [14]. The study of non–degenerate quadratic systems could be done using normal forms and applying the invariant theory.

The main goal of this paper is to investigate non–degenerate quadratic systems having invariant hyperbolas and this study is done applying the invariant theory. More precisely in this paper we give necessary and sufficient conditions for a quadratic system in QS to have invariant hyperbolas. We also determine the invariant criteria which provide the number and multiplicity of such hyperbolas.

Definition 1.1. We say that an invariant conic $\Phi(x, y) = p + qx + ry + sx^2 + 2txy + uy^2 = 0$, $(s, t, u) \neq (0, 0, 0)$, $(p, q, r, s, t, u) \in \mathbb{C}^6$ for a quadratic vector field X has *multiplicity* m if there exists a sequence of real quadratic vector fields X_k converging to X, such that each X_k has m distinct (complex) invariant conics $\Phi_k^1 = 0, \dots, \Phi_k^m = 0$, converging to $\Phi = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$. In the case when an invariant conic $\Phi(x, y) = 0$ has multiplicity one we call it *simple*.

Our main results are stated in the following theorem.

Theorem 1.2.
(A) The conditions $\gamma_1 = \gamma_2 = 0$ and either $\eta \geq 0$, $M \neq 0$ or $C_2 = 0$ are necessary for a quadratic system in the class QS to possess at least one invariant hyperbola.
(B) Assume that for a system in the class QS the condition $\gamma_1 = \gamma_2 = 0$ is satisfied.

(1) *If $\eta > 0$ then the necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in Figure 1, where we can also find the number and multiplicity of such hyperbolas.*
(2) *In the case $\eta = 0$ and either $M \neq 0$ or $C_2 = 0$ the corresponding necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in Figure 2, where we can also find the number and multiplicity of such hyperbolas.*

(C) Figures 1 and 2 actually contain the global bifurcation diagram in the 12-dimensional space of parameters of the systems belonging to family QS, which possess at least one invariant hyperbola. The corresponding conditions are given in terms of invariant polynomials with respect to the group of affine transformations and time rescaling.

Remark 1.3. An invariant hyperbola is denoted by $\mathcal{H}$ if it is real and by $\overset{c}{\mathcal{H}}$ if it is complex. In the case we have two such hyperbolas then it is necessary to distinguish whether they have parallel or non-parallel asymptotes in which case we denote them by $\mathcal{H}^p$ ($\overset{c}{\mathcal{H}}{}^p$) if their asymptotes are parallel and by $\mathcal{H}$ if there exists at least one pair of non-parallel asymptotes. We denote by $\mathcal{H}_k$ $(k = 2, 3)$ a hyperbola with multiplicity k; by $\mathcal{H}_2^p$ a double hyperbola, which after perturbation splits into two $\mathcal{H}^p$; and by $\mathcal{H}_3^p$ a triple hyperbola which splits into two $\mathcal{H}^p$ and one $\mathcal{H}$.

The term "complex invariant hyperbolas" of a real system requires some explanation. Indeed the term hyperbola is reserved for a real irreducible affine conic which has two real points at infinity. This distinguishes it from the other two irreducible real conics: the parabola with just one real point at infinity and the ellipse which has two complex points at infinity. We call "complex hyperbola" an irreducible affine conic $\Phi(x, y) = 0$, with $\Phi(x, y) = p + qx + ry + sx^2 + 2txy + uy^2 = 0$ over $\mathbb{C}$, such that there does not exist a non-zero complex number λ with $\lambda(p, q, r, s, t, u) \in \mathbb{R}^6$ and in addition this conic has two real points at infinity.

The invariants and comitants of differential equations (see Subsection 2.2) used for proving our main result are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples (see for instance [18, 19, 12, 1, 4]).

2. Preliminaries

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x, y) + p_2(x, y) \equiv P(x, y), \\ \frac{dy}{dt} &= q_0 + q_1(x, y) + q_2(x, y) \equiv Q(x, y) \end{aligned} \tag{2.1}$$

with homogeneous polynomials p_i and q_i $(i = 0, 1, 2)$ of degree i in x, y:

$$\begin{aligned} p_0 = a_{00}, \quad p_1(x, y) = a_{10}x + a_{01}y, \quad p_2(x, y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 = b_{00}, \quad q_1(x, y) = b_{10}x + b_{01}y, \quad q_2(x, y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Such a system (2.1) can be identified with a point in $\mathbb{R}^{12}$. Let

$$\tilde{a} = \big(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}\big)$$

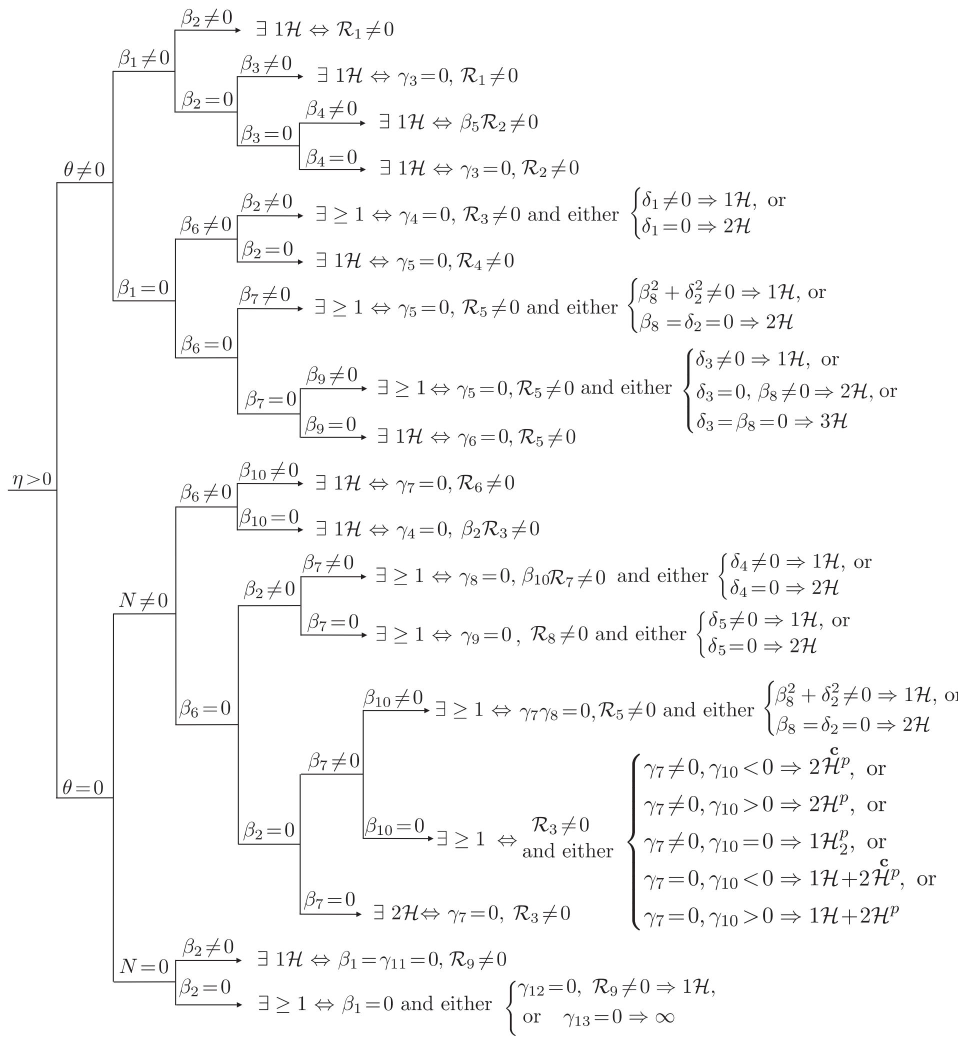

FIGURE 1. Existence of invariant hyperbolas: the case $\eta>0$

and consider the ring $\mathbb{R}[a_{00}, a_{10}, \dots, a_{02}, b_{00}, b_{10}, \dots, b_{02}, x, y]$ which we shall denote $\mathbb{R}[\tilde{a}, x, y]$.

2.1. **Group actions on quadratic systems** (2.1) **and invariant polynomials with respect to these actions.** On the set QS of all quadratic differential systems (2.1) acts the group $\mathrm{Aff}(2,\mathbb{R})$ of affine transformations on the plane. Indeed for every $g\in \mathrm{Aff}(2,\mathbb{R})$, $g:\mathbb{R}^2\to\mathbb{R}^2$ we have:

$$g: \begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = M\begin{pmatrix}x\\ y\end{pmatrix} + B;\quad g^{-1}: \begin{pmatrix}x\\ y\end{pmatrix} = M^{-1}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} - M^{-1}B.$$

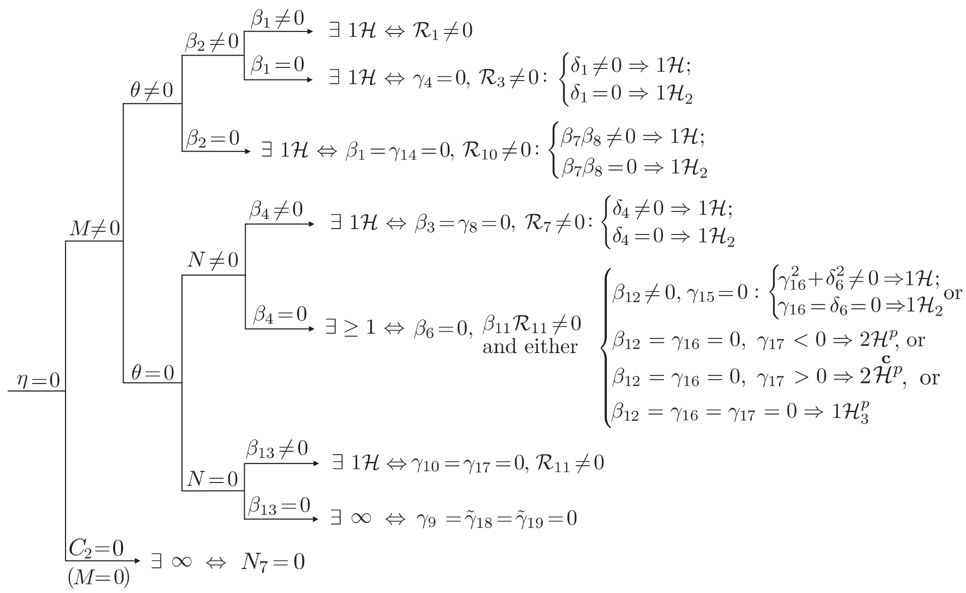

FIGURE 2. Existence of invariant hyperbolas: the case $\eta = 0$

where $M = \|M_{ij}\|$ is a 2×2 nonsingular matrix and B is a 2×1 matrix over $\mathbb{R}$. For every $S \in QS$ we can form its transformed system $\tilde{S} = gS$:

$$\frac{d\tilde{x}}{dt} = \tilde{P}(\tilde{x}, \tilde{y}), \quad \frac{d\tilde{y}}{dt} = \tilde{Q}(\tilde{x}, \tilde{y}), \tag{2.2}$$

where

$$\begin{pmatrix}\tilde{P}(\tilde{x}, \tilde{y})\\ \tilde{Q}(\tilde{x}, \tilde{y})\end{pmatrix} = M \begin{pmatrix}(P \circ g^{-1})(\tilde{x}, \tilde{y})\\ (Q \circ g^{-1})(\tilde{x}, \tilde{y})\end{pmatrix}.$$

The map $\mathrm{Aff}(2, \mathbb{R}) \times QS \to QS$ defined by

$$(g, S) \to \tilde{S} = gS$$

satisfies the axioms for a left group action. For every subgroup $G \subseteq \mathrm{Aff}(2, \mathbb{R})$ we have an induced action of G on QS. We can identify the set QS of systems (2.1) with a subset of $\mathbb{R}^{12}$ via the embedding $QS \hookrightarrow \mathbb{R}^{12}$ which associates to each system (2.1) the 12-tuple $(a_{00}, \dots, b_{02})$ of its coefficients.

On systems (S) such that $\max(\deg(p), \deg(q)) \leq 2$ we consider the action of the group $\mathrm{Aff}(2, \mathbb{R})$ which yields an action of this group on $\mathbb{R}^{12}$. For every $g \in \mathrm{Aff}(2, \mathbb{R})$ let $r_g : \mathbb{R}^{12} \to \mathbb{R}^{12}$ be the map which corresponds to g via this action. We know (cf. [18]) that r_g is linear and that the map $r : \mathrm{Aff}(2, \mathbb{R}) \to GL(12, \mathbb{R})$ thus obtained is a group homomorphism. For every subgroup G of $\mathrm{Aff}(2, \mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(12, \mathbb{R})$.

We shall denote a polynomial U in the ring $\mathbb{R}[\tilde{a}, x, y]$ by $U(\tilde{a}, x, y)$.

Definition 2.1. A polynomial $U(\tilde{a}, x, y) \in \mathbb{R}[\tilde{a}, x, y]$ is a *comitant* for systems (2.1) with respect to a subgroup G of $\mathrm{Aff}(2, \mathbb{R})$, if there exists $\chi \in \mathbb{Z}$ such that for every $(g, \tilde{a}) \in G \times \mathbb{R}^{12}$ and for every $(x, y) \in \mathbb{R}^2$ the following relation holds:

$$U(r_g(\tilde{a}), g(x, y)) \equiv (\det g)^{-\chi} U(\tilde{a}, x, y).$$

If the polynomial U does not explicitly depend on x and y then it is an *invariant*. The number $\chi \in \mathbb{Z}$ is the *weight* of the comitant $U(\tilde{a}, x, y)$. If $G = GL(2, \mathbb{R})$ (or $G = \mathrm{Aff}(2, \mathbb{R})$) then the comitant $U(\tilde{a}, x, y)$ of systems (2.1) is called *GL-comitant* (respectively, *affine comitant*).

Definition 2.2. A subset $X \subset \mathbb{R}^{12}$ will be called *G-invariant*, if for every $g \in G$ we have $r_g(X) \subseteq X$.

Let $T(2, \mathbb{R})$ be the subgroup of $\mathrm{Aff}(2, \mathbb{R})$ formed by translations. Consider the linear representation of $T(2, \mathbb{R})$ into its corresponding subgroup $\mathcal{T} \subset GL(12, \mathbb{R})$, i.e. for every $\tau \in T(2, \mathbb{R})$, $\tau:\ x = \tilde{x} + \alpha,\ y = \tilde{y} + \beta$ we consider as above $r_\tau : \mathbb{R}^{12} \to \mathbb{R}^{12}$.

Definition 2.3. A GL-comitant $U(\tilde{a}, x, y)$ of systems (2.1) is a *T-comitant* if for every $(\tau, \tilde{a}) \in T(2, \mathbb{R}) \times \mathbb{R}^{12}$ the relation $U(r_\tau(\tilde{a}), \tilde{x}, \tilde{y}) = U(\tilde{a}, \tilde{x}, \tilde{y})$ holds in $\mathbb{R}[\tilde{x}, \tilde{y}]$.

Consider s homogeneous polynomials $U_i(\tilde{a}, x, y) \in \mathbb{R}[\tilde{a}, x, y]$, $i = 1, \dots, s$:

$$U_i(\tilde{a}, x, y) = \sum_{j=0}^{d_i} U_{ij}(\tilde{a}) x^{d_i - j} y^j, \quad i = 1, \dots, s,$$

and assume that the polynomials U_i are GL-comitants of a system (2.1) where d_i denotes the degree of the binary form $U_i(\tilde{a}, x, y)$ in x and y with coefficients in $\mathbb{R}[\tilde{a}]$. We denote by

$$\mathcal{U} = \{U_{ij}(\tilde{a}) \in \mathbb{R}[\tilde{a}] : i = 1, \dots, s,\ j = 0, 1, \dots, d_i\},$$

the set of the coefficients in $\mathbb{R}[\tilde{a}]$ of the GL-comitants $U_i(\tilde{a}, x, y)$, $i = 1, \dots, s$, and by $V(\mathcal{U})$ its zero set:

$$V(\mathcal{U}) = \{\, \tilde{a} \in \mathbb{R}^{12} : U_{ij}(\tilde{a}) = 0,\ \forall U_{ij}(\tilde{a}) \in \mathcal{U}\}.$$

Definition 2.4. Let $U_1, \dots, U_s$ be GL-comitants of a system (2.1) . A GL-comitant $U(\tilde{a}, x, y)$ of this system is called a *conditional T-comitant* (or *CT-comitant*) modulo the ideal generated by $U_{ij}(\tilde{a})$ $(i = 1, \dots, s; j = 0, 1, \dots, d_i)$ in the ring $\mathbb{R}[\tilde{a}]$ if the following two conditions are satisfied:

(i) the algebraic subset $V(\mathcal{U}) \subset \mathbb{R}^{12}$ is affinely invariant (see Definition 2.2);
(ii) for every $(\tau, \tilde{a}) \in T(2, \mathbb{R}) \times V(\mathcal{U})$ we have $U(r_\tau(\tilde{a}), \tilde{x}, \tilde{y}) = U(\tilde{a}, \tilde{x}, \tilde{y})$ in $\mathbb{R}[\tilde{x}, \tilde{y}]$.

In other words a CT-comitant $U(\tilde{a}, x, y)$ is a T-comitant on the algebraic subset $V(\mathcal{U}) \subset \mathbb{R}^{12}$.

Definition 2.5. A homogeneous polynomial $U(\tilde{a}, x, y) \in \mathbb{R}[\tilde{a}, x, y]$ of even degree in x, y has *well determined sign* on $V \subset \mathbb{R}^{12}$ with respect to x, y if for every $\tilde{a} \in V$, the binary form $u(x, y) = U(\tilde{a}, x, y)$ yields a function of constant sign on $\mathbb{R}^2$ except on a set of zero measure where it vanishes.

Remark 2.6. We draw attention to the fact that if a CT-comitant $U(\tilde{a}, x, y)$ of even weight is a binary form of even degree in x and y, of even degree in $\tilde{a}$ and has well determined sign on some affine invariant algebraic subset V, then its sign is conserved after an affine transformation and time rescaling.

2.2. **Main invariant polynomials associated with invariant hyperbolas.** We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2.1):

$$\begin{gathered} C_i(\tilde{a},x,y) = yp_i(x,y) - xq_i(x,y),\ (i=0,1,2) \\ D_i(\tilde{a},x,y) = \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y},\ (i=1,2). \end{gathered} \tag{2.3}$$

As it was shown in [18] these polynomials of degree one in the coefficients of systems (2.1) are GL-comitants of these systems. Let $f,\ g \in \mathbb{R}[\tilde{a},x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f,g)^{(k)} \in \mathbb{R}[\tilde{a},x,y]$ is called *the transvectant of index k of (f,g)* (cf. [8, 11]).

Theorem 2.7 ([19])**.** *Any GL-comitant of systems* (2.1) *can be constructed from the elements* (2.3) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 2.8. We point out that the elements (2.3) generate the whole set of GL-comitants and hence also the set of affine comitants as well as the set of T-comitants.

We construct the following GL-comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{aligned} &T_1 = (C_0,C_1)^{(1)}, \quad T_2 = (C_0,C_2)^{(1)}, \quad T_3 = (C_0,D_2)^{(1)}, \\ &T_4 = (C_1,C_1)^{(2)}, \quad T_5 = (C_1,C_2)^{(1)}, \quad T_6 = (C_1,C_2)^{(2)}, \\ &T_7 = (C_1,D_2)^{(1)}, \quad T_8 = (C_2,C_2)^{(2)}, \quad T_9 = (C_2,D_2)^{(1)}. \end{aligned} \tag{2.4}$$

Using these GL-comitants as well as the polynomials (2.3) we construct the additional invariant polynomials. In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T-comitants expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$:

$$\begin{aligned} \hat{A} &= (C_1, T_8 - 2T_9 + D_2^2)^{(2)}/144, \\ \widehat{D} &= \big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1,T_5)^{(1)} \\ &\quad + 6D_1C_1D_2 - T_5) - 9D_1^2C_2\big]/36, \\ \widehat{E} &= [D_1(2T_9 - T_8) - 3(C_1,T_9)^{(1)} - D_2(3T_7 + D_1D_2)]/72, \\ \widehat{F} &= [6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2,T_9)^{(1)} \\ &\quad - 9D_2^2T_4 + 288D_1\widehat{E} - 24(C_2,\widehat{D})^{(2)} + 120(D_2,\widehat{D})^{(1)} \\ &\quad - 36C_1(D_2,T_7)^{(1)} + 8D_1(D_2,T_5)^{(1)}]/144, \end{aligned}$$

$$\begin{aligned} \widehat{B} = \Big\{ &16D_1(D_2,T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) \\ &+ 32C_0(D_2,T_9)^{(1)}(3D_1D_2 - 5T_6 + 9T_7) \\ &+ 2(D_2,T_9)^{(1)}(27C_1T_4 - 18C_1D_1^2. - 32D_1T_2 + 32(C_0,T_5)^{(1)}) \\ &+ 6(D_2,T_7)^{(1)}\Big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) \end{aligned}$$

$$
\begin{aligned}
&+ C_2(9T_4+96T_3)\Big] + 6(D_2,T_6)^{(1)}\Big[32C_0T_9 - C_1(12T_7+52D_1D_2)\\
&- 32C_2D_1^2\Big] + 48D_2(D_2,T_1)^{(1)}(2D_2^2-T_8)\\
&- 32D_1T_8(D_2,T_2)^{(1)} + 9D_2^2T_4(T_6-2T_7) - 16D_1(C_2,T_8)^{(1)}(D_1^2+4T_3)\\
&+ 12D_1(C_1,T_8)^{(2)}(C_1D_2-2C_2D_1) + 6D_1D_2T_4(T_8-7D_2^2-42T_9)\\
&+ 12D_1(C_1,T_8)^{(1)}(T_7+2D_1D_2) + 96D_2^2[D_1(C_1,T_6)^{(1)} + D_2(C_0,T_6)^{(1)}]-\\
&- 16D_1D_2T_3(2D_2^2+3T_8) - 4D_1^3D_2(D_2^2+3T_8+6T_9) + 6D_1^2D_2^2(7T_6+2T_7)\\
&- 252D_1D_2T_4T_9\Big\}/(2^8 3^3),
\end{aligned}
$$

$$
\widehat{K} = (T_8+4T_9+4D_2^2)/72, \quad \widehat{H} = (8T_9-T_8+2D_2^2)/72.
$$

These polynomials in addition to (2.3) and (2.4) will serve as bricks in constructing affine invariant polynomials for systems (2.1).

