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SINGULARITIES OF SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS IN A COMPLEX DOMAIN

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Abstract. We give an explicit representation of the solution of the following singular Cauchy problem with analytic data,

$$u_{tt} - xu_{xx} + Cu_x - B(t^2 - 4x)^{-1}u = 0.$$

We also study the singularities of this solution.

1. Introduction

In this article