The following 42 affine invariants $A_1,\dots,A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [2] by constructing $A_1,\dots,A_{42}$ using the above bricks.

$$
\begin{gathered}
A_1 = \hat{A}, \quad A_2 = (C_2,\widehat{D})^{(3)}/12, \quad A_3 = \big[C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48,\\
A_4 = (\widehat{H},\widehat{H})^{(2)}, \quad A_5 = (\widehat{H},\widehat{K})^{(2)}/2, \quad A_6 = (\widehat{E},\widehat{H})^{(2)}/2,\\
A_7 = \big[C_2,\widehat{E})^{(2)},D_2\big)^{(1)}/8, \quad A_8 = \big[\widehat{D},\widehat{H})^{(2)},D_2\big)^{(1)}/48, \quad A_9 = \big[\widehat{D},D_2)^{(1)},D_2\big)^{(1)},\\
A_{10} = \big[\widehat{D},\widehat{K})^{(2)},D_2\big)^{(1)}/8, \quad A_{11} = (\widehat{F},\widehat{K})^{(2)}/4, \quad A_{12} = (\widehat{F},\widehat{H})^{(2)}/4,\\
A_{13} = \big[C_2,\widehat{H})^{(1)},\widehat{H}\big)^{(2)},D_2\big)^{(1)}/24, \quad A_{14} = (\widehat{B},C_2)^{(3)}/36, \quad A_{15} = (\widehat{E},\widehat{F})^{(2)}/4,\\
A_{16} = \big[\widehat{E},D_2)^{(1)},C_2\big)^{(1)},\widehat{K}\big)^{(2)}/16, \quad A_{17} = \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{18} = \big[\widehat{D},\widehat{F})^{(2)},D_2\big)^{(1)}/16, \quad A_{19} = \big[\widehat{D},\widehat{D})^{(2)},\widehat{H}\big)^{(2)}/16,\\
A_{20} = \big[C_2,\widehat{D})^{(2)},\widehat{F}\big)^{(2)}/16, \quad A_{21} = \big[\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(2)}/16,\\
A_{22} = \frac{1}{1152}\big[C_2,\widehat{D})^{(1)},D_2\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}D_2\big)^{(1)}, \quad A_{23} = \big[\widehat{F},\widehat{H})^{(1)},\widehat{K}\big)^{(2)}/8,\\
A_{24} = \big[C_2,\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/32, \quad A_{25} = \big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(2)}/16,\\
A_{26} = (\widehat{B},\widehat{D})^{(3)}/36, \quad A_{27} = \big[\widehat{B},D_2)^{(1)},\widehat{H}\big)^{(2)}/24,\\
A_{28} = \big[C_2,\widehat{K})^{(2)},\widehat{D}\big)^{(1)},\widehat{E}\big)^{(2)}/16, \quad A_{29} = \big[\widehat{D},\widehat{F})^{(1)},\widehat{D}\big)^{(3)}/96,\\
A_{30} = \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/288, \quad A_{31} = \big[\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{32} = \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{H}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{33} = \big[\widehat{D},D_2)^{(1)},\widehat{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{34} = \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{K}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{35} = \big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{36} = \big[\widehat{D},\widehat{E})^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/16, \quad A_{37} = \big[\widehat{D},\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/576,\\
A_{38} = \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/64, \quad A_{39} = \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{H}\big)^{(2)}/64,
\end{gathered}
$$

$$A_{40} = \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{K}\big)^{(2)}/64, \quad A_{41} = \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/64,$$
$$A_{42} = \big[\widehat{D},\widehat{F})^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/16.$$

In the above list, the bracket “[” is used in order to avoid placing the otherwise necessary up to five parentheses “(”.

Using the elements of the minimal polynomial basis given above we construct the affine invariant polynomials

$$\gamma_1(\tilde{a}) = A_1^2(3A_6+2A_7) - 2A_6(A_8+A_{12}),$$
$$\begin{aligned}\gamma_2(\tilde{a}) = {}& 9A_1^2A_2(23252A_3+23689A_4) - 1440A_2A_5(3A_{10}+13A_{11})\\
&- 1280A_{13}(2A_{17}+A_{18}+23A_{19}-4A_{20}) - 320A_{24}(50A_8+3A_{10}\\
&+45A_{11}-18A_{12}) + 120A_1A_6(6718A_8+4033A_9+3542A_{11}\\
&+2786A_{12}) + 30A_1A_{15}(14980A_3-2029A_4-48266A_5)\\
&- 30A_1A_7(76626A_1^2-15173A_8+11797A_{10}+16427A_{11}-30153A_{12})\\
&+ 8A_2A_7(75515A_6-32954A_7) + 2A_2A_3(33057A_8-98759A_{12})\\
&- 60480A_1^2A_{24} + A_2A_4(68605A_8-131816A_9+131073A_{10}+129953A_{11})\\
&- 2A_2(141267A_6^2-208741A_5A_{12}+3200A_2A_{13}),\end{aligned}$$
$$\begin{aligned}\gamma_3(\tilde{a}) = {}& 843696A_5A_6A_{10} + A_1(-27(689078A_8+419172A_9-2907149A_{10}\\
&- 2621619A_{11})A_{13} - 26(21057A_3A_{23}+49005A_4A_{23}-166774A_3A_{24}\\
&+ 115641A_4A_{24})).\end{aligned}$$
$$\begin{aligned}\gamma_4(\tilde{a}) = {}& -9A_4^2(14A_{17}+A_{21}) + A_5^2(-560A_{17}-518A_{18}+881A_{19}-28A_{20}\\
&+ 509A_{21}) - A_4(171A_8^2+3A_8(367A_9-107A_{10})+4(99A_9^2+93A_9A_{11}\\
&+ A_5(-63A_{18}-69A_{19}+7A_{20}+24A_{21}))) + 72A_{23}A_{24},\end{aligned}$$
$$\begin{aligned}\gamma_5(\tilde{a}) = {}& -488A_2^3A_4 + A_2(12(4468A_8^2+32A_9^2-915A_{10}^2+320A_9A_{11}-3898A_{10}A_{11}\\
&- 3331A_{11}^2+2A_8(78A_9+199A_{10}+2433A_{11})) + 2A_5(25488A_{18}\\
&- 60259A_{19}-16824A_{21}) + 779A_4A_{21}) + 4(7380A_{10}A_{31}\\
&- 24(A_{10}+41A_{11})A_{33} + A_8(33453A_{31}+19588A_{32}-468A_{33}-19120A_{34})\\
&+ 96A_9(-A_{33}+A_{34}) + 556A_4A_{41} - A_5(27773A_{38}+41538A_{39}\\
&- 2304A_{41}+5544A_{42})),\end{aligned}$$
$$\gamma_6(\tilde{a}) = 2A_{20} - 33A_{21},$$
$$\begin{aligned}\gamma_7(\tilde{a}) = {}& A_1(64A_3-541A_4)A_7 + 86A_8A_{13} + 128A_9A_{13} - 54A_{10}A_{13}\\
&- 128A_3A_{22} + 256A_5A_{22} + 101A_3A_{24} - 27A_4A_{24},\end{aligned}$$
$$\begin{aligned}\gamma_8(\tilde{a}) = {}& 3063A_4A_9^2 - 42A_7^2(304A_8+43(A_9-11A_{10})) - 6A_3A_9(159A_8\\
&+ 28A_9+409A_{10}) + 2100A_2A_9A_{13} + 3150A_2A_7A_{16}\\
&+ 24A_3^2(34A_{19}-11A_{20}) + 840A_5^2A_{21} - 932A_2A_3A_{22} + 525A_2A_4A_{22}\\
&+ 844A_{22}^2 - 630A_{13}A_{33},\end{aligned}$$
$$\gamma_9(\tilde{a}) = 2A_8 - 6A_9 + A_{10}, \quad \gamma_{10}(\tilde{a}) = 3A_8 + A_{11},$$
$$\gamma_{11}(\tilde{a}) = -5A_7A_8 + A_7A_9 + 10A_3A_{14}, \quad \gamma_{12}(\tilde{a}) = 25A_2^2A_3 + 18A_{12}^2,$$
$$\gamma_{13}(\tilde{a}) = A_2, \quad \gamma_{14}(\tilde{a}) = A_2A_4 + 18A_2A_5 - 236A_{23} + 188A_{24},$$

$$
\begin{gathered}
\gamma_{15}(\tilde{a},x,y)=144T_1T_7^2-T_1^3(T_{12}+2T_{13})-4(T_9T_{11}+4T_7T_{15}+50T_3T_{23}\\
+2T_4T_{23}+2T_3T_{24}+4T_4T_{24}),\\
\gamma_{16}(\tilde{a},x,y)=T_{15},\quad \gamma_{17}(\tilde{a},x,y)=-(T_{11}+12\,T_{13}),\\
\tilde{\gamma}_{18}(\tilde{a},x,y)=C_1(C_2,C_2)^{(2)}-2C_2(C_1,C_2)^{(2)},\\
\tilde{\gamma}_{19}(\tilde{a},x,y)=D_1(C_1,C_2)^{(2)}-((C_2,C_2)^{(2)},C_0)^{(1)},
\end{gathered}
$$

$$
\begin{gathered}
\delta_1(\tilde{a})=9A_8+31A_9+6A_{10},\delta_2(\tilde{a})=41A_8+44A_9+32A_{10},\\
\delta_3(\tilde{a})=3A_{19}-4A_{17},\delta_4(\tilde{a})=-5A_2A_3+3A_2A_4+A_{22},\\
\delta_5(\tilde{a})=62A_8+102A_9-125A_{10},\delta_6(\tilde{a})=2T_3+3T_4,\\
\beta_1(\tilde{a})=3A_1^2-2A_8-2A_{12},\beta_2(\tilde{a})=2A_7-9A_6,\\
\beta_3(\tilde{a})=A_6,\beta_4(\tilde{a})=-5A_4+8A_5,\\
\beta_5(\tilde{a})=A_4,\beta_6(\tilde{a})=A_1,\\
\beta_7(\tilde{a})=8A_3-3A_4-4A_5,\beta_8(\tilde{a})=24A_3+11A_4+20A_5,\\
\beta_9(\tilde{a})=-8A_3+11A_4+4A_5,\beta_{10}(\tilde{a})=8A_3+27A_4-54A_5,\\
\beta_{11}(\tilde{a},x,y)=T_1^2-20T_3-8T_4,\beta_{12}(\tilde{a},x,y)=T_1,\\
\beta_{13}(\tilde{a},x,y)=T_3,
\end{gathered}
$$

$$
\begin{gathered}
\mathcal{R}_1(\tilde{a})=-2A_7(12A_1^2+A_8+A_{12})+5A_6(A_{10}+A_{11})-2A_1(A_{23}-A_{24})\\
+2A_5(A_{14}+A_{15})+A_6(9A_8+7A_{12}),\\
\mathcal{R}_2(\tilde{a})=A_8+A_9-2A_{10},\quad \mathcal{R}_3(\tilde{a})=A_9,\\
\mathcal{R}_4(\tilde{a})=-3A_1^2A_{11}+4A_4A_{19},\\
\mathcal{R}_5(\tilde{a},x,y)=(2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6)-(C_1,T_5)^{(1)}\\
+6D_1(C_1D_2-T_5)-9D_1^2C_2),\\
\mathcal{R}_6(\tilde{a})=-213A_2A_6+A_1(2057A_8-1264A_9+677A_{10}+1107A_{12})\\
+746(A_{27}-A_{28}),\\
\mathcal{R}_7(\tilde{a})=-6A_7^2-A_4A_8+2A_3A_9-5A_4A_9+4A_4A_{10}-2A_2A_{13},\\
\mathcal{R}_8(\tilde{a})=A_{10},\quad \mathcal{R}_9(\tilde{a})=-5A_8+3A_9,\\
\mathcal{R}_{10}(\tilde{a})=7A_8+5A_{10}+11A_{11},\quad \mathcal{R}_{11}(\tilde{a},x,y)=T_{16}.
\end{gathered}
$$

$$
\begin{gathered}
H_{12}(\tilde{a},x,y)=(\widehat{D},\widehat{D})^{(2)},\\
N_7(\tilde{a})=12D_1(C_0,D_2)^{(1)}+2D_1^3+9D_1(C_1,C_2)^{(2)}+36\big[C_0,C_1)^{(1)},D_2)^{(1)}.
\end{gathered}
$$

We remark the the last two invariant polynomials $H_{12}(\tilde{a},x,y)$ and $N_7(\tilde{a})$ are constructed in [15].

2.3. **Preliminary results involving polynomial invariants.** Considering the GL-comitant $C_2(\tilde{a},x,y)=yp_2(\tilde{a},x,y)-xq_2(\tilde{a},x,y)$ as a cubic binary form of x and y we calculate

$$
\eta(\tilde{a})=\mathrm{Discrim}[C_2,\xi],\quad M(\tilde{a},x,y)=\mathrm{Hessian}[C_2],
$$

where $\xi=y/x$ or $\xi=x/y$. According to [17] we have the next result.

Lemma 2.9 ([17]). *The number of infinite singularities (real and imaginary) of a quadratic system in QS is determined by the following conditions:*

- (i) 3 *real if* $\eta > 0$*;*
- (ii) 1 *real and* 2 *imaginary if* $\eta < 0$*;*
- (iii) 2 *real if* $\eta = 0$ *and* $M \neq 0$*;*
- (iv) 1 *real if* $\eta = M = 0$ *and* $C_2 \neq 0$*;*
- (v) ∞ *if* $\eta = M = C_2 = 0$.

Moreover, for each one of these cases the quadratic systems (2.1) *can be brought via a linear transformation to one of the following canonical systems:*

$$(\mathbf{S}_I)\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2; \end{cases}$$

$$(\mathbf{S}_{II})\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} = b + ex + fy - x^2 + gxy + hy^2; \end{cases}$$

$$(\mathbf{S}_{III})\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy, \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2; \end{cases}$$

$$(\mathbf{S}_{IV})\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy, \\ \dot{y} = b + ex + fy - x^2 + gxy + hy^2; \end{cases}$$

$$(\mathbf{S}_{V})\quad \begin{cases} \dot{x} = a + cx + dy + x^2, \\ \dot{y} = b + ex + fy + xy. \end{cases}$$

Lemma 2.10. *If a quadratic system* (2.6) *possesses a non-parabolic irreducible conic then the conditions* $\gamma_1 = \gamma_2 = 0$ *hold.*

Proof. According to [5] a system (2.6) possessing a second order non-parabolic irreducible curve as an algebraic particular integral can be written in the form

$$\dot{x} = a\Phi(x,y) + \Phi'_y(gx + hy + k), \quad \dot{y} = b\Phi(x,y) - \Phi'_x(gx + hy + k),$$

where a, b, g, h, k are real parameters and $\Phi(x,y)$ is the conic

$$\Phi(x,y) \equiv p + qx + ry + sx^2 + 2txy + uy^2 = 0. \tag{2.5}$$

A straightforward calculation gives $\gamma_1 = \gamma_2 = 0$ for the above systems and this completes the proof. $\square$

Assume that a conic (2.5) is an affine algebraic invariant curve for quadratic systems (2.1), which we rewrite in the form:

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + 2hxy + ky^2 \equiv P(x,y), \\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x,y). \end{aligned} \tag{2.6}$$

Remark 2.11. Following [9] we construct the determinant

$$\Delta = \begin{vmatrix} s & t & q/2 \\ t & u & r/2 \\ q/2 & r/2 & p \end{vmatrix},$$

associated to the conic (2.5). By [9] this conic is irreducible (i.e. the polynomial Φ defining the conic is irreducible over $\mathbb{C}$) if and only if $\Delta \neq 0$.

To detect if an invariant conic (2.5) of a system (2.6) has the multiplicity greater than one, we shall use the notion of *k-th extactic curve* $\mathscr{E}_k(X)$ of the vector field X (see (1.2)), associated to systems (2.6). This curve is defined in the paper [6, Definition5.1] as follows:

$$\mathscr{E}_k(X) = \det \begin{pmatrix} v_1 & v_2 & \dots & v_l \\ X(v_1) & X(v_2) & \dots & X(v_l) \\ \vdots & & & \vdots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \dots & X^{l-1}(v_l) \end{pmatrix},$$

where $v_1, v_2, \dots, v_l$ is the basis of $\mathbb{C}_n[x,y]$, the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}_n[x,y]$ and $l = (k+1)(k+2)/2$. Here $X^0(v_i) = v_i$ and $X^j(v_1) = X(X^{j-1}(v_1))$.

Considering the Definition 1.1 of a multiplicity of an invariant curve, according to [6] the following statement holds:

Lemma 2.12. *If an invariant curve $\Phi(x,y) = 0$ of degree k has multiplicity m, then $\Phi(x,y)^m$ divides $\mathscr{E}_k(X)$.*

We shall apply this lemma in order to detect additional conditions for a conic to be multiple. According to definition of an invariant curve (see page 2) considering the cofactor $K = Ux + Vy + W \in \mathbb{C}[x,y]$ the following identity holds:

$$\frac{\partial \Phi}{\partial x} P(x,y) + \frac{\partial \Phi}{\partial y} Q(x,y) = \Phi(x,y)(Ux + Vy + W).$$

This identity yields a system of 10 equations for determining the 9 unknown parameters p, q, r, s, t, u, U, V, W:

$$\begin{aligned}
Eq_1 &\equiv s(2g-U) + 2lt = 0, \\
Eq_2 &\equiv 2t(g+2m-U) + s(4h-V) + 2lu = 0, \\
Eq_3 &\equiv 2t(2h+n-V) + u(4m-U) + 2ks = 0, \\
Eq_4 &\equiv u(2n-V) + 2kt = 0, \\
Eq_5 &\equiv q(g-U) + s(2c-W) + 2et + lr = 0, \\
Eq_6 &\equiv r(2m-U) + q(2h-V) + 2t(c+f-W) + 2(ds+eu) = 0, \\
Eq_7 &\equiv r(n-V) + u(2f-W) + 2dt + kq = 0, \\
Eq_8 &\equiv q(c-W) + 2(as+bt) + er - pU = 0, \\
Eq_9 &\equiv r(f-W) + 2(bu+at) + dq - pV = 0, \\
Eq_{10} &\equiv aq + br - pW = 0.
\end{aligned} \tag{2.7}$$

3. Proof of the main theorem

Assuming that a quadratic system (2.6) in QS has an invariant hyperbola (2.5), we conclude that this system must possess at least two real distinct infinite singularities. So according to Lemmas 2.9 and 2.10 the conditions $\gamma_1 = \gamma_2 = 0$ and either $\eta \geq 0$ and $M \neq 0$ or $C_2 = 0$ have to be fulfilled.

In what follows, supposing that the conditions $\gamma_1 = \gamma_2 = 0$ hold, we shall examine three families of quadratic systems (2.6): systems with three real distinct infinite singularities (corresponding to the condition $\eta > 0$); systems with two real distinct infinite singularities (corresponding to the conditions $\eta = 0$ and $M \neq 0$) and systems with infinite number of singularities at infinity, i.e. with degenerate infinity (corresponding to the condition $C_2 = 0$).

3.1. **Systems with three real infinite singularities and $\theta \neq 0$.** In this case according to Lemma 2.9 systems (2.6) via a linear transformation could be brought to the following family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + (h-1)xy, \\ \frac{dy}{dt} &= b + ex + fy + (g-1)xy + hy^2. \end{aligned} \tag{3.1}$$

For this systems we calculate

$$C_2(x,y) = xy(x-y), \quad \theta = -(g-1)(h-1)(g+h)/2 \tag{3.2}$$

and we shall prove the next lemma.

Lemma 3.1. *Assume that for a system* (3.1) *the conditions $\theta \neq 0$ and $\gamma_1 = 0$ hold. Then this system via an affine transformation could be brought to the form*

$$\frac{dx}{dt} = a + cx + gx^2 + (h-1)xy, \quad \frac{dy}{dt} = b - cy + (g-1)xy + hy^2. \tag{3.3}$$

Proof. Since $\theta \neq 0$ the condition $(g-1)(h-1)(g+h) \neq 0$ holds and by a translation we may assume $d = e = 0$ for systems (3.1). Then we calculate

$$\gamma_1 = \frac{1}{64}(g-1)^2(h-1)^2\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3,$$

where

$$\begin{gathered} \mathcal{D}_1 = c + f, \quad \mathcal{D}_2 = c(g + 4h - 1) + f(1 + g - 2h), \\ \mathcal{D}_3 = c(1 - 2g + h) + f(4g + h - 1). \end{gathered}$$

Since $\theta \neq 0$ (i.e. $(g-1)(h-1) \neq 0$) the condition $\gamma_1 = 0$ is equivalent to $\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3 = 0$. We claim that without loss of generality we may assume $\mathcal{D}_1 = c + f = 0$, as other cases could be brought to this one via an affine transformation.

Indeed, assume first $\mathcal{D}_1 \neq 0$ and $\mathcal{D}_2 = 0$. Then as $g + h \neq 0$ (due to $\theta \neq 0$) we apply to systems (3.1) with $d = e = 0$ the affine transformation

$$x' = y - x - (c-f)/(g+h), \quad y' = -x \tag{3.4}$$

and we obtain the systems

$$\frac{dx'}{dt} = a' + c'x' + g'x'^2 + (h'-1)x'y', \quad \frac{dy'}{dt} = b' + f'y' + (g'-1)x'y' + h'y'^2. \tag{3.5}$$

These systems have the following new parameters:

$$\begin{gathered} a' = \left[c^2h - f^2g + cf(g-h) - (a-b)(g+h)^2\right]/(g+h)^2, \\ b' = -a, \quad c' = (cg - 2fg - ch)/(g+h), \\ f' = (c - f - cg + 2fg + fh)/(g+h), \quad g' = h, \quad h' = 1 - g - h. \end{gathered} \tag{3.6}$$

A straightforward computation gives

$$\mathcal{D}_1' = c' + f' = \left[c(g + 4h - 1) + f(1 + g - 2h)\right]/(g+h) = \mathcal{D}_2/(g+h) = 0$$

and hence, the condition $\mathcal{D}_2 = 0$ is replaced with $\mathcal{D}_1 = 0$ via an affine transformation.

Suppose now $\mathcal{D}_1 \neq 0$ and $\mathcal{D}_3 = 0$. Then we apply to systems (3.1) the affine transformation

$$x'' = -y, \quad y'' = x - y + (c-f)/(g+h)$$

and we obtain the systems

$$\frac{dx''}{dt} = a'' + c''x'' + g''x''^2 + (h''-1)x''y'', \quad \frac{dy''}{dt} = b'' + f''y'' + (g''-1)x''y'' + h''y''^2,$$

having the following new parameters:

$$a'' = -b, \quad b'' = \big[f^2g - c^2h + cf(-g+h) + (a-b)(g+h)^2\big]/(g+h)^2,$$
$$c'' = (c - f - cg + 2fg + fh)/(g+h),$$
$$f'' = (cg - 2fg - ch)/(g+h), \quad g'' = 1 - g - h, \quad h'' = g.$$

We calculate

$$\mathcal{D}_1'' = c'' + f'' = \big[c(1-2g+h) + f(4g+h-1)\big]/(g+h) = \mathcal{D}_3/(g+h) = 0.$$

Thus our claim is proved and this completes the proof of the lemma. $\square$

Lemma 3.2. *A system* (3.3) *possesses an invariant hyperbola of the indicated form if and only if the corresponding conditions indicated on the right hand side are satisfied:*

(I) $\Phi(x,y) = p + qx + ry + 2xy \quad \Leftrightarrow \quad \mathcal{B}_1 \equiv b(2h-1) - a(2g-1) = 0$, $(2h-1)^2 + (2g-1)^2 \neq 0$, $a^2 + b^2 \neq 0$;

(II) $\Phi(x,y) = p + qx + ry + 2x(x-y) \quad \Leftrightarrow \quad$ *either*
 (i) $c = 0$, $\mathcal{B}_2 \equiv b(1-2h) + 2a(g+2h-1) = 0$, $(2h-1)^2 + (g+2h-1)^2 \neq 0$, $a^2 + b^2 \neq 0$, *or*
 (ii) $h = 1/3$, $\mathcal{B}_2' \equiv (1+3g)^2(b - 2a + 6ag) + 6c^2(1-3g) = 0$, $a \neq 0$;

(III) $\Phi(x,y) = p + qx + ry + 2y(x-y) \quad \Leftrightarrow \quad$ *either*
 (i) $c = 0$, $\mathcal{B}_3 \equiv a(1-2g) + 2b(2g+h-1) = 0$, $(2g-1)^2 + (2g+h-1)^2 \neq 0$, $a^2 + b^2 \neq 0$, *or*
 (ii) $g = 1/3$, $\mathcal{B}_3' \equiv (1+3h)^2(a - 2b + 6bh) + 6c^2(1-3h) = 0$, $b \neq 0$.

Proof. Since for systems (3.3) we have $C_2 = xy(x-y)$ (i.e. the infinite singularities are located at the "ends" of the lines $x = 0$, $y = 0$ and $x - y = 0$) it is clear that if a hyperbola is invariant for these systems, then its homogeneous quadratic part has one of the following forms: (i) kxy, (ii) $kx(x-y)$, (iii) $ky(x-y)$, where k is a real nonzero constant. Obviously we may assume $k = 2$ (otherwise instead of hyperbola (2.5) we could consider $2\Phi(x,y)/k = 0$).

Considering the equations (2.7) we examine each one of the above mentioned possibilities.

(i) $\Phi(x,y) = p + qx + ry + 2xy$; in this case we obtain

$$t = 1,\ q = r = s = u = 0,\ U = 2g-1,\ V = 2h-1,\ W = 0,$$
$$Eq_8 = p(1-2g) + 2b, \quad Eq_9 = p(1-2h) + 2a,$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_7 = Eq_{10} = 0.$$

Calculating the resultant of the non-vanishing equations with respect to the parameter p we obtain

$$\operatorname{Res}_p(Eq_8, Eq_9) = a(1-2g) + b(2h-1) = \mathcal{B}_1.$$

So if $(2h-1)^2 + (2g-1)^2 \neq 0$ then the hyperbola exists if and only if $\mathcal{B}_1 = 0$. We may assume $2h-1 \neq 0$, otherwise the change $(x,y,a,b,c,g,h) \mapsto (y,x,b,a,-c,h,g)$ (which preserves systems (3.3)) could be applied. Then we obtain

$$p = 2a/(2h-1), \quad b = a(2g-1)/(2h-1), \quad \Phi(x,y) = \frac{2a}{2h-1} + 2xy = 0$$

and clearly for the irreducibility of the hyperbola the condition $a^2+b^2\neq 0$ must hold. This completes the proof of the statement (I) of the lemma.

(ii) $\Phi(x,y)=p+qx+ry+2x(x-y)$; since $g+h\neq 0$ (because $\theta\neq 0$) we obtain

$$s=2,\ t=-1,\ r=u=0,\ q=4c/(g+h),\ U=2g,\ V=2h-1,\ W=-hq/2,$$
$$Eq_8=4a-2b-2gp+4c^2(g-h)/(g+h)^2,$$
$$Eq_9=p(1-2h)-2a,\quad Eq_{10}=2c(2a-hp)/(g+h),$$
$$Eq_1=Eq_2=Eq_3=Eq_4=Eq_5=Eq_6=Eq_7=0.$$

(1) Assume first $c\neq 0$. Then considering the equations $Eq_9=0$ and $Eq_{10}=0$ we obtain $p(3h-1)=0$. Taking into account the relations above we obtain the hyperbola

$$\Phi(x,y)=p+4cx/(g+h)+2x(x-y)=0$$

which evidently is reducible if $p=0$. So $p\neq 0$ and this implies $h=1/3$. Then from the equation $Eq_9=0$ we obtain $p=6a$. Since $\theta=(g-1)(3g+1)/9\neq 0$ we have $Eq_9=Eq_{10}=0$, $Eq_8=-2\mathcal{B}_2'/(3g+1)^2$. So the equation $Eq_8=0$ gives $\mathcal{B}_2'=0$ and then systems (3.3) with $h=1/3$ possess the hyperbola

$$\Phi(x,y)=6a+\frac{12c}{3g+1}x+2x(x-y)=0,$$

which is irreducible if and only if $a\neq 0$.

(2) Suppose now $c=0$. In this case there remain only two non–vanishing equations:

$$Eq_8=4a-2b-2gp=0,\quad Eq_9=p(1-2h)-2a=0.$$

Calculating the resultant of these equations with respect to the parameter p we obtain

$$\mathrm{Res}_p(Eq_8,Eq_9)=b(1-2h)+2a(g+2h-1)=\mathcal{B}_2.$$

If $(1-2h)^2+(g+2h-1)^2\neq 0$ (which is equivalent to $(1-2h)^2+g^2\neq 0$) then the condition $\mathcal{B}_2'=0$ is necessary and sufficient for a system (3.3) with $c=0$ to possess the invariant hyperbola

$$\Phi(x,y)=p+2x(x-y)=0,$$

where p is the parameter determined from the equation $Eq_9=0$ (if $2h-1\neq 0$), or $Eq_8=0$ (if $g\neq 0$). We observe that the hyperbola is irreducible if and only if $p\neq 0$ which due to the mentioned equations is equivalent to $a^2+b^2\neq 0$.

Thus the statement II of the lemma is proved.

(iii) $\Phi(x,y)=p+qx+ry+2y(x-y)$; we observe that due to the change $(x,y,a,b,c,g,h)\mapsto(y,x,b,a,-c,h,g)$ (which preserves systems (3.3)) this case could be brought to the previous one and hence, the conditions could be constructed directly applying this change. This completes the proof of Lemma 3.2. □

In what follows the next remark will be useful.

Remark 3.3. Consider systems (3.3).

(i) The change $(x,y,a,b,c,g,h)\mapsto(y,x,b,a,-c,h,g)$ which preserves these systems replaces the parameter g by h and h by g.

(ii) Moreover if $c=0$ then having the relation $(2h-1)(2g-1)(1-2g-2h)=0$ (respectively $(4h-1)(4g-1)(3-4g-4h)=0$) due to a change we may assume $2h-1=0$ (respectively $4h-1=0$).

To prove the statement (ii) it is sufficient to observe that in the case $2g-1=0$ (respectively $4g-1=0$) we could apply the change given in the statement (i) (with $c=0$), whereas in the case $1-2g-2h=0$ (respectively $3-4g-4h=0$) we apply the change $(x,y,a,b,g,h)\mapsto(y-x,-x,b-a,-a,h,1-g-h)$, which conserves systems (3.3) with $c=0$.

Next we determine the invariant criteria which are equivalent to the conditions given by Lemma 3.2.

Lemma 3.4. *Assume that for a quadratic system* (2.6) *the conditions* $\eta>0$, $\theta\neq 0$ *and* $\gamma_1=\gamma_2=0$ *hold. Then this system possesses at least one invariant hyperbola if and only if one of the following sets of the conditions are satisfied:*

- (i) *If* $\beta_1\neq 0$ *then either*
  - (i.1) $\beta_2\neq 0$, $\mathcal{R}_1\neq 0$, *or*
  - (i.2) $\beta_2=0$, $\beta_3\neq 0$, $\gamma_3=0$, $\mathcal{R}_1\neq 0$, *or*
  - (i.3) $\beta_2=\beta_3=0$, $\beta_4\beta_5\mathcal{R}_2\neq 0$, *or*
  - (i.4) $\beta_2=\beta_3=\beta_4=0$, $\gamma_3=0$, $\mathcal{R}_2\neq 0$;
- (ii) *If* $\beta_1=0$ *then either*
  - (ii.1) $\beta_6\neq 0$, $\beta_2\neq 0$, $\gamma_4=0$, $\mathcal{R}_3\neq 0$, *or*
  - (ii.2) $\beta_6\neq 0$, $\beta_2=0$, $\gamma_5=0$, $\mathcal{R}_4\neq 0$, *or*
  - (ii.3) $\beta_6=0$, $\beta_7\neq 0$, $\gamma_5=0$, $\mathcal{R}_5\neq 0$, *or*
  - (ii.4) $\beta_6=0$, $\beta_7=0$, $\beta_9\neq 0$, $\gamma_5=0$, $\mathcal{R}_5\neq 0$, *or*
  - (ii.5) $\beta_6=0$, $\beta_7=0$, $\beta_9=0$, $\gamma_6=0$, $\mathcal{R}_5\neq 0$.

Proof. Assume that for a quadratic system (2.6) the conditions $\eta>0$, $\theta\neq 0$ and $\gamma_1=0$ are fulfilled. According to Lemma 3.1 due to an affine transformation and time rescaling this system could be brought to the canonical form (3.3), for which we calculate

$$\begin{gathered}\gamma_2=-1575c^2(g-1)^2(h-1)^2(g+h)(3g-1)(3h-1)(3g+3h-4)\mathcal{B}_1,\\ \beta_1=-c^2(g-1)(h-1)(3g-1)(3h-1)/4,\\ \beta_2=-c(g-h)(3g+3h-4)/2.\end{gathered}\tag{3.7}$$

3.1.1. *Case* $\beta_1\neq 0$. According to Lemma 2.10 the condition $\gamma_2=0$ is necessary for the existence of a hyperbola. Since $\theta\beta_1\neq 0$ in this case the condition $\gamma_2=0$ is equivalent to $(3g+3h-4)\mathcal{B}_1=0$.

Subcase $\beta_2\neq 0$. Then $(3g+3h-4)\neq 0$ and the condition $\gamma_2=0$ gives $\mathcal{B}_1=0$. Moreover the condition $\beta_2\neq 0$ yields $g-h\neq 0$ and this implies $(2h-1)^2+(2g-1)^2\neq 0$. According to Lemma 3.2 systems (3.3) possess an invariant hyperbola, which is irreducible if and only if $a^2+b^2\neq 0$.

On the other hand for these systems we calculate

$$\mathcal{R}_1=-3c(a-b)(g-1)^2(h-1)^2(g+h)(3g-1)(3h-1)/8$$

and we claim that for $\mathcal{B}_1=0$ the condition $\mathcal{R}_1=0$ is equivalent to $a=b=0$. Indeed, as the equation $\mathcal{B}_1=0$ is linear homogeneous in a and b, as well as the second equation $a-b=0$, calculating the respective determinant we obtain $-2(g+h)\neq 0$ due to $\theta\neq 0$. This proves our claim and hence the statement $(i.1)$ of Lemma 3.4 is proved.

Subcase $\beta_2 = 0$. Since $\beta_1 \neq 0$ (i.e. $c \neq 0$) we obtain $(g-h)(3g+3h-4) = 0$. On the other hand for systems (3.3) we have

$$\beta_3 = -c(g-h)(g-1)(h-1)/4$$

and we consider two possibilities: $\beta_3 \neq 0$ and $\beta_3 = 0$.

Possibility $\beta_3 \neq 0$. In this case we have $g-h \neq 0$ and the condition $\beta_2 = 0$ implies $3g+3h-4 = 0$, i.e. $g = 4/3 - h$. So the condition $(2h-1)^2 + (2g-1)^2 \neq 0$ for systems (3.3) becomes $(2h-1)^2 + (6h-5)^2 \neq 0$ and obviously this condition is satisfied.

For systems (3.3) with $g = 4/3 - h$ we calculate

$$\gamma_3 = 22971c(h-1)^3(3h-1)^3\mathcal{B}_1, \quad \mathcal{R}_1 = (a-b)c(h-1)^3(3h-1)^3/6,$$
$$\beta_1 = -c^2(h-1)^2(3h-1)^2/4, \quad \beta_3 = -c(h-1)(3h-2)(3h-1)/18.$$

So because $\beta_1 \neq 0$ the condition $\gamma_3 = 0$ is equivalent to $\mathcal{B}_1 = 0$. Moreover if in addition $\mathcal{R}_1 = 0$ (i.e. $a-b=0$) we obtain $a = b = 0$, because the determinant of the systems of linear equations

$$3\mathcal{B}_1 = a(5-6h) - 3b(2h-1) = 0, \quad a - b = 0$$

with respect to the parameters a and b equals $4(3h-2) \neq 0$ due to the condition $\beta_3 \neq 0$. So the statement (i.2) of the lemma is proved.

Possibility $\beta_3 = 0$. Since $\beta_1 \neq 0$ (i.e. $c(g-1)(h-1) \neq 0$) we obtain $g = h$ and for systems (3.3) we calculate

$$\gamma_2 = 6300c^2h(h-1)^4(3h-2)(3h-1)^2\mathcal{B}_1, \quad \theta = -h(h-1)^2,$$
$$\beta_1 = -c^2(h-1)^2(3h-1)^2/4, \quad \beta_4 = 2h(3h-2), \quad \beta_5 = -2h^2(2h-1).$$

So by the condition $\theta\beta_1 \neq 0$ we obtain that the necessary condition $\gamma_2 = 0$ is equivalent to $\mathcal{B}_1(3h-2) = 0$ and we shall consider two cases: $\beta_4 \neq 0$ and $\beta_4 = 0$.

(1) *Case* $\beta_4 \neq 0$. Therefore $3h-2 \neq 0$ and this implies $\mathcal{B}_1 = 0$. Considering Lemma 3.2 the condition $(2h-1)^2 + (2g-1)^2 \neq 0$ for $g = h$ becomes $2h-1 \neq 0$. So for the existence of a invariant hyperbola the condition $\beta_5 \neq 0$ is necessary. Moreover this hyperbola is irreducible if and only if $a^2 + b^2 \neq 0$. Since for these systems we have

$$\mathcal{R}_2 = (a+b)(h-1)^2(3h-1)/2, \quad \mathcal{B}_1 = -(2h-1)(a-b)$$

we conclude, that when $\mathcal{B}_1 = 0$ the condition $\mathcal{R}_2 \neq 0$ is equivalent to $a^2 + b^2 \neq 0$ and this completes the proof of the statement (i.3) of the lemma.

(2) *Case* $\beta_4 = 0$. Then by $\theta \neq 0$ we obtain $h = 2/3$ and arrive at the 3-parameter family of systems

$$\frac{dx}{dt} = a + cx + 2x^2/3 - xy/3, \quad \frac{dy}{dt} = b - cy - xy/3 + 2y^2/3, \tag{3.8}$$

For these systems we calculate $\gamma_3 = 7657c\mathcal{B}_1/9$, $\beta_1 = -c^2/36$, $\mathcal{R}_2 = (a+b)/18$, where $\mathcal{B}_1 = (b-a)/3$. Since for these systems the condition $(2h-1)^2 + (2g-1)^2 = 2/9 \neq 0$ holds, according to Lemma 3.2 we conclude that the statement (i.4) of the lemma is proved.

3.1.2. *Case* $\beta_1 = 0$. Considering (3.7) and the condition $\theta \neq 0$ we obtain $c(3g-1)(3h-1) = 0$. On the other hand for systems (3.3) we calculate

$$\beta_6 = -c(g-1)(h-1)/2$$

and we shall consider two subcases: $\beta_6 \neq 0$ and $\beta_6 = 0$.

Subcase $\beta_6 \neq 0$. Then $c \neq 0$ and the condition $\beta_1 = 0$ implies $(3g-1)(3h-1) = 0$. Therefore by Remark 3.3 we may assume $h = 1/3$ and this leads to the following 4-parameter family of systems

$$\frac{dx}{dt} = a + cx + gx^2 - 2xy/3, \quad \frac{dy}{dt} = b - cy + (g-1)xy + y^2/3, \tag{3.9}$$

which is a subfamily of (3.3). According to Lemma 3.2 the above systems possess a hyperbola if and only if either $\mathcal{B}_1 = a(1-2g) - b/3 = 0$ and $a^2 + b^2 \neq 0$ (the statement I), or $\mathcal{B}_2' = (1+3g)^2(b - 2a + 6ag) + 6c^2(1-3g) = 0$ and $a \neq 0$ (the statement II). We observe that in the first case, when $a(1-2g) - b/3 = 0$ the condition $a^2 + b^2 \neq 0$ is equivalent to $a \neq 0$.

On the other hand for these systems we calculate

$$\gamma_4 = -16(g-1)^2(3g-1)^2\mathcal{B}_1\mathcal{B}_2'/81, \quad \beta_6 = c(g-1)/3,$$
$$\beta_2 = c(g-1)(3g-1)/2, \quad \mathcal{R}_3 = a(3g-1)^3/18.$$

So we consider two possibilities: $\beta_2 \neq 0$ and $\beta_2 = 0$.

Possibility $\beta_2 \neq 0$. In this case $(g-1)(3g-1) \neq 0$ and the conditions $\gamma_4 = 0$ and $\mathcal{R}_3 \neq 0$ are equivalent to $\mathcal{B}_1\mathcal{B}_2' = 0$ and $a \neq 0$, respectively. This completes the proof of the statement (ii.1).

Possibility $\beta_2 = 0$. From the condition $\beta_6 \neq 0$ we obtain $g = 1/3$ and this leads to the following 3-parameter family of systems:

$$\frac{dx}{dt} = a + cx + x^2/3 - 2xy/3, \quad \frac{dy}{dt} = b - cy - 2xy/3 + y^2/3. \tag{3.10}$$

Since $c \neq 0$ (because $\beta_6 \neq 0$) according to Lemma 3.2 these systems possess an invariant hyperbola if and only if one of the following sets conditions are fulfilled:

$$\mathcal{B}_1 = (a-b)/3 = 0, \quad a^2 + b^2 \neq 0;$$
$$\mathcal{B}_2' = 4b = 0, \quad a \neq 0; \quad \mathcal{B}_3' = 4a = 0, \quad b \neq 0.$$

We observe that the last two conditions are equivalent to $ab = 0$ and $a^2 + b^2 \neq 0$.

On the other hand for systems (3.10) we calculate

$$\gamma_5 = 16\mathcal{B}_1\mathcal{B}_2'\mathcal{B}_3'/27, \quad \mathcal{R}_4 = 128(a^2 - ab + b^2)/6561.$$

It is clear that the condition $\mathcal{R}_4 = 0$ is equivalent to $a^2 + b^2 = 0$. So the statement (ii.2) is proved.

Subcase $\beta_6 = 0$. Since $\theta \neq 0$ (i.e. $(g-1)(h-1) \neq 0$) the condition $\beta_6 = 0$ yields $c = 0$. Therefore according to Lemma 3.2 systems (3.3) with $c = 0$ possess an invariant hyperbola if and only if one of the following sets of conditions holds:

$$\mathcal{B}_1 \equiv b(2h-1) - a(2g-1) = 0, \quad (2h-1)^2 + (2g-1)^2 \neq 0,\ a^2 + b^2 \neq 0;$$
$$\mathcal{B}_2 \equiv b(1-2h) + 2a(g+2h-1) = 0, \quad (2h-1)^2 + (g+2h-1)^2 \neq 0,\ a^2 + b^2 \neq 0;$$
$$\mathcal{B}_3 \equiv a(1-2g) + 2b(2g+h-1) = 0, \quad (2g-1)^2 + (2g+h-1)^2 \neq 0,\ a^2 + b^2 \neq 0.$$

Considering the following three expressions

$$\alpha_1 = 2g-1, \quad \alpha_2 = 2h-1, \quad \alpha_3 = 1-2g-2h$$

we observe that the condition $(2h-1)^2 + (2g-1)^2 \neq 0$ (respectively $(2h-1)^2 + (g+2h-1)^2 \neq 0$; $(2g-1)^2 + (2g+h-1)^2 \neq 0$) is equivalent to $\alpha_1^2 + \alpha_2^2 \neq 0$ (respectively $\alpha_2^2 + \alpha_3^2 \neq 0$; $\alpha_1^2 + \alpha_3^2 \neq 0$).

On the other hand for these systems we calculate

$$\gamma_5 = -288(g-1)(h-1)(g+h)\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3,$$
$$\theta = -(g-1)(h-1)(g+h)/2,$$
$$\beta_7 = 2\alpha_1\alpha_2\alpha_3, \quad \beta_9 = 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3),$$
$$\mathcal{R}_5 = 36(bx-ay)\big[(g-1)^2x^2 + 2(g+h+gh-1)xy + (h-1)^2y^2\big].$$

We observe that if $\alpha_1 = \alpha_2 = 0$ (respectively $\alpha_2 = \alpha_3 = 0$; $\alpha_1 = \alpha_3 = 0$) then the factor $\mathcal{B}_1$ (respectively $\mathcal{B}_2$; $\mathcal{B}_3$) vanishes identically. Considering the values of the invariant polynomials β_7 and β_9 we conclude that two of the factors α_i $(i = 1,2,3)$ vanish if and only if $\beta_7 = \beta_9 = 0$. So we have to consider two subcases: $\beta_7^2 + \beta_9^2 \neq 0$ and $\beta_7^2 + \beta_9^2 = 0$.

Possibility $\beta_7^2 + \beta_9^2 \neq 0$. In this case by $\theta \neq 0$ the conditions $\gamma_5 = 0$ and $\mathcal{R}_5 \neq 0$ are equivalent to $\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3 = 0$ and $a^2 + b^2 \neq 0$, respectively. So by Lemma 3.2 there exists at least one hyperbola and hence the statements $(ii.3)$ and $(ii.4)$ are valid.

Possibility $\beta_7^2 + \beta_9^2 = 0$. As it was mentioned above, in this case two of the factors α_i $(i = 1,2,3)$ vanish. Considering Remark 3.3, without loss of generality we may assume $\alpha_1 = \alpha_2 = 0$.

Thus we have $g = h = 1/2$ and we obtain the family of systems

$$\frac{dx}{dt} = a + x^2/2 - xy/2, \quad \frac{dy}{dt} = b - xy/2 + y^2/2. \tag{3.11}$$

Since $c = 0$ and the conditions of the statement I of Lemma 3.2 are not satisfied for these systems, according to Lemma 3.2 the above systems possess an invariant hyperbola if and only if $a^2 + b^2 \neq 0$ and either $\mathcal{B}_2 = a = 0$ or $\mathcal{B}_3 = b = 0$. For systems (3.11) we calculate

$$\gamma_6 = -9\mathcal{B}_2\mathcal{B}_3, \quad \mathcal{R}_5 = 9(bx-ay)(x+y)^2$$

and we conclude that the statement (ii.5) of the lemma holds.

As all the cases are examined, Lemma 3.4 is proved. □

The next lemma is related to the number of the invariant hyperbolas that quadratic systems with $\eta > 0$ and $\theta \neq 0$ could have.

Lemma 3.5. *Assume that for a quadratic system* (2.6) *the conditions* $\eta > 0$, $\theta \neq 0$ *and* $\gamma_1 = \gamma_2 = 0$ *are satisfied. Then this system possesses:*

(A) *two invariant hyperbolas if and only if either*
 (A1) $\beta_1 = 0$, $\beta_6 \neq 0$, $\beta_2 \neq 0$, $\gamma_4 = 0$, $\mathcal{R}_3 \neq 0$ *and* $\delta_1 = 0$, *or*
 (A2) $\beta_1 = 0$, $\beta_6 = 0$, $\beta_7 \neq 0$, $\gamma_5 = 0$, $\mathcal{R}_5 \neq 0$ *and* $\beta_8 = \delta_2 = 0$, *or*
 (A3) $\beta_1 = 0$, $\beta_6 = \beta_7 = 0$, $\beta_9 \neq 0$, $\gamma_5 = 0$, $\mathcal{R}_5 \neq 0$ *and* $\delta_3 = 0$, $\beta_8 \neq 0$*;*

(B) *three invariant hyperbolas if and only if* $\beta_1 = 0$, $\beta_6 = \beta_7 = 0$, $\beta_9 \neq 0$, $\gamma_5 = 0$, $\mathcal{R}_5 \neq 0$ *and* $\delta_3 = \beta_8 = 0$.

Proof. For systems (3.3) we have

$$\beta_6 = -c(g-1)(h-1)/2, \quad \theta = -(g-1)(h-1)(g+h)/2,$$
$$\beta_1 = -c^2(g-1)(h-1)(3g-1)(3h-1)/4.$$

3.1.3. *Case* $\beta_6 \neq 0$. Then $c \neq 0$ and according to Lemma 3.2 we could have at least two hyperbolas only if the conditions given either by the statements I and II; (ii) (i.e. $\mathcal{B}_1 = \mathcal{B}_2' = 0$ and $h = 1/3$), or by the statements I and III; (ii) (i.e. $\mathcal{B}_1 = \mathcal{B}_3' = 0$ and $g = 1/3$) are satisfied. Therefore the condition $(3g-1)(3h-1) = 0$ is necessary. This condition is governed by the invariant polynomial β_1. So we assume $\beta_1 = 0$ and due to Remark 3.3 we may consider $h = 1/3$. Then we calculate

$$\gamma_4 = -16(g-1)^2(3g-1)^2\mathcal{B}_1\mathcal{B}_2'/81, \quad \beta_1 = 0,$$
$$\theta = (g-1)(1+3g)/9 \neq 0, \quad \beta_2 = c(g-1)(3g-1)/2.$$

Solving the systems of equations $\mathcal{B}_1\big|_{h=1/3} = \mathcal{B}_2' = 0$ with respect to a and b we obtain

$$a = \frac{6c^2(3g-1)}{(1+3g)^2} \equiv A_0, \quad b = -\frac{18c^2(2g-1)(3g-1)}{(1+3g)^2} \equiv B_0.$$

In this case we obtain the family of systems

$$\frac{dx}{dt} = A_0 + cx + gx^2 - 2xy/3, \quad \frac{dy}{dt} = B_0 - cy + (g-1)xy + y^2/3, \tag{3.12}$$

which possess two invariant hyperbolas:

$$\Phi_1(x,y) = -\frac{36c^2(3g-1)}{(1+3g)^2} + 2xy = 0,$$
$$\Phi_2(x,y) = -\frac{36c^2(3g-1)}{(1+3g)^2} + \frac{12c}{1+3g}x + 2x(x-y) = 0,$$

where $c(3g-1) \neq 0$ due to $a \neq 0$. Thus for the irreducibility of the hyperbolas above, the condition $c(3g-1) \neq 0$ (i.e. $\beta_2 \neq 0$) is necessary.

Since the condition $\gamma_4 = 0$ gives $\mathcal{B}_1\mathcal{B}_2' = 0$ it remains to find out the invariant polynomial which in addition to γ_4 is responsible for the relation $\mathcal{B}_1 = \mathcal{B}_2' = 0$. We observe that in the case $\mathcal{B}_1 = 0$ (i.e. $b = 3a(1-2g)$) we have

$$\delta_1 = (3g-1)\big[a(1+3g)^2 - 6c^2(3g-1)\big]/18 = (3g-1)\mathcal{B}_2'/18.$$

It remains to observe that in the case considered we have $\mathcal{R}_3 = a(3g-1)^3/18 \neq 0$ and that due to the condition $\beta_2 \neq 0$ (i.e. $c(3g-1) \neq 0$) by Lemma 3.2 we could not have a third hyperbola of the form $\Phi(x,y) = p + qx + ry + 2y(x-y) = 0$. This completes the proof of the statement $(\mathcal{A}_1)$ of the lemma.

3.1.4. *Case* $\beta_6 = 0$. Then $c = 0$ and we calculate for systems (3.3)

$$\beta_7 = 2\alpha_1\alpha_2\alpha_3, \quad \beta_9 = 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3), \quad \beta_8 = 2(4g-1)(4h-1)(3-4g-4h),$$

where $\alpha_1 = 2g-1$, $\alpha_2 = 2h-1$ and $\alpha_3 = 1-2g-2h$.

Subcase $\beta_7 \neq 0$. Then $\alpha_1\alpha_2\alpha_3 \neq 0$ and we consider two possibilities: $\beta_8 \neq 0$ and $\beta_8 = 0$.

Possibility $\beta_8 \neq 0$. We claim that in this case we could not have more than one hyperbola. Indeed, as $c = 0$ we observe that all five polynomials $\mathcal{B}_i$ $(i = 1, 2, 3)$, $\mathcal{B}_2'$ and $\mathcal{B}_3'$ are linear (and homogeneous) with respect to a and b and the condition $a^2+b^2 \neq 0$ must hold. So in order to have nonzero solutions in (a, b) of the equations

$$\mathcal{U} = \mathcal{V} = 0, \quad \mathcal{U}, \mathcal{V} \in \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_2', \mathcal{B}_3'\}, \quad \mathcal{U} \neq \mathcal{V}$$

it is necessary that the corresponding determinants $\det(\mathcal{U}, \mathcal{V}) = 0$. We have for each couple, respectively:

$$\begin{aligned}
&(\omega_1)\ \det(\mathcal{B}_1, \mathcal{B}_2) = && -(2h-1)(4h-1) = 0;\\
&(\omega_2)\ \det(\mathcal{B}_1, \mathcal{B}_3) = && -(2g-1)(4g-1) = 0;\\
&(\omega_3)\ \det(\mathcal{B}_2, \mathcal{B}_3) = && (1-2g-2h)(3-4g-4h) = 0;\\
&(\omega_4)\ \det(\mathcal{B}_1, \mathcal{B}_2')\big|_{h=1/3} = && (3g+1)^2/3;\\
&(\omega_5)\ \det(\mathcal{B}_1, \mathcal{B}_3')\big|_{g=1/3} = && (3h+1)^2/3;\\
&(\omega_6)\ \det(\mathcal{B}_2', \mathcal{B}_3)\big|_{\{c=0,\, h=1/3\}} = && (1+3g)^2(6g-1)(12g-5)/3 = 0;\\
&(\omega_7)\ \det(\mathcal{B}_2, \mathcal{B}_3')\big|_{\{c=0,\, g=1/3\}} = && (1+3h)^2(6h-1)(12h-5)/3 = 0;\\
&(\omega_8)\ \det(\mathcal{B}_2', \mathcal{B}_3')\big|_{\{h=1/3,\, g=1/3\}} = && -16 \neq 0.
\end{aligned} \tag{3.13}$$

We observe that the determinant (ω_8) is not zero. Moreover since $\beta_7 \neq 0$ and $\beta_8 \neq 0$ we deduce that none of the determinants (ω_i) $(i = 1, 2, 3)$ could vanish.

On the other hand for systems (3.3) with $c = 0$ we have $\theta = (g-1)(3g+1)/9$ in the case $h = 1/3$ and $\theta = (h-1)(3h+1)/9$ in the case $g = 1/3$. Therefore due to $\theta \neq 0$ in the cases (ω_4) and (ω_5) we also could not have zero determinants.

Thus it remains to consider the cases (ω_6) and (ω_7). Considering Remark 3.3 we observe that the case (ω_7) could be brought to the case (ω_6). So assuming $h = 1/3$ we calculate

$$\beta_7 = 2(2g-1)(6g-1)/9, \quad \beta_8 = -2(4g-1)(12g-5)/9, \quad \theta = (g-1)(3g+1)/9$$

and hence the determinant corresponding to the case (ω_6) could not be zero due to $\theta\beta_7\beta_8 \neq 0$. This completes the proof of our claim.

Possibility $\beta_8 = 0$. In this case we obtain $(4g-1)(4h-1)(3-4g-4h) = 0$ and due to Remark 3.3 we may assume $h = 1/4$. Then $\det(\mathcal{B}_1, \mathcal{B}_2) = 0$ (see the case (ω_1)) and we obtain $\mathcal{B}_1 = (2a - b - 4ag)/2 = -\mathcal{B}_2 = 0$. Since in this case we have

$$\delta_2 = 2(2g-1)(4g-1)(b-2a+4ag), \quad \beta_7 = (2g-1)(4g-1)/2$$

we conclude that due $\beta_7 \neq 0$ the condition $2a - b - 4ag = 0$ is equivalent to $\delta_2 = 0$. So setting $b = 2a(1-2g)$ we arrive at the family of systems

$$\frac{dx}{dt} = a + gx^2 - 3xy/4, \quad \frac{dy}{dt} = 2a(1-2g) + (g-1)xy + y^2/4. \tag{3.14}$$

These systems possess the invariant hyperbolas

$$\Phi_1''(x, y) = -4a + 2xy = 0, \quad \Phi_2''(x, y) = 4a + 2x(x-y) = 0,$$

which are irreducible if and only if $a \neq 0$. Since for these systems we have

$$\mathcal{R}_5 = 9a(2x - 4gx - y)\big[16(g-1)^2x^2 + 8(5g-3)xy + 9y^2\big]/4$$

the condition $a \neq 0$ is equivalent to $\mathcal{R}_5 \neq 0$. On the other hand for these systems we calculate

$$\mathcal{B}_3 = -2a(2g-1)(4g-1), \quad \mathcal{B}_3'\big|_{h=1/4} = 49a/24$$

and because $\beta_7\mathcal{R}_5 \neq 0$ we obtain $\mathcal{B}_3\mathcal{B}_3' \neq 0$, i.e. systems (3.14) could not possess a third hyperbola. This completes the proof of the statement (A2).

Subcase $\beta_7 = 0$. Then $(2g-1)(2h-1)(1-2g-2h) = 0$ and due to Remark 3.3 we may assume $h = 1/2$. Then by Lemma 3.2 we must have $g(2g-1) \neq 0$ and this is equivalent to $\beta_9 = -4g(2g-1) \neq 0$. Herein we have $\det(\mathcal{B}_1, \mathcal{B}_2) = 0$ and we obtain $\mathcal{B}_1 = a(1-2g) = 0$ and $\mathcal{B}_2 = 2ag = 0$. This implies $a = 0$, which due to $\beta_9 \neq 0$ is equivalent to $\delta_3 = 16a^2g^2(2g-1)^2 = 0$. So we obtain the family of systems

$$\frac{dx}{dt} = gx^2 - xy/2, \quad \frac{dy}{dt} = b + (g-1)xy + y^2/2 \tag{3.15}$$

which possess the following two hyperbolas

$$\Phi_1(x,y) = -\frac{2b}{2g-1} + 2xy = 0, \quad \Phi_2(x,y) = -\frac{b}{g} + 2x(x-y) = 0.$$

These hyperbolas are irreducible if and only if $b \neq 0$ which is equivalent to $\mathcal{R}_5 = 9bx\big[4(g-1)^2x^2 + 4(3g-1)xy + y^2\big] \neq 0$.

For the above systems we have $\mathcal{B}_3 = b(4g-1)$ and $\mathcal{B}_3' = 25b/4$. Since $b \neq 0$ only the condition $\mathcal{B}_3 = 0$ could be satisfied and this implies $g = 1/4$. It is not too hard to find out that in this case we obtain the third hyperbola:

$$\Phi_3(x,y) = -4b + 2y(x-y) = 0.$$

We observe that for the systems above $\beta_8 = -2(4g-1)^2$ and hence the third hyperbola exists if and only if $\beta_8 = 0$. So the statements $(\mathcal{A}_3)$ and $(\mathcal{B})$ are proved.

Since all the possibilities are examined, Lemma 3.5 is proved. □

3.2. **Systems with three real infinite singularities and $\theta = 0$.** Considering (3.2) for systems (3.1) we obtain $(g-1)(h-1)(g+h) = 0$ and we may assume $g = -h$, otherwise in the case $g = 1$ (respectively $h = 1$) we apply the change $(x,y,g,h) \mapsto (-y, x-y, 1-g-h, g)$ (respectively $(x,y,g,h) \mapsto (y-x, -x, h, 1-g-h)$) which preserves the quadratic parts of systems (3.1).

So $g = -h$ and for systems (3.1) we calculate $N = 9(h^2-1)(x-y)^2$. We consider two cases: $N \neq 0$ and $N = 0$.

3.2.1. *Case* $N \neq 0$. Then $(h-1)(h+1) \neq 0$ and due to a translation we may assume $d = e = 0$ and this leads to the family of systems

$$\frac{dx}{dt} = a + cx - hx^2 + (h-1)xy, \quad \frac{dy}{dt} = b + fy - (h+1)xy + hy^2. \tag{3.16}$$

Remark 3.6. We observe that by changing $(x,y,a,b,c,f,h) \mapsto (y,x,b,a,f,c,-h)$ which conserves systems (3.16) we can change the sign of the parameter h.

Lemma 3.7. *A system* (3.16) *with* $(h-1)(h+1) \neq 0$ *possesses at least one invariant hyperbola of the indicated form if and only if the following conditions are satisfied, respectively:*

(I) $\Phi(x,y) = p + qr + ry + 2xy \;\Leftrightarrow\; c + f = 0$, $\mathcal{E}_1 \equiv a(2h+1) + b(2h-1) = 0$, $a^2 + b^2 \neq 0$;

(II) $\Phi(x,y) = p + qr + ry + 2x(x-y) \;\Leftrightarrow\; c - f = 0$ *and either*

(i) $(2h-1)(3h-1)\neq 0$, $\mathcal{E}_2 \equiv 2c^2(h-1)(2h-1)+(3h-1)^2(b-2a+2ah-2bh)=0$, $a\neq 0$, *or*
(ii) $h=1/3$, $c=0$, $a\neq 0$, *or*
(iii) $h=1/2$, $a=0$, $b+4c^2\neq 0$;

(III) $\Phi(x,y)=p+qr+ry+2y(x-y) \;\Leftrightarrow\; c-f=0$ *and either*
(i) $(2h+1)(3h+1)\neq 0$, $\mathcal{E}_3 \equiv 2c^2(h+1)(2h+1)+(3h+1)^2(a-2b-2bh+2ah)=0$, $b\neq 0$, *or*
(ii) $h=-1/3$, $c=0$, $b\neq 0$, *or*
(iii) $h=-1/2$, $b=0$, $a+4c^2\neq 0$.

Proof. As it was mentioned in the proof of Lemma 3.2 (see page 14) we may assume that the quadratic part of an invariant hyperbola has one of the following forms: (i) $2xy$, (ii) $2x(x-y)$, (iii) $2y(x-y)$. Considering the equations (2.7) we examine each one of these possibilities.

(i) $\Phi(x,y)=p+qx+ry+2xy$; in this case because $N\neq 0$ (i.e. $(h-1)(h+1)\neq 0$) we obtain

$$\begin{gathered}
t=1,\quad q=r=s=u=0,\quad U=-2h-1,\quad V=2h-1,\quad W=c+f,\\
Eq_8=p(1+2h)+2b,\quad Eq_9=p(1-2h)+2a,\quad Eq_{10}=-p(c+f),\\
Eq_1=Eq_2=Eq_3=Eq_4=Eq_5=Eq_6=Eq_7=0.
\end{gathered}$$

Since in this case the hyperbola has the form $\Phi(x,y)=p+2xy$ it is clear that $p\neq 0$, otherwise we obtain a reducible hyperbola. So the condition $c+f=0$ is necessary.

Calculating the resultant of the non-vanishing equations with respect to the parameter p we obtain

$$\mathrm{Res}_p(Eq_8,Eq_9)=2[a(2h+1)+b(2h-1)]=2\mathcal{E}_1.$$

Since $(2h-1)^2+(2h+1)^2\neq 0$ we conclude that an invariant hyperbola exists if and only if $\mathcal{E}_1=0$. Due to Remark 3.6 we may assume $2h-1\neq 0$. Then we obtain

$$p=2a/(2h-1),\quad b=a(2h+1)/(2h-1),\quad \Phi(x,y)=\frac{2a}{2h-1}+2xy=0$$

and clearly for the irreducibility of the hyperbola the condition $a\neq 0$ must hold.

This completes the proof of the statement I of the lemma.

(ii) $\Phi(x,y)=p+qx+ry+2x(x-y)$; since $(h-1)(h+1)\neq 0$ (because $N\neq 0$) we obtain

$$\begin{gathered}
s=2,\quad t=-1,\quad r=u=0,\quad U=-2h,\quad V=2h-1,\quad W=(4c+hq)/2,\\
Eq_6=2(c-f),\quad Eq_8=4a-2b+2hp-cg-hq^2/2,\\
Eq_9=p(1-2h)-2a,\quad Eq_{10}=-2cp+aq-hpq/2,\\
Eq_1=Eq_2=Eq_3=Eq_4=Eq_5=Eq_7=0.
\end{gathered}\tag{3.17}$$

We observe that the equation $Eq_6=0$ implies the condition $c-f=0$.

(1) Assume first $(2h-1)(3h-1)\neq 0$. Then considering the equation $Eq_9=0$ we obtain $p=2a/(1-2h)$. As the hyperbola $\Phi(x,y)=p+qx+2x(x-y)=0$ has to be irreducible the condition $p\neq 0$ holds and this implies $a\neq 0$. Therefore from

$$Eq_{10}=\frac{a(4c-q+3hq)}{2h-1}=0$$

From $3h-1 \neq 0$ we obtain $q = 4c/(1-3h)$ and then we obtain

$$Eq_8 = \frac{2\mathcal{E}_2}{(2h-1)(3h-1)^2} = 0.$$

So we deduce that the conditions $c - f = 0$, $\mathcal{E}_2 = 0$ and $a \neq 0$ are necessary and sufficient for the existence of a hyperbola of systems (3.16) in the case $(2h-1)(3h-1) \neq 0$.

(2) Suppose now $h = 1/3$. Then considering (3.17) we have $Eq_9 = (p-6a)/3 = 0$, i.e. $p = 6a \neq 0$ (otherwise we obtain a reducible hyperbola). Therefore the equation $Eq_{10} = -12ac = 0$ yields $c = 0$. Herein the equation $Eq_8 = 0$ becomes $Eq_8 = [12(4a-b) - q^2]/6 = 0$, i.e. $q = \pm 2\sqrt{3(4a-b)}$ and obviously we obtain at leas one real hyperbola if $4a - b \geq 0$ and two complex if $4a - b < 0$.

Thus in the case $h = 1/3$ we have at least one hyperbola if and only if the conditions $f = c = 0$ and $a \neq 0$ hold.

(3) Assume finally $h = 1/2$. In this case we obtain $Eq_9 = -2a = 0$, i.e. $a = 0$ and we have

$$Eq_8 = -2b + p - cq - q^2/4 = 0, \quad Eq_{10} = -p(8c+q)/4 = 0,$$
$$\Phi(x,y) = p + qx + 2x(x-y).$$

Therefore $p \neq 0$ and we obtain $q = -8c$ and $p = 2(b + 4c^2) \neq 0$. This completes the proof of the statement II of the lemma.

(iii) $\Phi(x,y) = p + qx + ry + 2y(x-y)$; we observe that due to the change $(x, y, a, b, c, f, h) \mapsto (y, x, b, a, c, f, -h)$ (which preserves systems (3.16)) this case could be brought to the previous one and hence, the conditions could be constructed directly applying this change.

Thus Lemma 3.7 is proved. □

We shall construct now the affine invariant conditions for the existence of at least one invariant hyperbola for quadratic systems in the considered family.

Lemma 3.8. *Assume that for a quadratic system* (2.6) *the conditions* $\eta > 0$, $\theta = 0$, $N \neq 0$, *and* $\gamma_1 = \gamma_2 = 0$ *hold. Then this system possesses at least one invariant hyperbola if and only if one of the following sets of the conditions is satisfied:*

- (i) *If* $\beta_6 \neq 0$ *then either*
  - (i.1) $\beta_{10} \neq 0$, $\gamma_7 = 0$, $\mathcal{R}_6 \neq 0$, *or*
  - (i.2) $\beta_{10} = 0$, $\gamma_4 = 0$, $\beta_2\mathcal{R}_3 \neq 0$;
- (ii) *If* $\beta_6 = 0$ *then either*
  - (ii.1) $\beta_2 \neq 0$, $\beta_7 \neq 0$, $\gamma_8 = 0$, $\beta_{10}\mathcal{R}_7 \neq 0$, *or*
  - (ii.2) $\beta_2 \neq 0$, $\beta_7 = 0$, $\gamma_9 = 0$, $\mathcal{R}_8 \neq 0$, *or*
  - (ii.3) $\beta_2 = 0$, $\beta_7 \neq 0$, $\beta_{10} \neq 0$, $\gamma_7\gamma_8 = 0$, $\mathcal{R}_5 \neq 0$, *or*
  - (ii.4) $\beta_2 = 0$, $\beta_7 \neq 0$, $\beta_{10} = 0$, $\mathcal{R}_3 \neq 0$, $\gamma_7 \neq 0$, *or*
  - (ii.5) $\beta_2 = 0$, $\beta_7 \neq 0$, $\beta_{10} = 0$, $\mathcal{R}_3 \neq 0$, $\gamma_7 = 0$, *or*
  - (ii.6) $\beta_2 = 0$, $\beta_7 = 0$, $\gamma_7 = 0$, $\mathcal{R}_3 \neq 0$.

Proof. Assume that for a quadratic system (2.6) the conditions $\eta > 0$, $\theta = 0$ and $N \neq 0$ are fulfilled. As it was mentioned earlier due to an affine transformation and time rescaling this system could be brought to the canonical form (3.16), for which we calculate

$$\gamma_1 = (c-f)^2(c+f)(h-1)^2(h+1)^2(3h-1)(3h+1)/64,$$

$$\beta_6 = (c-f)(h-1)(h+1)/4, \quad \beta_{10} = -2(3h-1)(3h+1).$$

Subcase $\beta_6 \neq 0$. By Lemma 2.10 for the existence of an invariant hyperbola of systems (3.16) the condition $\gamma_1 = 0$ is necessary and this condition is equivalent to $(c+f)(3h-1)(3h+1) = 0$. We examine two possibilities: $\beta_{10} \neq 0$ and $\beta_{10} = 0$.

Possibility $\beta_{10} \neq 0$. Then we obtain $f = -c$ (this implies $\gamma_2 = 0$) and we have

$$\gamma_7 = 8(h-1)(h+1)\,\mathcal{E}_1.$$

Therefore becuase $\beta_6 \neq 0$ the condition $\gamma_7 = 0$ is equivalent to $\mathcal{E}_1 = 0$. So we have $a = \lambda(2h-1)$, $b = -\lambda(2h+1)$ (where $\lambda \neq 0$ is an arbitrary parameter) and then we calculate

$$\mathcal{R}_6 = -632\lambda c(h-1)(h+1).$$

Since $\beta_6 \neq 0$ we deduce that the condition $\mathcal{R}_6 \neq 0$ is equivalent to $a^2 + b^2 \neq 0$. This completes the proof of the statement (i.1) of the lemma.

Possibility $\beta_{10} = 0$. Then we have $(3h-1)(3h+1) = 0$ and by Remark 3.6 we may assume $h = 1/3$. Then we obtain the 4-parameter family of systems

$$\frac{dx}{dt} = a + cx - x^2/3 - 2xy/3, \quad \frac{dy}{dt} = b + fy - 4xy/3 + y^2/3, \tag{3.18}$$

for which we calculate $\gamma_1 = 0$ and

$$\gamma_2 = 44800(c-f)^2(c+f)(2c-f)/243,$$
$$\beta_6 = -2(c-f)/9, \quad \beta_2 = -4(2c-f)/9.$$

Since $\beta_6 \neq 0$ (i.e. $c - f \neq 0$) by Lemma 2.10 the necessary condition $\gamma_2 = 0$ gives $(c+f)(2c-f) = 0$. We claim that for the existence of an invariant hyperbola the condition $2c - f \neq 0$ (i.e. $\beta_2 \neq 0$) must be satisfied. Indeed, setting $f = 2c$ we obtain $\beta_6 = 2c/9 \neq 0$. However, according to the Lemma 3.7, for the existence of a hyperbola of systems (3.18), the condition $(c+f)(c-f) = 0$ is necessary , which for $f = 2c$ becomes $-3c^2 = 0$. The contradiction obtained proves our claim.

Thus the condition $\beta_2 \neq 0$ is necessary and then we have $f = -c$. By Lemma 3.7 in the case $h = 1/3$ we have an invariant hyperbola (which is of the form $\Phi(x,y) = p + qx + ry + 2xy = 0$) if and only if $\mathcal{E}_1 = (5a-b)/3 = 0$ and $a^2 + b^2 \neq 0$.

On the other hand for systems (3.18) with $f = -c$ we calculate

$$\gamma_4 = -4096c^2\mathcal{E}_1/243, \quad \beta_6 = -4c/9, \quad \mathcal{R}_3 = -4a/9.$$

So the statement (i.2) of the lemma is proved.

Subcase $\beta_6 = 0$. Then $f = c$ (this implies $\gamma_2 = 0$) and we calculate

$$\gamma_8 = 42(h-1)(h+1)\mathcal{E}_2\mathcal{E}_3, \quad \beta_2 = c(h-1)(h+1)/2,$$
$$\beta_7 = -2(2h-1)(2h+1), \quad \beta_{10} = -2(3h-1)(3h+1),$$
$$\mathcal{R}_7 = -(h-1)(h+1)U(a,b,c,h)/4,$$

where $U(a,b,c,h) = 2c^2(h-1)(h+1) - b(h+1)(3h-1)^2 + a(h-1)(3h+1)^2$.

Possibility $\beta_2 \neq 0$. Then $c \neq 0$ and we shall consider two cases: $\beta_7 \neq 0$ and $\beta_7 = 0$.

(1) *Case* $\beta_7 \neq 0$. We observe that in this case for the existence of a hyperbola the condition $\beta_{10} \neq 0$ is necessary. Indeed, since $f = c \neq 0$ and $(2h-1)(2h+1) \neq 0$, according to Lemma 3.7 (see the statements II and III for the existence of at least one invariant hyperbola it is necessary and sufficient $(3h-1)(3h+1) \neq 0$ and either $\mathcal{E}_2 = 0$ and $a \neq 0$, or $\mathcal{E}_3 = 0$ and $b \neq 0$.

We claim that the condition $a \neq 0$ (when $\mathcal{E}_2 = 0$) as well as the condition $b \neq 0$ (when $\mathcal{E}_3 = 0$) is equivalent to $U(a, b, c, h) \neq 0$. Indeed, as $\mathcal{E}_2$ as well as $\mathcal{E}_3$ and $U(a, b, c, h)$ are linear polynomials in a and b, then the equations $\mathcal{E}_2 = U(a, b, c, h) = 0$ (respectively $\mathcal{E}_2 = U(a, b, c, h) = 0$) with respect to a and b gives $a = 0$ and $b = 2c^2(h-1)/(3h-1)^2$ (respectively $b = 0$ and $a = -2c^2(h+1)/(3h+1)^2$). This proves our claim.

It remains to observe that the condition $\mathcal{E}_2\mathcal{E}_3 = 0$ is equivalent to $\gamma_8 = 0$. So this completes the proof of the statement *(ii.1)* of the lemma.

(2) *Case* $\beta_7 = 0$. Then by Remark 3.6 we may assume $h = 1/2$ and since $f = c$, by Lemma 3.7 for the existence of a hyperbola of systems (3.16) (with $h = 1/2$ and $f = c$) the conditions $a = 0$ and $b + 4c^2 \neq 0$. On the other hand we calculate

$$\gamma_9 = 3a/2, \quad \mathcal{R}_8 = (7a + b + 4c^2)/8$$

and clearly these invariant polynomials govern the above conditions. So the statement (ii.2) of the lemma is proved.

Possibility $\beta_2 = 0$. In this case we have $f = c = 0$.

(1) *Case* $\beta_7 \neq 0$. Then $(2h-1)(2h+1) \neq 0$.

(a) *Subcase* $\beta_{10} \neq 0$. In this case $(3h-1)(3h+1) \neq 0$. By Lemma 3.7 we could have an invariant hyperbola if and only if $\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3 = 0$. On the other hand for systems (3.16) with $f = c = 0$ we have

$$\gamma_7\gamma_8 = -336(h-1)^2(1+h)^2\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3,$$
$$\mathcal{R}_5 = 36(bx - ay)(x - y)\big[(1+h)^2x - (h-1)^2y\big]$$

and therefore the condition $\mathcal{R}_5 \neq 0$ is equivalent to $a^2 + b^2 \neq 0$. This completes the proof of the statement (ii.3) of the lemma.

(b) *Subcase* $\beta_{10} = 0$. Then we have $(3h-1)(3h+1) = 0$ and by Remark 3.6 we may assume $h = 1/3$. By Lemma 3.7 we could have an invariant hyperbola if and only if either the conditions I or II; (ii) of Lemma 3.7 are satisfied. In this case we calculate

$$\gamma_7 = -64\mathcal{E}_1/9, \quad \mathcal{R}_3 = -4a/9$$

and hence, the condition $\mathcal{R}_3 \neq 0$ implies the irreducibility of the hyperbola. Therefore in the case $\gamma_7 \neq 0$ we arrive at the statement *(ii.4)* of the lemma, whereas for $\gamma_7 = 0$ the statement *(ii.5)* of the lemma holds.

(2) *Case* $\beta_7 = 0$. Then $(2h-1)(2h+1) = 0$ and by Remark 3.6 we may assume $h = 1/2$. By Lemma 3.7 we could have an invariant hyperbola if and only if either the conditions $\mathcal{E}_1 = 2a = 0$ and $b \neq 0$ (see statement I) or $a = 0$ and $b \neq 0$ (see statement II; (iii) of the lemma) are fulfilled. As we could see the conditions coincide and hence by this lemma we have two hyperbolas: the asymptotes of one of them are parallel to the lines $x = 0$ and $y = 0$, whereas the asymptotes of the other hyperbola are parallel to the lines $x = 0$ and $y = x$.

On the other hand for systems (3.16) (with $h = 1/2$ and $f = c = 0$) we calculate

$$\gamma_7 = -12a, \quad \mathcal{R}_3 = (5a - b)/16$$

and this leads to the statement (ii.6) of the lemma.

Since all the possibilities are considered, Lemma 3.8 is proved. $\square$

Lemma 3.9. *Assume that for a quadratic system* (2.6) *the conditions* $\eta > 0$, $\theta = 0$, $N \neq 0$ *and* $\gamma_1 = \gamma_2 = 0$ *are satisfied. Then this system possesses:*

(A) *three distinct invariant hyperbolas if and only if* $\beta_6 = \beta_2 = \beta_{10} = \gamma_7 = 0$, $\beta_7\mathcal{R}_3 \neq 0$ *and* $\gamma_{10} \neq 0$; *more precisely all three hyperbolas are real (1* $\mathcal{H}$ *and 2* $\mathcal{H}^p$*) if* $\gamma_{10} > 0$ *and one is real and two are complex (1* $\mathcal{H}$ *and 2* $\overset{c}{\mathcal{H}}{}^p$*) if* $\gamma_{10} < 0$;

(B) *two distinct invariant hyperbolas if and only if* $\beta_6 = 0$ *and either*
- (B1) $\beta_2 \neq 0$, $\beta_7 \neq 0$, $\gamma_8 = 0$, $\beta_{10}\mathcal{R}_7 \neq 0$ *and* $\delta_4 = 0$ *(*$\Rightarrow$ *2* $\mathcal{H}$*), or*
- (B2) $\beta_2 \neq 0$, $\beta_7 = 0$, $\gamma_9 = 0$, $\mathcal{R}_8 \neq 0$ *and* $\delta_5 = 0$ *(*$\Rightarrow$ *2* $\mathcal{H}$*), or*
- (B3) $\beta_2 = 0$, $\beta_7 \neq 0$, $\beta_{10} \neq 0$, $\gamma_7\gamma_8 = 0$, $\mathcal{R}_5 \neq 0$ *and* $\beta_8 = \delta_2 = 0$ *(*$\Rightarrow$ *2* $\mathcal{H}$*), or*
- (B4) $\beta_2 = 0$, $\beta_7 \neq 0$, $\beta_{10} = 0$, $\gamma_7 \neq 0$, $\mathcal{R}_3 \neq 0$ *and* $\gamma_{10} < 0$ *(*$\Rightarrow$ *2* $\overset{c}{\mathcal{H}}{}^p$*)), or*
- (B5) $\beta_2 = 0$, $\beta_7 \neq 0$, $\beta_{10} = 0$, $\gamma_7 \neq 0$, $\mathcal{R}_3 \neq 0$ *and* $\gamma_{10} > 0$ *(*$\Rightarrow$ *2* $\mathcal{H}^p$*), or*
- (B6) $\beta_2 = 0$, $\beta_7 = 0$, $\gamma_7 = 0$, $\mathcal{R}_3 \neq 0$ *(*$\Rightarrow$ *2* $\mathcal{H}$*);*

(C) *one double (*$\mathcal{H}_2^p$*) invariant hyperbola if and only if* $\beta_6 = \beta_2 = 0$, $\beta_7 \neq 0$, $\beta_{10} = 0$, $\gamma_7 \neq 0$, $\mathcal{R}_3 \neq 0$ *and* $\gamma_{10} = 0$.

Proof. For systems (3.16) we calculate

$$\begin{gathered}\beta_6 = (c - f)(h - 1)(h + 1)/4, \quad \beta_7 = -2(2h + 1)(2h - 1),\\ \beta_{10} = -2(3h + 1)(3h - 1), \quad \beta_2 = \big[(c + f)(h^2 - 1) - 8(c - f)h\big]/4.\end{gathered} \tag{3.19}$$

According to Lemma 3.7 in order to have at least two invariant hyperbolas the condition $c - f = 0$ must hold. This condition is governed by the invariant polynomial β_6 and in what follows we assume $\beta_6 = 0$ (i.e. $f = c$).

Case $\beta_2 \neq 0$. Then we have $c \neq 0$ and the conditions given by the statement I of Lemma 3.7 could not be satisfied.

Case $\beta_7 \neq 0$. We observe that in this case due to $c \neq 0$ we could have two invariant hyperbolas if and only if $(3h-1)(3h+1) \neq 0$ (i.e $\beta_{10} \neq 0$), $\mathcal{E}_2 = \mathcal{E}_3 = 0$ and $ab \neq 0$. The system of equations $\mathcal{E}_2 = \mathcal{E}_3 = 0$ with respect to the parameters a and b gives the solution

$$a = -\frac{2c^2(1 + h)^3(2h - 1)}{(3h - 1)^2(1 + 3h)^2} \equiv a_0, \quad b = -\frac{2c^2(h - 1)^3(1 + 2h)}{(3h - 1)^2(1 + 3h)^2} \equiv b_0, \tag{3.20}$$

which exists and $ab \neq 0$ by the condition $(2h - 1)(2h + 1)(3h - 1)(3h + 1) \neq 0$.

In this case systems (3.16) with $a = a_0$ and $b = b_0$ possess the two hyperbolas

$$\begin{aligned}\Phi_1^{(1)}(x, y) &= \frac{4c^2(1 + h)^3}{(3h - 1)^2(1 + 3h)^2} - \frac{4c}{3h - 1}\,x + 2x(x - y) = 0,\\ \Phi_2^{(1)}(x, y) &= \frac{4c^2(h - 1)^3}{(3h - 1)^2(1 + 3h)^2} - \frac{4c}{1 + 3h}\,y + 2y(x - y) = 0.\end{aligned}$$

Since $c \neq 0$ by Lemma 3.7 we could not have a third invariant hyperbola.

Now we need the invariant polynomials which govern the condition $\mathcal{E}_2 = \mathcal{E}_3 = 0$. First we recall that for these systems we have $\gamma_8 = 42(h-1)(h+1)\mathcal{E}_2\mathcal{E}_3$, and hence the condition $\gamma_8 = 0$ is necessary. In order to set $\mathcal{E}_2 = 0$ we use the following parametrization:

$$c = c_1(3h-1)^2, \quad a = a_1(2h-1)$$

and then the condition $\mathcal{E}_2 = 0$ gives $b = 2(h-1)(a_1 + c_1^2)$. Herein for systems (3.16) with

$$f = c = c_1(3h-1)^2, \quad a = a_1(2h-1), \quad b = 2(h-1)(a_1 + c_1^2)$$

we calculate

$$\mathcal{E}_3 = 3\big[2c_1^2(1+h)^3 + a_1(1+3h)^2\big], \quad \delta_4 = (h-1)(2h-1)\mathcal{E}_3/2$$

and hence the condition $\mathcal{E}_3 = 0$ is equivalent to $\delta_4 = 0$.

It remains to observe that in this case $\mathcal{R}_7 = -3a_1(h-1)^4(h+1)/4 \neq 0$, otherwise $a_1 = 0$ and then the condition $\delta_4 = 0$ implies $c_1 = 0$, i.e. $c = 0$ and this contradicts to $\beta_2 \neq 0$. So we arrive at the statement (B1) of the lemma.

Case $\beta_7 = 0$. Then $(2h-1)(2h+1) = 0$ and by Remark 3.6 we may assume $h = 1/2$. In this case by Lemma 3.7 in order to have at least two hyperbolas the conditions II; (iii) and III (i) have to be satisfied simultaneously. Therefore we arrive at the conditions

$$a = 0, \quad b + 4c^2 \neq 0, \quad \mathcal{E}_3 = (50a - 75b + 24c^2)/4 = 0$$

and as $a = 0$ we have $b = 24c^2/75$ and $b + 4c^2 = 108c^2/25 \neq 0$ due to $\beta_2 \neq 0$. So we obtain the family of systems

$$\frac{dx}{dt} = cx - x(x+y)/2, \quad \frac{dy}{dt} = 8c^2/25 + cy - y(3x-y)/2 \tag{3.21}$$

which possess the two invariant hyperbolas

$$\Phi_1^{(2)}(x,y) = 216c^2/25 - 8cx + 2x(x-y) = 0,$$
$$\Phi_2^{(2)}(x,y) = -8c^2/25 - 8cy/5 + 2y(x-y) = 0.$$

These hyperbolas are irreducible due to $\beta_2 \neq 0$ (i.e. $c \neq 0$).

We need to determine the affine invariant conditions which are equivalent to $a = \mathcal{E}_3 = 0$. For systems (3.16) with $f = c$ and $h = 1/2$ we calculate

$$\gamma_9 = 3a/2, \quad \delta_5 = -3(25b - 8c^2)/2$$

and obviously these invariant polynomials govern the conditions mentioned before. It remains to observe that for systems (3.21) we have $\mathcal{R}_8 = 108c^2/25 \neq 0$ due to $\beta_2 \neq 0$. This completes the proof of the statement (B2) of the lemma.

Case $\beta_2 = 0$. Then $c = 0$ and by Lemma 3.7 systems (3.16) with $f = c = 0$ could possess at least two invariant hyperbolas if and only if one of the following sets of conditions holds:

$$
\begin{aligned}
&(\phi_1)\ \ \mathcal{E}_1=\mathcal{E}_2=0,\quad (2h-1)(3h-1)\neq 0,\quad a\neq 0;\\
&(\phi_2)\ \ \mathcal{E}_1=\mathcal{E}_3=0,\quad (2h+1)(3h+1)\neq 0,\quad b\neq 0;\\
&(\phi_3)\ \ \mathcal{E}_2=\mathcal{E}_3=0,\quad (2h-1)(2h+1)(3h-1)(3h+1)\neq 0,\quad ab\neq 0;\\
&(\phi_4)\ \ \mathcal{E}_1=0,\quad h=1/3,\quad a\neq 0;\\
&(\phi_5)\ \ \mathcal{E}_1=a=0,\quad h=1/2,\quad b\neq 0;\\
&(\phi_6)\ \ \mathcal{E}_1=0,\quad h=-1/3,\quad b\neq 0;\\
&(\phi_7)\ \ \mathcal{E}_1=b=0,\quad h=-1/2,\quad a\neq 0.
\end{aligned}
\tag{3.22}
$$

As for systems (3.16) with $f=c=0$ we have

$$\beta_7=-2(2h+1)(2h-1),\quad \beta_{10}=-2(3h+1)(3h-1)$$

we consider two subcases: $\beta_7\neq 0$ and $\beta_7=0$.

Subcase $\beta_7\neq 0$. Then $(2h+1)(2h-1)\neq 0$ and we examine two possibilities: $\beta_{10}\neq 0$ and $\beta_{10}=0$.

(1) *Possibility* $\beta_{10}\neq 0$. In this case $(3h+1)(3h-1)\neq 0$. We observe that due to $f=c=0$ all tree polynomials $\mathcal{E}_i$ are linear (homogeneous) with respect to the parameters a and b. So each one of the sets of conditions (ϕ_1)–(ϕ_3) could be compatible only if the corresponding determinant vanishes, i.e.

$$
\begin{gathered}
\det(\mathcal{E}_1,\mathcal{E}_2)\Rightarrow -(2h-1)(3h-1)^2(4h-1)=0,\\
\det(\mathcal{E}_1,\mathcal{E}_3)\Rightarrow (2h+1)(3h+1)^2(4h+1)=0,\\
\det(\mathcal{E}_2,\mathcal{E}_3)\Rightarrow -3(3h-1)^2(3h+1)^2=0,
\end{gathered}
\tag{3.23}
$$

otherwise we obtain the trivial solution $a=b=0$. Clearly the third determinant could not be zero due to the condition $\beta_{10}\neq 0$, i.e. the conditions in the set (ϕ_3) are incompatible in this case. As regard the conditions (ϕ_1) (respectively (ϕ_2)) we observe that they could be compatible only if $4h-1=0$ (respectively $4h+1=0$).

On the other hand we have $\beta_8=-6(4h-1)(4h+1)$ and we conclude that for the existence of two hyperbolas in these case the condition $\beta_8=0$ is necessary.

Assuming $\beta_8=0$ we may consider $h=1/4$ due to Remark 3.6 and we obtain

$$\mathcal{E}_1=(3a-b)/2=-16\mathcal{E}_2=0.$$

So we obtain $b=3a$ and we arrive at the systems

$$\frac{dx}{dt}=a-x^2/4-3xy/4,\quad \frac{dy}{dt}=3a-5xy/4+y^2/4,\tag{3.24}$$

which possess the two invariant hyperbolas

$$\Phi_1^{(3)}(x,y)=-4a+2xy=0,\quad \Phi_2^{(3)}(x,y)=4a+2x(x-y)=0.$$

Clearly these hyperbolas are irreducible if and only if $a\neq 0$.

On the other hand for systems (3.16) with $f=c=0$ and $h=1/4$ we have

$$
\begin{gathered}
\gamma_7=-15(3a-b),\quad \gamma_8=15435(3a-5b)(3a-b))/8192,\\
\delta_2=-6(3a-b),\quad \mathcal{R}_5=9(bx-ay)(25x-9y)(x-y)/4.
\end{gathered}
$$

We observe that the conditions $\mathcal{E}_1=\mathcal{E}_2=0$ and $a\neq 0$ are equivalent to $\gamma_7=0$ and $\mathcal{R}_5\neq 0$. However to insert this possibility in the generic diagram (see Figure 1) we remark that these conditions are equivalent to $\gamma_7\gamma_8=\delta_2=0$ and $\mathcal{R}_5\neq 0$.

It remains to observe that for the systems above we have $\mathcal{E}_3 = 147a/8 \neq 0$ and, hence we could not have a third hyperbola. So the statement (B3) of the lemma is proved.

(2) *Possibility* $\beta_{10} = 0$. In this case $(3h+1)(3h-1) = 0$ and without loss of generality we may assume $h = 1/3$ due to the change $(x, y, a, b, h) \mapsto (y, x, b, a, -h)$, which conserves systems (3.16) with $f = c = 0$ and transfers the conditions (ϕ_6) to (ϕ_4).

So $h = 1/3$ and we arrive at the following 2-parameter family of systems

$$\frac{dx}{dt} = a - x^2/3 - 2xy/3, \quad \frac{dy}{dt} = b - 4xy/3 + y^2/3, \tag{3.25}$$

for which we have $\mathcal{E}_1 = (5a-b)/3$ and we shall prove the next statements:

- if $\mathcal{E}_1 \neq 0$, $4a - b < 0$ and $a \neq 0$ we have 2 complex invariant hyperbolas $\overset{c}{\mathcal{H}^p}$;
- if $\mathcal{E}_1 \neq 0$, $4a - b > 0$ and $a \neq 0$ we have 2 real invariant hyperbolas $\mathcal{H}^p$;
- if $\mathcal{E}_1 \neq 0$, $4a - b = 0$ and $a \neq 0$ we have one double invariant hyperbola $\mathcal{H}^p_2$.
- if $\mathcal{E}_1 = 0$, $4a - b > 0$ and $a \neq 0$ we have 3 real invariant hyperbolas (two of them being $\mathcal{H}^p$);
- if $\mathcal{E}_1 = 0$, $4a - b < 0$ and $a \neq 0$ we have 1 real and two complex invariant hyperbolas (of the type $\overset{c}{\mathcal{H}^p}$).

So we consider two cases: $\mathcal{E}_1 \neq 0$ and $\mathcal{E}_1 = 0$

(a) *Case* $\mathcal{E}_1 \neq 0$. In this case by Lemma 3.7 we could not have an invariant hyperbola with the quadratic part of the form xy. However systems (3.25) possess the following two invariant hyperbola:

$$\Phi^{(4)}_{1,2}(x, y) = 3a \pm \sqrt{3(4a-b)}\,x + x(x-y) = 0$$

and these conics are irreducible if and only if $a \neq 0$. Moreover the above hyperbolas have parallel asymptotes and they are real if $4a - b > 0$ (i.e . we have two $\mathcal{H}^p$) and complex if $4a - b < 0$ (i.e. we have two $\overset{c}{\mathcal{H}^p}$). We observe that in the case $4a - b = 0$ the hyperbola $\Phi^{(4)}_{1,2}(x, y) = 0$ collapse and we obtain a hyperbola of multiplicity two (i.e . we have $\mathcal{H}^p_2$).

(b) *Case* $\mathcal{E}_1 = 0$. Then $b = 5a$ and we obtain the following 1-parameter family of systems

$$\frac{dx}{dt} = a - x^2/3 - 2xy/3, \quad \frac{dy}{dt} = 5a - 4xy/3 + y^2/3. \tag{3.26}$$

which possess three invariant hyperbolas

$$\Phi^{(4)}_{1,2}(x, y) = 3a \pm \sqrt{-3a}\,x + x(x-y) = 0, \quad \Phi^{(4)}_3(x, y) = 3a - xy = 0.$$

These conics are irreducible if and only if $a \neq 0$. Also the hyperbolas $\Phi^{(4)}_{1,2}(x, y) = 0$ have parallel asymptotes and they are real if $a < 0$ and complex if $a > 0$.

Thus the above statements are proved and in order to determine the corresponding invariant conditions, for systems (3.16) with $c = f = 0$ and $h = 1/3$ we calculate

$$\gamma_7 = -64(5a-b)/27, \quad \gamma_{10} = 8(4a-b)/27, \quad \mathcal{R}_3 = -4a/9.$$

Considering the conditions given by the above statements it is easy to observe that the corresponding invariant conditions are given by the statements (B4), (B5), (C) and (A) of Lemma 3.9, respectively.

Subcase $\beta_7 = 0$. Then $(2h+1)(2h-1) = 0$ and by Remark 3.6 we may assume $h = 1/2$. Considering (3.23) we conclude that only the case (ϕ_5) could be satisfied and we obtain the additional conditions: $a = 0$, $b \neq 0$. Therefore we arrive at the family of systems

$$\frac{dx}{dt} = -x^2/2 - xy/2, \quad \frac{dy}{dt} = b - 3xy/2 + y^2/2, \tag{3.27}$$

which possess the following two hyperbolas

$$\Phi_1^{(5)}, (x,y) = -b + 2xy = 0, \quad \Phi_2^{(5)}(x,y) = 2b + 2x(x-y) = 0.$$

We observe that the condition $a = 0$ is equivalent to $\gamma_7 = -12a = 0$. Regarding the condition $b \neq 0$, in the case $a = 0$ it is equivalent to $\mathcal{R}_3 = -b/16 \neq 0$. Since for these systems we have $\mathcal{E}_3 = 75b/4 \neq 0$ we deduce that we could not have a third invariant hyperbola. This completes the proof of the statement (B6) of the lemma.

Since all the cases are examined, Lemma 3.9 is proved. □

3.2.2. *Case* $N = 0$. As $\theta = -(g-1)(h-1)(g+h)/2 = 0$ we observe that the condition $N = 0$ implies the vanishing of two factors of θ. We may assume $g = 1 = h$, otherwise in the case $g + h = 0$ and $g - 1 \neq 0$ (respectively $h - 1 \neq 0$) we apply the change $(x, y, g, h) \mapsto (-y, x - y, 1 - g - h, g)$ (respectively $(x, y, g, h) \mapsto (y - x, -x, h, 1 - g - h)$) which preserves the form of systems (3.1).

So $g = h = 1$ and from an additional translation, systems (3.1) become

$$\frac{dx}{dt} = a + dy + x^2, \quad \frac{dy}{dt} = b + ex + y^2. \tag{3.28}$$

Lemma 3.10. *A system* (3.28) *possesses at least one invariant hyperbola of the indicated form if and only if the corresponding conditions on the right hand side are satisfied:*

- I $\Phi(x,y) = p + qr + ry + 2xy \;\Leftrightarrow\; d = e = 0$ *and* $a - b = 0$*;*
- II $\Phi(x,y) = p + qr + ry + 2x(x-y) \;\Leftrightarrow\; d = 0$, $\mathcal{M}_1 \equiv 64a - 16b - e^2 = 0$, $16a + e^2 \neq 0$*;*
- III $\Phi(x,y) = p + qr + ry + 2y(x-y) \;\Leftrightarrow\; e = 0$, $\mathcal{M}_2 \equiv 64b - 16a - d^2 = 0$, $16b + d^2 \neq 0$.

Proof. As it was mentioned in the proof of Lemma 3.2 (see page 14) we may assume that the quadratic part of an invariant hyperbola has one of the following forms: (i) $2xy$, (ii) $2x(x-y)$, (iii) $2y(x-y)$. Considering the equations (2.7) we examine each one of these possibilities.

(i) $\Phi(x,y) = p + qx + ry + 2xy$; in this case we obtain

$$t = 1, \quad s = u = 0, \quad p = (4b + q^2 + qr)/2, \quad U = 1, \quad V = 1, \quad W = -(q+r)/2,$$
$$Eq_9 = (4a - 4b - q^2 + r^2)/2, \quad Eq_{10} = 4aq + 4b(q+2r) + q(q+r)^2,$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_7 = Eq_8 = 0.$$

Calculating the resultant of the non-vanishing equations with respect to the parameter r we obtain

$$\mathrm{Res}_r(Eq_9, Eq_{10}) = (a-b)(4b+q^2)^2/4.$$

If $b = -q^2/4$ then we obtain the hyperbola $\Phi(x,y) = (r+2x)(q+2y)/2 = 0$, which is reducible.

Thus $b = a$ and we obtain

$$Eq_9 = -(q-r)(q+r)/2 = 0, \quad Eq_{10} = (q+r)(8a+q^2+qr)/4 = 0.$$

It is not too difficult to observe that the case $q+r \neq 0$ (then $q = r$) leads to reducible hyperbola (as we obtain $b = a = -q^2/4$, see the case above). So $q = -r$ and the above equations are satisfied. This leads to the invariant hyperbola $\Phi(x,y) = 2a - rx + ry + 2xy = 0$. Considering Remark 2.11 we calculate $\Delta = -(4a+r^2)/2$. So the hyperbola above is irreducible if and only if $4a + r^2 \neq 0$. Thus any system belonging to the family

$$\frac{dx}{dt} = a + x^2, \quad \frac{dy}{dt} = a + y^2 \tag{3.29}$$

possesses one-parameter family of invariant hyperbolas $\Phi(x,y) = 2a - r(x-y) + 2xy = 0$, where $r \in \mathbb{R}$ is a parameter satisfying the relation $4a + r^2 \neq 0$. This completes the proof of the statement **I** of the lemma.

(ii) $\Phi(x,y) = p + qx + ry + 2x(x-y)$; in this case we obtain

$$s = 2, \quad t = -1, \quad u = 0, \quad p = (8a - 4b + 4de - 2e^2 + q^2)/4,$$
$$r = 2d - e - q, \quad U = 2, \quad V = 1, \quad W = -(2e+q)/2, \quad Eq_7 = -2d$$

and hence the condition $d = 0$ is necessary. Then we calculate

$$\begin{gathered} Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_7 = Eq_8 = 0, \\ Eq_9 = -4a + b - (2e^2 + 6eq + 3q^2)/4, \\ Eq_{10} = \big[16a(e+q) - 4b(4e+3q) + (2e+q)(q^2 - 2e^2)\big]/8, \\ \operatorname{Res}_q(Eq_9, Eq_{10}) = -(64a - 16b - e^2)(4a - 4b - e^2)^2/256. \end{gathered}$$

(1) Assume first $64a - 16b - e^2 = 0$. Then $b = 4a - e^2/16$ and we obtain

$$Eq_9 = -3(e+2q)(3e+2q)/16 = 0, \quad Eq_{10} = -(3e+2q)(64a+4e^2-eq-2q^2)/32 = 0.$$

(1a) If $q = -3e/2$ all the equations vanish and we arrive at the invariant hyperbola

$$\Phi(x,y) = -2a + e^2/8 + e(-3x+y)/2 + 2x(x-y) = 0$$

for which we calculate $\Delta = (16a + e^2)/8$. Therefore this hyperbola is irreducible if and only if $16a + e^2 \neq 0$.

(1b) In the case $3e + 2q \neq 0$ we have $q = -e/2 \neq 0$ and the equation $Eq_{10} = 0$ implies $e(16a + e^2) = 0$. Therefore because $e \neq 0$ we obtain $16a + e^2 = 0$. However in this case we have the hyperbola

$$\Phi(x,y) = -(16a + 3e^2)/8 - e(x+y)/2 + 2x(x-y) = 0,$$

the determinant of which equals $(16a + e^2)/8$ and hence the condition above leads to an irreducible hyperbola.

(2) Suppose now $4a - 4b - e^2 = 0$, i.e. $b = a - e^2/4$. Herein we obtain

$$Eq_9 = -3\big[4a + (e+q)^2\big]/4 = 0, \quad Eq_{10} = q\big[4a + (e+q)^2\big]/8 = 0$$

and the hyperbola

$$\Phi(x,y) = 2x(x-y) + qx - (e+q)y + (4a - e^2 + q^2)/4 = 0,$$

for which we calculate $\Delta = -[4a + (e+q)^2\big]/4$. Obviously the condition $Eq_9 = 0$ implies $\Delta = 0$ and hence the invariant hyperbola is reducible. So in the case $d = 0$

and $4a-4b-e^2=0$ systems (3.28) could not possess an invariant hyperbola and the statement II of the lemma is proved.

(iii) $\Phi(x,y) = p+qx+ry+2y(x-y)$; we observe that because the change $(x,y,a,b,d,e) \mapsto (y,x,b,a,e,d)$ (which preserves systems (3.28)) this case could be brought to the previous one and hence, the conditions could be constructed directly applying this change. Thus Lemma 3.10 is proved. □

Lemma 3.11. *Assume that for a quadratic system* (2.6) *the conditions $\eta>0$ and $\theta=N=0$ hold. Then this system could possess either a single invariant hyperbola or a family of invariant hyperbolas. More precisely, it possesses:*

- (i) *one invariant hyperbola if and only if $\beta_1=0$, $\mathcal{R}_9\neq 0$ and either $(i.1)$ $\beta_2\neq 0$ and $\gamma_{11}=0$, or $(i.2)$ $\beta_2=\gamma_{12}=0$;*
- (ii) *a family of such hyperbolas if and only if $\beta_1=\beta_2=\gamma_{13}=0$.*

Proof. For systems (3.28) we calculate

$$\beta_1 = 4de, \quad \beta_2=-2(d+e), \quad \gamma_{11}=19de(d+e)+e\mathcal{M}_1+d\mathcal{M}_2,$$
$$\mathcal{R}_9\big|_{d=0}=\big[5(16a+e^2)-\mathcal{M}_1\big]/2, \quad \mathcal{R}_9\big|_{e=0}=\big[5(16b+d^2)-\mathcal{M}_2\big]/2.$$

By Lemma 3.10 the condition $de=0$ (i.e. $\beta_1=0$) is necessary for a system (3.28) to possess an invariant hyperbola.

Subcase $\beta_2\neq 0$. Then $d^2+e^2\neq 0$ and considering the values of the above invariant polynomials by Lemma 3.10 we deduce that the statement $(i.1)$ of the lemma is proved.

Subcase $\beta_2=0$. In this case we obtain $d=e=0$ and we calculate

$$\gamma_{13}=4(a-b), \quad \mathcal{R}_9=8(a+b), \quad \gamma_{12}=-128(a-4b)(4a-b)=\mathcal{M}_1\mathcal{M}_2/2.$$

Therefore by Lemma 3.10 in the case $\gamma_{12}=0$ we arrive at the statement $(i.2)$, whereas for $\gamma_{13}=0$ we arrive at the statement (ii) of the lemma.

It remains to observe that if the systems (3.28) possess the family we mentioned of invariant hyperbolas, then they have the form (3.29), depending on the parameter a. We may assume $a\in\{-1,0,1\}$ due to the rescaling $(x,y,t)\mapsto(|a|^{1/2}x,|a|^{1/2}y,|a|^{-1/2}t)$.

3.3. **Systems with two real distinct infinite singularities and $\theta\neq 0$.** For this family of systems by Lemma 2.9 the conditions $\eta=0$ and $M\neq 0$ are satisfied and then via a linear transformation and time rescaling systems (2.6) could be brought to the following family of systems:

$$\begin{aligned} \frac{dx}{dt} &= a+cx+dy+gx^2+hxy,\\ \frac{dy}{dt} &= b+ex+fy+(g-1)xy+hy^2. \end{aligned} \tag{3.30}$$

For this systems we calculate

$$C_2(x,y)=x^2y, \quad \theta=-h^2(g-1)/2 \tag{3.31}$$

and since $\theta \neq 0$ due to a translation we may assume $d = e = 0$. So in what follows we consider the family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + gx^2 + hxy, \\ \frac{dy}{dt} &= b + fy + (g-1)xy + hy^2. \end{aligned} \tag{3.32}$$

□

Lemma 3.12. *A system* (3.32) *could not posses more than one invariant hyperbola. And it possesses one such hyperbola if and only if* $c+f=0$, $\mathcal{G}_1 \equiv a(1-2g)+2bh = 0$ *and* $a \neq 0$.

Proof. Since $C_2 = x^2y$ we may assume that the quadratic part of an invariant hyperbola has the form $2xy$. Considering the equations (2.7) and the condition $\theta \neq 0$ (i.e. $h(g-1) \neq 0$) for systems (3.32) we obtain

$$\begin{gathered} t = 1, \quad s = u = q = r = 0, \quad p = a/h, \quad U = 2g-1, \quad V = 2h, \quad W = c+f, \\ Eq_8 = (a - 2ag + 2bh)/h = \mathcal{G}_1/h, \quad Eq_{10} = -a(c+f)/h, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_7 = Eq_9 = 0. \end{gathered}$$

Since the hyperbola (2.5) in this case becomes $\Phi(x,y) = a/h + 2xy = 0$ the condition $a \neq 0$ is necessary in order to have an invariant hyperbola. Then the equation $Eq_{10} = 0$ implies $c+f=0$ and the condition $Eq_8/h = 0$ yields $\mathcal{G}_1 = 0$. Since $h \neq 0$ we set $b = a(2g-1)/(2h)$ and this leads to the family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + gx^2 + hxy, \\ \frac{dy}{dt} &= \frac{a(2g-1)}{2h} - cy + (g-1)xy + hy^2, \end{aligned} \tag{3.33}$$

which possess the invariant hyperbola

$$\Phi(x,y) = \frac{a}{h} + 2xy = 0.$$

This completes the proof of the lemma. □

Next we determine the corresponding affine invariant conditions.

Lemma 3.13. *Assume that for a quadratic system* (2.6) *the conditions* $\eta = 0$, $M \neq 0$ *and* $\theta \neq 0$ *hold. Then this system possesses a single invariant hyperbola (which could be simple or double) if and only if one of the following sets of the conditions hold, respectively:*

(i) $\beta_2\beta_1 \neq 0$, $\gamma_1 = \gamma_2 = 0$, $\mathcal{R}_1 \neq 0$*: simple;*
(ii) $\beta_2 \neq 0$, $\beta_1 = \gamma_1 = \gamma_4 = 0$, $\mathcal{R}_3 \neq 0$*: simple if* $\delta_1 \neq 0$ *and double if* $\delta_1 = 0$*;*
(iii) $\beta_2 = \beta_1 = \gamma_{14} = 0$, $\mathcal{R}_{10} \neq 0$*: simple if* $\beta_7\beta_8 \neq 0$ *and double if* $\beta_7\beta_8 = 0$.

Proof. For systems (3.32) we calculate

$$\gamma_1 = (2c-f)(c+f)^2h^4(g-1)^2/32, \quad \beta_2 = h^2(2c-f)/2.$$

According to Lemma 2.10 for the existence of an invariant hyperbola the condition $\gamma_1 = 0$ is necessary and therefore we consider two cases: $\beta_2 \neq 0$ and $\beta_2 = 0$.

3.3.1. *Case* $\beta_2 \neq 0$. Then $2c - f \neq 0$ and the condition $\gamma_1 = 0$ implies $f = -c$. Then we calculate

$$\gamma_2 = 14175c^2h^5(g-1)^2(3g-1)\mathcal{G}_1, \quad \beta_2 = 3ch^2/2,$$
$$\beta_1 = -3c^2h^2(g-1)(3g-1)/4, \ \ \mathcal{R}_1 = -9ach^4(g-1)^2(3g-1)/8$$

and we examine two subcases $\beta_1 \neq 0$ and $\beta_1 = 0$.

Subcase $\beta_1 \neq 0$. Then the necessary condition $\gamma_2 = 0$ (see Lemma 2.10) gives $\mathcal{G}_1 = 0$ and by Lemma 3.12 systems (3.32) possess an invariant hyperbola. We claim that this hyperbola could not be double. Indeed, since the condition $\theta \neq 0$ holds we apply Lemma 3.5 which provides necessary and sufficient conditions in order to have at least two hyperbolas. According to this lemma the condition $\beta_1 = 0$ is necessary for the existence of at least two hyperbolas. So it is clear that in this case the hyperbola of systems (3.33) could not be double due to $\beta_1 \neq 0$. This completes the proof of the statement (i) of the lemma.

Subcase $\beta_1 = 0$. Because $\beta_2 \neq 0$ (i.e. $c \neq 0$) this implies $g = 1/3$ and then $\gamma_2 = 0$ and

$$\gamma_4 = 16h^6(a+6bh)^2/3 = 48h^6\mathcal{G}_1^2, \quad \mathcal{R}_3 = 3bh^3/2.$$

Therefore the condition $\gamma_4 = 0$ is equivalent to $\mathcal{G}_1 = 0$ and in this case $\mathcal{R}_3 \neq 0$ gives $b \neq 0$ which is equivalent to $a \neq 0$. By Lemma 3.12 systems (3.32) possess a hyperbola. We claim that this hyperbola is double if and only if the condition $a = -12c^2$ holds.

Indeed, as we would like after some perturbation to have two hyperbolas, then the respective conditions provided by Lemma 3.5 must hold. We calculate:

$$\beta_1 = 0, \quad \beta_2 = 3ch^2/2, \quad \beta_6 = ch/3, \quad \gamma_4 = 0, \quad \delta_1 = -(a+12c^2)h^2/4$$

and since $\beta_6 \neq 0$ (because $\beta_2 \neq 0$) we could have a double hyperbola only if the identities provided by the statement (A1) are satisfied. Therefore the condition $\delta_1 = 0$ is necessary and due to $\theta \neq 0$ (i.e. $h \neq 0$) we obtain $a = -12c^2$.

So our claim is proved and we obtain the family of systems

$$\frac{dx}{dt} = -12c^2 + cx + x^2/3 + hxy, \quad \frac{dy}{dt} = 2c^2/h - cy - 2xy/3 + hy^2, \tag{3.34}$$

which possess the hyperbola $\Phi(x,y) = -12c^2/h + 2xy = 0$. The perturbed systems

$$\begin{aligned} \frac{dx}{dt} &= -\frac{18c^2(2h+\varepsilon)(3h+\varepsilon)}{(3h-\varepsilon)^2} + cx + x^2/3 + (h+\varepsilon)xy, \\ \frac{dy}{dt} &= \frac{6c^2(3h+\varepsilon)}{(3h-\varepsilon)^2} - cy - 2xy/3 + hy^2, \quad |\varepsilon| \ll 1 \end{aligned} \tag{3.35}$$

possess the two distinct invariant hyperbolas:

$$\Phi_1^{\varepsilon}(x,y) = -\frac{36c^2(3h+\varepsilon)}{(3h-\varepsilon)^2} + 2xy = 0,$$
$$\Phi_2^{\varepsilon}(x,y) = -\frac{36c^2(3h+\varepsilon)}{(3h-\varepsilon)^2} - \frac{12c\varepsilon}{3h-\varepsilon}y + 2y(x+\varepsilon y) = 0.$$

It remains to observe that the hyperbola $\Phi(x,y) = -12c^2/h + 2xy = 0$ could not be triple, because in this case for systems (3.34) the necessary conditions provided by the statement $(\mathcal{B})$ of Lemma 3.5 to have three invariant hyperbolas are not satisfied: we have $\beta_6 \neq 0$.

Thus the statement (ii) of the lemma is proved.

3.3.2. *Case* $\beta_2 = 0$. Then $f = 2c$ and this implies $\gamma_1 = 0$. On the other hand we calculate

$$\gamma_2 = -14175ac^2(g-1)^3(1+3g)h^5, \quad \beta_1 = -9c^2(g-1)^2h^2/16$$

and since $f = 2c$, according to Lemma 3.12 the condition $c = 0$ is necessary in order to have an invariant hyperbola. The condition $c = 0$ is equivalent to $\beta_1 = 0$ and this implies $\gamma_2 = 0$. It remains to detect invariant polynomials which govern the conditions $\mathcal{G}_1 = 0$ and $a \neq 0$. For $c = 0$ we have

$$\gamma_{14} = 80h^3\big[a(1-2g) + 2bh\big] = 80h^3\mathcal{G}_1, \quad \mathcal{R}_{10} = -4ah^2.$$

So for $\beta_1 = \beta_2 = 0$, $\gamma_{14} = 0$ and $\mathcal{R}_{10} \neq 0$ systems (3.33) (with $c = 0$) possess the invariant hyperbola $\Phi(x,y) = a/h + 2xy = 0$.

Next we shall determine the conditions under which this hyperbola is simple or double. In accordance with Lemma 3.5 we calculate:

$$\beta_1 = \beta_6 = 0, \beta_7 = -8(2g-1)h^2.$$

We examine two possibilities: $\beta_7 \neq 0$ and $\beta_7 = 0$.

Possibility $\beta_7 \neq 0$. According to Lemma 3.5 for systems (3.33) with $c = 0$ could be satisfied only the identities given by the statement (A2). So we have to impose the following conditions:

$$\gamma_5 = \beta_8 = \delta_2 = 0.$$

We have $\beta_8 = -32(4g-1)h^2 = 0$ which implies $g = 1/4$. Then we obtain $\gamma_5 = \delta_2 = 0$ and we obtain the family of systems

$$\frac{dx}{dt} = a + x^2/4 + hxy, \quad \frac{dy}{dt} = -a/(4h) - 3xy/4 + hy^2, \tag{3.36}$$

which possess the hyperbola $\Phi(x,y) = a/h + 2xy = 0$. On the other hand we observe that the perturbed systems

$$\frac{dx}{dt} = a + \frac{\varepsilon}{2h} + x^2/4 + (h+\varepsilon)xy, \quad \frac{dy}{dt} = -a/(4h) - 3xy/4 + hy^2, \tag{3.37}$$

which possess the two distinct invariant hyperbolas:

$$\Phi_1^{\varepsilon}(x,y) = a/h + 2xy = 0, \quad \Phi_2^{\varepsilon}(x,y) = a/h + 2y(x+\varepsilon y) = 0.$$

Since $\beta_7 \neq 0$, according to Lemma 3.5 the hyperbola $\Phi(x,y) = a/h + 2xy = 0$ could not be triple.

Possibility $\beta_7 = 0$. In this case we obtain $g = 1/2$ and this implies $\gamma_8 = \delta_3 = 0$. Hence the identities given by the statement (A3) of Lemma 3.5 are satisfied. In this case we obtain the family of systems

$$\frac{dx}{dt} = a + x^2/2 + hxy, \quad \frac{dy}{dt} = -xy/2 + hy^2, \tag{3.38}$$

which possess the hyperbola $\Phi(x,y) = a/h + 2xy = 0$. On the other hand we observe that the perturbed systems

$$\frac{dx}{dt} = a + x^2/2 + (h+\varepsilon)xy, \quad \frac{dy}{dt} = -xy/2 + hy^2, \tag{3.39}$$

possess the two distinct invariant hyperbolas:

$$\Phi_1^\varepsilon(x,y) = \frac{2a}{2h+\varepsilon} + 2xy = 0, \quad \Phi_2^\varepsilon(x,y) = a/h + 2y(x+\varepsilon y) = 0.$$

Since for systems (3.38) we have $\beta_8 = -32h^2 \neq 0$, according to Lemma 3.5 the hyperbola $\Phi(x,y) = a/h + 2xy = 0$ could not be triple.

It remains to observe that the conditions of the statement (B) of Lemma 3.5 in order to have three invariant hyperbolas could not be satisfied for systems (3.33) (i.e. the necessary conditions for these systems to possess a triple hyperbola). Indeed for systems (3.33) we have

$$\beta_7 = -8(2g-1)h^2, \quad \beta_8 = -32(4g-1)h^2, \quad \theta = -(g-1)h^2/2$$

and hence the conditions $\beta_7 = 0$ and $\beta_8 = 0$ are incompatible due to $\theta \neq 0$. As all the cases are examined we deduce that Lemma 3.13 is proved. $\square$

3.4. **Systems with two real distinct infinite singularities and $\theta = 0$.** By Lemma 2.9 systems (2.6) via a linear transformation could be brought to the systems (3.30) for which we have

$$\theta = -h^2(g-1)/2, \quad \beta_4 = 2h^2, \quad N = (g^2-1)^2x^2 + 2h(g-1)xy + h^2y^2. \tag{3.40}$$

We shall consider to cases: $N \neq 0$ and $N = 0$.

3.4.1. *Case* $N \neq 0$. Since $\theta = 0$ we obtain $h(g-1) = 0$ and $(g^2-1)^2 + h^2 \neq 0$. So we examine two subcases: $\beta_4 \neq 0$ and $\beta_4 = 0$.

Subcase $\beta_4 \neq 0$. Then $h \neq 0$ (this implies $N \neq 0$) and we obtain $g = 1$. Applying a translation and the additional rescaling $y \to y/h$ we may assume $c = f = 0$ and $h = 1$. So in what follows we consider the family of systems

$$\frac{dx}{dt} = a + dy + x^2 + xy, \quad \frac{dy}{dt} = b + ex + y^2. \tag{3.41}$$

Lemma 3.14. *A system* (3.41) *possesses an invariant hyperbola if and only if* $e = 0$, $\mathcal{L}_1 \equiv 9a - 18b + d^2 = 0$ *and* $a + d^2 \neq 0$.

Proof. Since $C_2 = x^2y$ we determine that the quadratic part of an invariant hyperbola has the form $2xy$. Considering the equations (2.7) for systems (3.41) we obtain

$$t = 1, \quad s = u = 0, \quad r = 2d, \quad p = 2b + 2de + dq + q^2/2,$$
$$U = 1, \quad V = 2, \quad W = -(q+r)/2, \quad Eq_5 = e,$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_8 = 0.$$

Therefore the condition $Eq_5 = 0$ yields $e = 0$ and then we have

$$Eq_9 = 2a - 4b + 2d^2 - q^2, \quad Eq_{10} = aq + b(4d+q) + q(2d+q)^2/4.$$

Clearly in order to have a common solution of the equations $Eq_9 = Eq_{10} = 0$ with respect to the parameter q the condition

$$\operatorname{Res}_q(Eq_9, Eq_{10}) = (a+d^2)^2(9a - 18b + d^2)/2 = 0$$

is necessary. We claim that the condition $a + d^2 = 0$ leads to a hyperbola. Indeed, setting $a = -d^2$ we obtain $Eq_9 = -(4b+q^2) = 0$. On the other hand we obtain the hyperbola

$$\Phi(x,y) = 2b + dq + q^2/2 + qx + 2dy + 2xy = 0$$

for which by considering Remark 2.11 we calculate $\Delta = -(4b+q^2)/2$. Therefore the equation $Eq_9 = -(4b+q^2) = 0$ leads to an invariant hyperbola. This proves our claim.

So $a+d^2 \neq 0$ and we set $b = (9a+d^2)/18$. Then $Eq_9 = 0$ gives $(4d-3q)(4d+3q) = 0$ and we examine two subcases: $q = 4d/3$ and $q = -4d/3$.

(1) Assuming $q = 4d/3$ we obtain $Eq_{10} = 4d(a+d^2) = 0$. Since $a+d^2 \neq 0$ we have $d = 0$ and this leads to the family of systems

$$\frac{dx}{dt} = a + x^2 + xy, \quad \frac{dy}{dt} = a/2 + y^2. \tag{3.42}$$

These systems possess the invariant hyperbola $\Phi(x,y) = a + 2xy = 0$.

(2) Suppose now $q = -4d/3$. This implies $Eq_{10} = 0$ and we obtain the systems

$$\frac{dx}{dt} = a + dy + x^2 + xy, \quad \frac{dy}{dt} = (9a+d^2)/18 + y^2, \tag{3.43}$$

which possess the invariant hyperbola

$$\Phi_1(x,y) = (3a-d^2)/3 - 2d(2x-3y)/3 + 2xy = 0.$$

Its determinant Δ equals $-(a+d^2)$ and hence, the conic is irreducible if and only if $a+d^2 \neq 0$.

It remains to observe that the family of systems (3.42) is a subfamily of the family (3.43) (corresponding to $d = 0$) and this complete the proof of the lemma. □

Subcase $\beta_4 = 0$. This implies $h = 0$ and the condition $N \neq 0$ gives $g^2 - 1 \neq 0$. Using a translation we may assume $e = f = 0$ and we arrive at the family of systems

$$\frac{dx}{dt} = a + cx + dy + gx^2, \quad \frac{dy}{dt} = b + (g-1)xy. \tag{3.44}$$

Lemma 3.15. *A system (3.44) possesses at least one invariant hyperbola if and only if $d = 0$, $2g-1 \neq 0$ and either*

(i) $3g-1 \neq 0$, $\mathcal{K}_1 \equiv c^2(1-2g) + a(3g-1)^2 = 0$ *and* $b \neq 0$, *or*
(ii) $g = 1/3$, $c = 0$ *and* $b \neq 0$.

Moreover in the second case we have two real hyperbolas ($\mathcal{H}^p$) if $a < 0$; two complex hyperbolas ($\overset{c}{\mathcal{H}}{}^p$) if $a > 0$ and these hyperbolas coincide if $a = 0$.

Proof. As earlier we assumed that the quadratic part of an invariant hyperbola has the form $2xy$ and considering the equations (2.7) for systems (3.44) we obtain

$$t = 1, \quad s = u = q = 0, \quad U = 2g-1, \quad V = 0, \quad W = c - gr/2,$$
$$Eq_7 = 2d, \quad Eq_8 = 2b + p(1-2g), \quad Eq_9 = 2a - cr + gr^2/2,$$
$$Eq_{10} = br - cp + gpr/2, \quad Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = 0.$$

Therefore the condition $Eq_7 = 0$ yields $d = 0$ and we claim that the condition $2g-1 \neq 0$ must hold. Indeed, supposing $g = 1/2$ the equation $Eq_8 = 0$ yields $b = 0$ and then

$$Eq_9 = 2a + r(r-4c)/4 = 0, \quad Eq_{10} = p(r-4c)/4 = 0.$$

Since $p \neq 0$ (otherwise we obtain a reducible hyperbola) we obtain $r = 4c$, however in this case $Eq_9 = 0$ implies $a = 0$ and we arrive at degenerate systems. This completes the proof of our claim.

Thus we have $2g-1\neq 0$ and then the equation $Eq_8=0$ gives $p=2b/(2g-1)$ and we obtain:

$$Eq_{10}=b(2c+r-3gr))/(1-2g).$$

Since in this case the hyperbola is of the form

$$\Phi(x,y)=\frac{2b}{2g-1}+ry+2xy=0$$

it is clear that the condition $b\neq 0$ must hold and, therefore we obtain $2c+r(1-3g)=0$.

(1) Assume first $3g-1\neq 0$. Then we obtain $r=2c/(3g-1)$ and the equation $Eq_9=0$ becomes

$$Eq_9=\frac{2}{(3g-1)^2}\big[c^2(1-2g)+a(3g-1)^2\big]=\frac{2}{(3g-1)^2}\mathcal{K}_1=0.$$

The condition $\mathcal{K}_1=0$ implies $a=c^2(2g-1)/(3g-1)^2$ and we arrive at the family of systems

$$\frac{dx}{dt}=\frac{c^2(2g-1)}{(3g-1)^2}+cx+gx^2,\quad \frac{dy}{dt}=b+(g-1)xy, \tag{3.45}$$

possessing the invariant hyperbola

$$\Phi(x,y)=\frac{2b}{2g-1}+\frac{2c}{3g-1}y+2xy=0,$$

which is irreducible if and only if $b\neq 0$.

(2) Suppose now $g=1/3$. In this case the equation $Eq_{10}=0$ yields $c=0$ and then we obtain $p=-6b$ and the equation $Eq_9=0$ becomes $Eq_9=(12a+r^2)/6=0$. Therefore for the existence of an invariant hyperbola the condition $a\leq 0$ is necessary. In this case setting $a=-3z^2\leq 0$ we arrive at the family of systems

$$\frac{dx}{dt}=a+x^2/3,\quad \frac{dy}{dt}=b-2xy/3, \tag{3.46}$$

possessing the two invariant conics

$$\Phi_{1,2}(x,y)=3b\pm\sqrt{-3a}\,y-xy=0,$$

which are irreducible if and only if $b\neq 0$. Clearly these hyperbolas are real for $a<0$, they are complex for $a>0$ and coincide (and we obtain a double one) if $a=0$. □

Lemma 3.16. *Assume that for a quadratic system (2.6) the conditions $\eta=0$, $M\neq 0$, $\theta=0$ and $N\neq 0$ are satisfied. Then this system could possess either a single invariant hyperbola, or two distinct ($\mathcal{H}^p$) such hyperbolas, or one triple invariant hyperbola. More precisely, it possesses:*

- (i) *one invariant hyperbola if and only if either*
  - (i.1) *$\beta_4\neq 0$, $\beta_3=\gamma_8=0$ and $\mathcal{R}_7\neq 0$ (simple if $\delta_4\neq 0$ and double if $\delta_4=0$), or*
  - (i.2) *$\beta_4=\beta_6=0$, $\beta_{11}\mathcal{R}_{11}\neq 0$, $\beta_{12}\neq 0$ and $\gamma_{15}=0$ (simple if $\gamma_{16}^2+\delta_6^2\neq 0$ and double if $\gamma_{16}=\delta_6=0$);*
- (ii) *two distinct invariant hyperbolas (both simple) if and only if $\beta_4=\beta_6=0$, $\beta_{11}\mathcal{R}_{11}\neq 0$, $\beta_{12}=\gamma_{16}=0$ and $\gamma_{17}\neq 0$. Moreover these hyperbolas are real ($\mathcal{H}^p$) if $\gamma_{17}<0$ and they are complex ($\overset{c}{\mathcal{H}^p}$) if $\gamma_{17}>0$;*

(iii) *one triple invariant hyperbola (which splits into three distinct hyperbolas, two of them being* $(\mathcal{H}^p)$*) if and only if* $\beta_4=\beta_6=0$, $\beta_{11}\mathcal{R}_{11}\neq 0$, $\beta_{12}=\gamma_{16}=0$ *and* $\gamma_{17}=0$.

Proof. Assume that for a quadratic system (2.6) the conditions $\eta=0$, $M\neq 0$, $\theta=0$ and $N\neq 0$.

Case $\beta_4\neq 0$. As it was shown earlier in this case via an affine transformation and time rescaling the system could be brought to the form (3.41), for which we calculate

$$\gamma_1=-9de^2/8,\quad \beta_3=-e/4,$$

and by Lemma 3.14 the condition $\beta_3=0$ is necessary in order to have an invariant hyperbola. In this case we obtain

$$\gamma_8=42(9a-18b+d^2)^2=42\mathcal{L}_1^2,\quad \mathcal{R}_7=-\mathcal{L}_1/8-(a+d^2)/3$$

and considering Lemma 3.14 for $\beta_3=\gamma_8=0$ we obtain systems (3.43) possessing the hyperbola $\Phi(x,y)=(3a-d^2)/3-2d(2x-3y)/3+2xy=0$. To detect its multiplicity we apply Lemma 2.12 setting $k=2$. So in order to have the polynomial $\Phi(x,y)$ as a double factor in $\mathscr{E}_k$, we force its cofactor in $\mathscr{E}_2$ to be zero along the curve $\Phi(x,y)=0$ (i.e we set $y=(-3a+d^2+4dx)/(6(d+x))$). We obtain

$$\frac{\mathscr{E}_2}{\Phi(x,y)}=\frac{(a+d^2)^4(81a+17d^2)}{2^{11}3^{12}(d+x)^{10}}(7d+15x)(3a+d^2+4dx+6x^2)^{10}=0$$

and since $a+d^2\neq 0$ (see Lemma 3.14) we obtain $81a+17d^2=0$. So we obtain the family of systems

$$\frac{dx}{dt}=-17d^2/81+dy+x^2+xy,\quad \frac{dy}{dt}=-4d^2/81+y^2, \tag{3.47}$$

which possess the invariant hyperbola: $\Phi(x,y)=-44d^2/81-4dx/3+2dy+2xy=0$. The perturbed systems

$$\frac{dx}{dt}=-\frac{d^2(17-2\varepsilon+\varepsilon^2)}{(\varepsilon^2-9)^2}+dy+x^2+(1+\varepsilon)xy,\quad \frac{dy}{dt}=-\frac{4d^2}{(\varepsilon^2-9)^2}+y^2, \tag{3.48}$$

possess the two hyperbolas:

$$\begin{gathered}\Phi_1^\varepsilon(x,y)=-\frac{4d^2(11-4\varepsilon+\varepsilon^2}{(\varepsilon^2-9)^2(1+\varepsilon)}-\frac{4d}{(1+\varepsilon)(3+\varepsilon)}x+\frac{2d}{1+\varepsilon}y+2xy=0,\\ \Phi_2^\varepsilon(x,y)=\frac{4d^2(11+4\varepsilon+\varepsilon^2}{(\varepsilon^2-9)^2(\varepsilon-1)}-\frac{4d}{(1-\varepsilon)(3-\varepsilon)}x-\frac{6d}{\varepsilon-3}y+2y(x+\varepsilon y)=0,\end{gathered}$$

We observe that for systems (3.43) we have $\delta_4=(81a+17d^2)/6$ and $\beta_7=-8$. Therefore if $\delta_4=0$ the invariant hyperbola is double and by Lemma 3.5 it could not be triple due to $\beta_7\neq 0$. This completes the proof of the statement $(i.1)$ of the lemma.

Case $\beta_4=0$. Then we arrive at the family of systems (3.44), for which we have

$$\beta_6=d(g^2-1)/4,\quad N=4(g^2-1)x^2,\quad \beta_{11}=4(2g-1)^2x^2,\quad \beta_{12}=(3g-1)x,$$

So from $N\neq 0$ the necessary conditions $d=0$ and $2g-1\neq 0$ (see Lemma 3.15) are equivalent to $\beta_6=0$ and $\beta_{11}\neq 0$, respectively.

Subcase $\beta_{12} \neq 0$. In this case $3g-1 \neq 0$ and then by Lemma 3.15 an invariant hyperbola exists if and only if $\mathcal{K}_1 = 0$ and $b \neq 0$. On the other hand for systems (3.44) with $d = 0$ we calculate

$$\gamma_{15} = 4(g-1)^2(3g-1)\mathcal{K}_1 x^5, \quad \mathcal{R}_{11} = -3b(g-1)^2x^4$$

and hence the above conditions are governed by the invariant polynomials γ_{15} and $\mathcal{R}_{11}$. So we obtain systems (3.45) possessing the hyperbola $\Phi(x,y) = 2b/(2g-1) + 2cy/(3g-1) + 2xy = 0$.

By to Lemma 2.12 we calculate the polynomial $\mathscr{E}_2$ and we observe that $\mathscr{E}_2$ contains the polynomial $\Phi(x,y)$ as a simple factor.

To have this polynomial as a double factor in $\mathscr{E}_2$, we force its cofactor in $\mathscr{E}_2$ to be zero along the curve $\Phi(x,y) = 0$ (i.e we set $y = b(3g-1)/((2g-1)(c-x+3gx))$). We obtain

$$\begin{aligned}\frac{\mathscr{E}_2}{\Phi(x,y)} = \frac{288b^3(g-1)[c+(3g-1)x]^3}{(2g-1)^3(3g-1)^{16}}\big[c(2g-1)+g(3g-1)x\big]^{10}\big[c^2(31\\ -87g+62g^2)+6c(3g-2)(3g-1)^2x+(3g-1)^3(4g-1)x^2\big] = 0\end{aligned}$$

and since $(2g-1)(3g-1) \neq 0$ we obtain $c = 0$ and either $g = 1/4$ or $g = 0$. However in the second case we obtain degenerate systems. So $g = 1/4$ and we arrive at the family of systems

$$\frac{dx}{dt} = x^2/4, \quad \frac{dy}{dt} = b - 3xy/4, \tag{3.49}$$

which possess the hyperbola $\Phi(x,y) = -4b + 2xy = 0$. On the other hand the perturbed systems

$$\frac{dx}{dt} = -2b\varepsilon + \varepsilon xy + x^2/4, \quad \frac{dy}{dt} = b - 3xy/4 \tag{3.50}$$

possess the two invariant hyperbolas

$$\Phi_1^\varepsilon(x,y) = -2b + xy = 0, \quad \Phi_2^\varepsilon(x,y) = -2b + y(x+\varepsilon y) = 0.$$

It remains to determine the invariant polynomials which govern the conditions $c = 0$ and $g = 1/4$. We observe that for systems (3.45) we have $\gamma_{16} = -c(g-1)^2x^3/2$ and $\delta_6 = (g-1)(4g-1)x^2/2$.

To deduce that the hyperbola $\Phi(x,y) = -4b + 2xy = 0$ could not be triple it is sufficient to calculate $\mathscr{E}_2$ for systems (3.49):

$$\mathscr{E}_2 = -\frac{135x^{15}}{65536}\Phi(x,y)^2(5b-3xy)(17b-7xy)$$

and to observe that the cofactor of $\Phi(x,y)^2$ could not vanish along the curve $\Phi(x,y) = 0$. This leads to the statement $(i.2)$ of the lemma.

Subcase $\beta_{12} = 0$. Then $g = 1/3$ and by Lemma 3.15 at least one invariant hyperbola exists if and only if $c = 0$, $a \le 0$ and $b \neq 0$. On the other hand for systems (3.44) with $d = 0$ and $g = 1/3$ we calculate

$$\gamma_{16} = -2cx^3/9, \quad \gamma_{17} = 32ax^2/9, \quad \mathcal{R}_{11} = -4bx^4/3$$

Therefore the condition $c = 0$ (respectively $b \neq 0$) is equivalent to $\gamma_{16} = 0$ (respectively $\mathcal{R}_{11} \neq 0$). Considering the statement (ii) of Lemma 3.15 we examine two possibilities: $\gamma_{17} \neq 0$ (i.e. $a \neq 0$) and $\gamma_{17} = 0$ (i.e. $a = 0$).

(1) *Possibility* $\gamma_{17} \neq 0$. By Lemma 3.15 in this case we arrive at systems (3.46) possessing the two invariant hyperbolas $\Phi_{1,2}(x,y) = 3b \pm \sqrt{-3a}\,y - xy = 0$. We

claim that none of the hyperbolas could be double. Indeed calculating $\mathscr{E}_2$ (see Lemma 2.12) we obtain:

$$\mathscr{E}_2 = -\frac{2560(x^2 \pm 3a)^6}{177147}\Phi_1\Phi_2(2bx - x^2y \pm ay)\big[3bx^2 - x^3y \pm 9a(xy - b)\big].$$

So each hyperbola appears as a factor of degree one and we could not increase there degree because of $b \neq 0$. This proves our claim and we arrive at the statement (ii) of the lemma.

(2) *Possibility* $\gamma_{17} = 0$. In this case we have $a = 0$ and this leads to the systems

$$\frac{dx}{dt} = x^2/3, \quad \frac{dy}{dt} = b - 2xy/3, \tag{3.51}$$

possessing the hyperbola $\Phi(x, y) = -3b + xy = 0$. Calculating $\mathscr{E}_2$ for this systems we obtain that $\Phi(x, y)$ is a triple factor of $\mathscr{E}_2$. According to Lemma 2.12 this hyperbola is triple, as it is shown by the following perturbed systems:

$$\frac{dx}{dt} = -12b^2\varepsilon^2 + x^2/3, \quad \frac{dy}{dt} = b - 2xy/3 + 3b\varepsilon^2y^2, \tag{3.52}$$

possessing the three distinct invariant hyperbolas:

$$\Phi_{1,2} = -3b \pm 3b\varepsilon y + xy = 0, \quad \Phi_3 = -3b + y(x - 3b\varepsilon^2 y).$$

So we arrive at the statement (iii) of Lemma 3.16 and this completes the proof of this lemma. ☐

3.4.2. *Case* $N = 0$. Considering (3.40) the condition $N = 0$ implies $h = 0$ and $g = \pm 1$. On the other hand for (3.30) with $h = 0$ we have $\beta_{13} = (g - 1)^2x^2/4$ and we consider two cases: $\beta_{13} \neq 0$ and $\beta_{13} = 0$.

Subcase $\beta_{13} \neq 0$. Then $g - 1 \neq 0$ (this implies $g = -1$) and due to a translation we may assume $e = f = 0$. So we obtain the family of systems

$$\frac{dx}{dt} = a + cx + dy - x^2, \quad \frac{dy}{dt} = b - 2xy. \tag{3.53}$$

Lemma 3.17. *A system* (3.53) *possesses at least one invariant hyperbola if and only if* $d = 0$, $16a + 3c^2 = 0$ *and* $b \neq 0$.

Proof. We again assume that the quadratic part of an invariant hyperbola has the form $2xy$ and considering the equations (2.7) for systems (3.53) we obtain

$$\begin{gathered} t = 1, \quad s = u = q = 0, \quad p = -2b/3, \quad r = -c/2, \quad U = -3, \\ V = 0, \quad W = c + r/2, \quad Eq_7 = 2d, \quad Eq_9 = (16a + 3c^2)/8, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = Eq_{10} = 0. \end{gathered}$$

Therefore the conditions $Eq_7 = 0$ and $Eq_9 = 0$ yield $d = 0$ and $16a + 3c^2 = 0$. In this case we obtain the systems

$$\frac{dx}{dt} = -3c^2/16 + cx - x^2, \quad \frac{dy}{dt} = b - 2xy, \tag{3.54}$$

which possess the invariant hyperbola

$$\Phi(x, y) = -2b/3 - cy/2 + 2xy = 0.$$

Obviously this conic is irreducible if and only if $b \neq 0$. So Lemma 3.17 is proved. ☐

Subcase $\beta_{13} = 0$. Then $g = 1$ and due to a translation we may assume $c = 0$. So we obtain the following family of systems

$$\frac{dx}{dt} = a + dy + x^2, \quad \frac{dy}{dt} = b + ex + fy. \tag{3.55}$$

Lemma 3.18. *A system* (3.55) *could not possess a finite number of invariant hyperbolas. And it has 1-parameter family of invariant hyperbolas if and only if* $d = e = 0$ *and* $4a + f^2 = 0$.

Proof. Considering the equations (2.7) and the fact that the quadratic part of an invariant hyperbola has the form $2xy$, for systems (3.55) we calculate

$$t = 1, \quad s = u = 0, \quad U = 1, \quad V = 0, \quad W = f - r/2,$$
$$Eq_5 = 2e, \quad Eq_7 = 2d, \quad Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = 0.$$

Therefore the conditions $Eq_5 = 0$ and $Eq_7 = 0$ yield $d = e = 0$ and then we have

$$Eq_8 = 2b - p - fq + qr/2, \quad Eq_9 = (4a + r^2)/2, \quad Eq_{10} = aq + br - p(2f - r)/2.$$

The equations $Eq_8 = Eq_{10} = 0$ have a common solution with respect to the parameter q only if

$$\mathrm{Res}_q(Eq_8, Eq_{10}) = -2ab + p(a + f^2) - fr(b + p) + r^2(2b + p)/4 = 0.$$

On the other hand in order to have a common solution of the above equations with respect to r the following condition is necessary:

$$\mathrm{Res}_r\big(Eq_9, \mathrm{Res}_q(Eq_8, Eq_{10})\big) = (4a + f^2)(4ab^2 + f^2p^2)/4 = 0.$$

We claim, that the condition $4a + f^2 = 0$ is necessary for the existence of an invariant hyperbola.

Indeed, supposing $4a + f^2 \neq 0$ we deduce that the condition $4ab^2 + f^2p^2 = 0$ must hold.

(1) Assume first $f \neq 0$. If $b = 0$ then we obtain $p = 0$ and the equation $Eq_{10} = 0$ gives $aq = 0$. In the case $q = 0$ we obtain a reducible conic. If $a = 0$ then the equation $Eq_9 = 0$ implies $r = 0$ and we again get a reducible conic.

Thus $b \neq 0$ and hence $a \leq 0$. We set $a = -z^2 \leq 0$ and then $r = \pm 2z$ and $p = \pm 2bz/f$. It is not too hard to convince ourselves that all four possibilities lead either to reducible conics, or to the equality $4a + f^2 = 0$, which contradicts our assumption.

(2) Suppose now $f = 0$. This implies $ab = 0$ and since $b \neq 0$ (otherwise we obtain degenerate systems) we have $a = 0$ and this again contradicts to $4a + f^2 \neq 0$. This completes the proof of our claim.

Thus $4a + f^2 = 0$ and setting $a = -f^2/4$ we arrive at the family of systems

$$\frac{dx}{dt} = -f^2/4 + x^2, \quad \frac{dy}{dt} = b + fy, \tag{3.56}$$

which possess the family of invariant hyperbolas

$$\Phi(x, y) = (4b - fq)/2 + qx + fy + 2xy = 0,$$

depending on the free parameter q. Since the corresponding determinant Δ (see Remark 2.11) for this family equals $fq - 2b$, we conclude that all the conics are irreducible, except the hyperbola, for which the equality $fq - 2b = 0$ holds. Thus the lemma is proved. $\square$

We observe that in the above systems we may assume $b = 1$. Indeed, if $b = 0$ then $f \neq 0$ (otherwise we obtain a degenerate system) and therefore due to the translation $y \to y + b'/f$ with $b' \neq 0$ and the addition rescaling $y \to b'y$ we obtain $b' = 1$. Moreover, in this case we may assume $f \in \{0, 1\}$ due to rescaling $(x, y, t) \mapsto (fx, fy, t/f)$ in the case $f \neq 0$.

Lemma 3.19. *Assume that for a quadratic system* (2.6) *the conditions $\eta = 0$, $M \neq 0$, $\theta = 0$ and $N = 0$ hold. Then this system could possess either a single invariant hyperbola, or a family of such hyperbolas. More precisely this system possesses*

- (i) *one simple invariant hyperbola if and only if $\beta_{13} \neq 0$, $\gamma_{10} = \gamma_{17} = 0$ and $\mathcal{R}_{11} \neq 0$;*
- (ii) *one family of invariant hyperbolas if and only if $\beta_{13} = \gamma_9 = \tilde{\gamma}_{18} = \tilde{\gamma}_{19} = 0$.*

Proof. Assume that for a quadratic system (2.6) the conditions $\eta = 0$, $M \neq 0$ $\theta = 0$ and $N = 0$ hold.

Subcase $\beta_{13} \neq 0$. In this case we consider systems (3.53) for which we calculate

$$\gamma_{10} = 14d^2, \quad \mathcal{R}_{11} = -12bx^4 + 6dxy^2(cx + dy),$$
$$\gamma_{17} = 8(16a + 3c^2)x^2 - 4dy(14cx + 9dy).$$

So for $\gamma_{10} = \gamma_{17} = 0$ and $\mathcal{R}_{11} \neq 0$ we obtain systems (3.54) possessing the hyperbola $\Phi(x, y) = -2b/3 - cy/2 + 2xy = 0$. We claim that this hyperbola is a simple one. Indeed calculating $\mathscr{E}_2$ we obtain that the polynomial $\Phi(x, y)$ is a factor of degree one in $\mathscr{E}_2$. So setting $y = -4b/(3(c - 4x))$ (i.e. $\Phi(x, y) \equiv 0$) we obtain

$$\frac{\mathscr{E}_2}{\Phi(x, y)} = -2^{-24}5b^3(c - 4x)^3(3c - 4x)^{12}/3 \neq 0$$

because $b \neq 0$. So the hyperbola above could not be double and this proves our claim.

Thus the statement (i) of lemma is proved.

Subcase $\beta_{13} = 0$. Then we consider systems (3.55) and we calculate

$$\gamma_9 = -6d^2, \quad \tilde{\gamma}_{18} = 8ex^4, \quad \tilde{\gamma}_{19} = 4(4a + f^2)x.$$

So the conditions $d = e = 0$ are equivalent to $\gamma_9 = \tilde{\gamma}_{18} = 0$ and $4a + f^2 = 0$ is equivalent to $\tilde{\gamma}_{19} = 0$. Considering Lemma 3.18 we arrive at the statement (ii).

It remains to observe that for systems (3.55) with $d = e = 0$ and $a = -f^2/4$ we have $\gamma_{17} = 8f^2x^2$ and this invariant polynomial governs the condition $f = 0$. As all the cases are examined, Lemma 3.19 is proved. $\square$

To complete the proof of the Main Theorem we remark, that both generic families of quadratic systems (with three and with two distinct real infinite singularities) are examined and now we could compare the obtained results with the statements of the Main Theorem.

So comparing the statements of Lemmas 3.4, 3.5, 3.8, 3.9 and 3.11 with the conditions given by Figure 1, it is not too difficult to conclude that the statement (B)(1) of the Main Theorem is valid.

Analogously, comparing the statements of Lemmas 3.13, 3.16 and 3.19 with the conditions given by Figure 2 we deduce that the statement (B2) of the Main Theorem is valid.

3.5. **Systems with infinite number of singularities at infinity: $C_2 = 0$.** In this section we construct the conditions for a quadratic system with $C_2 = 0$ to possess at least one invariant hyperbola. So consider the family of quadratic systems (2.6) assuming $C_2 = 0$ and we prove the next assertion.

Lemma 3.20. *If for a quadratic system* (2.6) *the condition $C_2(x, y) = 0$ holds, then this system possesses invariant hyperbola if and only if $N_7 = 0$.*

Proof. Assume that for a quadratic system (2.6) the condition $C_2(x, y) = 0$ is satisfied. Then the line at infinity is filled up with singularities and according to Lemma 2.9 in this case via an affine transformation and time rescaling quadratic systems could be brought to the following systems

$$\dot{x} = \hat{a} + \hat{c}x + \hat{d}y + x^2, \quad \dot{y} = \hat{b} + xy. \tag{3.57}$$

We observe that for $\hat{d} = 0$ these systems possess two parallel invariant lines and we consider two subcases: $\hat{d} \neq 0$ and $\hat{d} = 0$.

3.5.1. *Subcase $\hat{d} \neq 0$.* As it was shown in [15, page 749] in this case via some parametrization and using an additional affine transformation and time rescaling we arrive at the following 2-parameter family of systems

$$\dot{x} = a + y + (x + c)^2, \quad \dot{y} = xy. \tag{3.58}$$

Considering (2.7) for these systems we obtain $Eq_1 = s(2 - U) = 0$. We claim that $U = 2$ due to the condition $s^2 + t^2 + u^2 \neq 0$. Indeed, supposing $U \neq 2$ we obtain $s = 0$ and then calculations yield

$$Eq_2 = 2t(2 - U) = 0 \quad Eq_3 = u(2 - U) - 2tV = 0.$$

Clearly because $U \neq 2$ we have $t = u = 0$ which contradicts to $s^2 + t^2 + u^2 \neq 0$ and this completes the proof of our claim. So we assume $U = 2$, and calculations yield $Eq_2 = -sV = 0$, $Eq_3 = -2tV = 0$, $Eq_4 = -uV = 0$. Since $\Phi(x, y) = 0$ must be a conic (i.e. $s^2 + t^2 + u^2 \neq 0$) the above relations imply $V = 0$. Then we have

$$Eq_5 = -q + 4cs - sW = 0, \quad Eq_6 = -r + 2s + 4ct - 2tW = 0,$$
$$Eq_7 = 2t - uW = 0, \quad Eq_8 = -2p + 2cq + 2as + 2c^2s - qW = 0$$

and this gives

$$q = s(4c - W), \quad r = 2s + 2cuW - uW^2, \quad t = uW/2,$$
$$p = s(2a + 10c^2 - 6cW + W^2)/2.$$

Considering the values of the parameters we detected we finally obtain

$$Eq_i = 0, \quad i = 1, 2, \dots, 8, \quad Eq_{10} = s(2c - W)(4a + 4c^2 - 4cW + W^2)/2 = 0,$$
$$Eq_9 = 4cs + (au - 3s + c^2u)W - 2cuW^2 + uW^3 = 0.$$

We observe that $s \neq 0$, otherwise we obtain $\Phi(x, y) = uy(2cW - W^2 + Wx + y)$, i.e. the conic becomes reducible. So we consider the two possibilities defined by the equality $(2c - W)\big[4a + (W - 2c)^2\big] = 0$.

Possibility $W = 2c$. Then we obtain $Eq_{10} = 0$ and $Eq_9 = 2c(au - s + c^2u) = 0$.

Case $c = 0$. In this case we obtain the 1-parameter family of systems

$$\dot{x} = a + y + x^2, \quad \dot{y} = xy \tag{3.59}$$

which possess the 2-parameter family of invariant conics $\Phi(x, y) = as + 2sy + sx^2 + uy^2 = 0$ which will be of hyperbolic type if and only if the condition $su < 0$ holds. Moreover following Remark 2.11 we calculate $\Delta = s^2(au - s)$ and this conic is irreducible if and only if $s(au - s) \neq 0$.

Since $su < 0$ we may set a new parameter $u = -sm^2$ and this leads to the 1-parameter family of hyperbolas

$$\widetilde{\Phi}(s, x, y) = a + 2y + x^2 - m^2y^2 = 0. \tag{3.60}$$

Case $au - s + au + c^2u = 0$. Then $s = (a + c^2)u$ and systems (3.58) possess the following invariant conic $\Phi(x, y) = (a+c^2)(a+c^2+2cx+2y)+(a+c^2)x^2+2cxy+y^2 = 0$ for which we calculate $\Delta = 0$, i.e. by Remark 2.11 this conic is reducible.

Possibility $4a + (W - 2c)^2 = 0$. If $a = 0$ then $W = 2c$ and as it was shown above for the existence of a hyperbola it is necessary $c = 0$. So we arrive at the particular case of the family of hyperbolas (3.60) defined by the condition $a = 0$. Therefore we consider two cases: $a < 0$ and $a > 0$.

Case $a < 0$. Then we may assume $a = -k^2$ and after the rescaling $(x, y, t) \mapsto (kx, k^2y, t/k)$ we obtain the systems

$$\dot{x} = y - 1 + (x + c)^2, \quad \dot{y} = xy, \tag{3.61}$$

for which we have $W = 2(c \pm 1)$ and we obtain $Eq_{10} = 0$, $Eq_9 = 2(c \pm 3)\big[(c \pm 1)^2 - s\big] = 0$. We consider the two subcases given by two factors of Eq_9.

(1) *Subcase* $c \pm 3 = 0$. We may assume $c > 0$ because of the rescaling $(x, y, t) \mapsto (-x, y, -t)$ in the above systems. Therefore we set $c = 3$ and then systems (3.61) could be brought to system (3.58) with $c = 0$ and $a = -1$ via the transformation $(x, y, t) \mapsto \big(2(x - 1),\ 4(y - x - 1),\ t/2\big)$. So we arrive at the system (3.59) with $a = -1$ and as it was shown above this system possesses the family of hyperbolas (3.60) with $a = -1$.

(2) *Subcase* $(c \pm 1)^2 - s = 0$. Then $s = (c \pm 1)^2$ and this leads to the reducible conics $\Phi(x, y) = (c^2 - 1 \pm x + cx + y)^2 = 0$.

Case $a > 0$. Then we may assume $a = k^2$ and applying the same rescaling as above we arrive at the family systems $\dot{x} = 1 + y + (x + c)^2$, $\dot{y} = xy$. So we have $W = 2(c \pm i)$ and we obtain $Eq_{10} = 0$ $Eq_9 = 2(c \pm 3i)\big[(c \pm i)^2 - s\big] = 0$. Since $c \in \mathbb{R}$ we obtain $s = (c \pm i)^2$ and this again leads to the reducible conics $\Phi(x, y) = (c^2 - 1 \pm ix + cx + y)^2 = 0$.

Thus we detect that in the case $d \neq 0$ a system (3.58) could possesses an invariant hyperbola if and only if either the conditions $c = 0$ or $a < 0$ (then $a = -1$) and $c^2 - 9 = 0$ hold. On the other hand for these systems we calculate $N_7 = c(9a + c^2)/2$ and we claim that the above conditions are equivalent to $N_7 = 0$. Indeed, if $c = 0$ or $a = -1$ and $c^2 - 9 = 0$ we obtain $N_7 = 0$. Conversely, assuming $N_7 = 0$ we have either $c = 0$ or $9a + c^2 = 0$. However in the second case the condition $a \leq 0$ must hold. If $a = 0$ we obtain $c = 0$ and we arrive at the first case. If $a < 0$ as it was mentioned earlier due to a rescaling we may assume $a = -1$ (see systems (3.61)) and then we obtain $c^2 + 9a = c^2 - 9 = 0$ and this completes the proof of our claim.

3.5.2. *Subcase* $\hat{d} = 0$. In this case systems (3.57) become as systems

$$\dot{x} = \hat{a} + \hat{c}x + x^2, \quad \dot{y} = \hat{b} + xy, \tag{3.62}$$

for which following [15] we calculate the value of invariant polynomial $H_{12} = -8\hat{a}^2x^2$ and we consider two possibilities: $\hat{a} \neq 0$ and $\hat{a} = 0$.

Possibility $\hat{a} \neq 0$. As it was shown in [15, page 750] in this case via an affine transformation and time rescaling after some additional parametrization we arrive at the following 2-parameter family of systems

$$\dot{x} = a + (x + c)^2, \quad \dot{y} = xy \tag{3.63}$$

for which the condition $H_{12} = -8(a + c^2)^2x^2 \neq 0$ must hold.

Next, to determine the conditions for the existence of a hyperbola as earlier we apply the equations (2.7). Since the quadratic parts of the above systems coincide with quadratic parts of systems (3.58) by the same reasons from the first four equations (2.7) we determine that $s \neq 0$, $U = 2$ and $V = 0$ and then calculations yield

$$Eq_5 = -q + 4cs - sW = 0, \quad Eq_6 = -r + 4ct - 2tW = 0,$$
$$Eq_7 = -uW = 0, \quad Eq_8 = -2p + 2cq + 2as + 2c^2s - qW = 0.$$

So we obtain $q = s(4c - W)$, $r = 2t(2c - W)$, $p = s(2a + 10c^2 - 6cW + W^2)/2$, $uW = 0$ and we consider two cases: $u = 0$ and $u \neq 0$.

Case $u = 0$. In this case we have $Eq_i = 0$, $i = 1, 2, \dots, 8$ and

$$Eq_9 = 2t(a + c^2 - 2cW + W^2) = 0,$$
$$Eq_{10} = s(2c - W)(4a + 4c^2 - 4cW + W^2) = 0$$

and we observe that $t \neq 0$ otherwise we obtain

$$\Phi(x, y) = s(2a + 10c^2 - 6cW + W^2 + 8cx - 2Wx + 2x^2)/2 = 0,$$

i.e. $\Phi(x, y)$ is a product of two parallel lines. It was mentioned above that the condition $s \neq 0$ also must hold, i.e. $st \neq 0$ and we calculate $\mathrm{Res}_W(Eq_9, Eq_{10}) = 2s^2t^3(a + c^2)^2(9a + c^2)$ and clearly for the existence of a common solution of the equations $Eq_9 = Eq_{10} = 0$ the condition $(a+c^2)^2(9a+c^2) = 0$ is necessary. However the condition $H_{12} \neq 0$ implies $a + c^2 \neq 0$ and therefore we obtain $9a + c^2 = 0$.

So $a = -c^2/9$ and we detect that in this case the polynomials Eq_9 and Eq_{10} have as a common factor $4c - 3W$. Therefore we obtain $W = 4c/3$ and we arrive at the systems

$$\dot{x} = (2c + 3x)(4c + 3x)/9, \quad \dot{y} = xy,$$

which possess the following family of hyperbolas $\Phi(x, y) = 16c^2s + 24csx + 9sx^2 + 12cty + 18txy = 0$. In order to have irreducible invariant conics we determine $\Delta = -324c^2st^2 \neq 0$. So $s \neq 0$ and setting a new parameter $m = 6t/s$ we arrive at the 1-parameter family of hyperbolas

$$\Phi(x, y) = 16c^2 + 24cx + 2cmy + 9x^2 + 3mxy = 0.$$

Case $u \neq 0$. Then we obtain $W = 0$ and we calculate

$$Eq_i = 0, \quad i = 1, 2, \dots, 8, \quad Eq_9 = 2(a + c^2)t = 0, \quad Eq_{10} = 4c(a + c^2)s = 0$$

and since $s \neq 0$ and $a + c^2 \neq 0$ (due to $H_{12} \neq 0$) we obtain $t = c = 0$. So we arrive at the 1-parameter family of systems $\dot{x} = a + x^2$, $\dot{y} = xy$, which possess the family of conics

$$\Phi(x, y) = as + sx^2 + uy^2 = 0.$$

Clearly these conics are of hyperbolic type if $su < 0$ and they are irreducible if in addition we have $a \neq 0$. So setting $u = -m^2 s$ we obtain the following 1-parameter family of hyperbolas:

$$\Phi(x, y) = a + x^2 - m^2y^2 = 0.$$

Thus we detect that in the case $\hat{d} = 0$ and $\hat{a} \neq 0$ a system (3.62) could be brought to (3.63) which possess an invariant hyperbola if and only if the condition $c(9a+c^2) = 0$ holds. On the other hand for these systems we have $N_7 = c(9a+c^2)/2$ and we deduce that in the case under consideration Lemma 3.20 is valid.

Possibility $\hat{a} = 0$. This condition implies $\hat{b} \neq 0$ (otherwise we obtain degenerate systems (3.62)). So we may assume $\hat{b} = 1$ due to the rescaling $y \to \hat{b}y$ and this leads to the 1-parameter family of systems (we set $\hat{c} = c$)

$$\dot{x} = cx + x^2, \quad \dot{y} = 1 + xy, \tag{3.64}$$

And again, since the quadratic parts of the above systems coincides with quadratic parts of systems (3.58) by the same reasons from the first four equations (2.7) we determine that $s \neq 0$, $U = 2$ and $V = 0$ and then calculations yield

$$Eq_5 = -q + 2cs - sW = 0, \quad Eq_6 = -r + 2ct - 2tW = 0,$$
$$Eq_7 = -uW = 0, \quad Eq_8 = -2p + cq + 2t - qW = 0.$$

So we obtain $q = s(2c-W)$, $r = 2t(c-W)$, $p = (2c^2s+2t-3csW+sW^2)/2$, $uW = 0$ and we claim that the condition $u = 0$ must hold. Indeed supposing $u \neq 0$ we obtain $W = 0$ and this implies $Eq_9 = 2u = 0$ and this contradiction proves our claim. So $u = 0$ and calculations yield $Eq_i = 0$, $i = 1, 2, \dots, 8$, $Eq_9 = -2t(c - W)W = 0$, and $Eq_{10} = (4ct - 2c^2sW - 6tW + 3csW^2 - sW^3)/2 = 0$. We observe that $t \neq 0$ otherwise we obtain $\Phi(x, y) = s(2c^2 - 3cW + W^2 + 4cx - 2Wx + 2x^2)/2 = 0$, i.e. $\Phi(x, y)$ is a product of two parallel lines. So we obtain $W(c - W) = 0$ and we have to consider the two subcases given by these two factors. However we obtain $Eq_{10} = 2ct = 0$ if $W = 0$ and $Eq_{10} = -ct = 0$ if $W = c$ and therefore due to $t \neq 0$ in both cases we obtain $c = 0$. So we arrive at the system

$$\dot{x} = x^2, \quad \dot{y} = 1 + xy,$$

which possess the following family of hyperbolas $\Phi(x, y) = t + sx^2 + 2txy = 0$ and for the irreducibility of these conics the condition $t \neq 0$ is necessary. Then setting $m = s/t$ we obtain the 1-parameter family of hyperbolas

$$\Phi(x, y) = 1 + mx^2 + 2xy = 0.$$

Thus in the case $\hat{d} = \hat{a} = 0$ a system (3.62) could be brought to (3.64) which possess an invariant hyperbola if and only if the condition $c = 0$ holds. On the other hand for these systems we have $N_7 = -16c^3$ and this completes the proof of Lemma 3.20. □

Then, we conclude that the Main Theorem is completely proved.

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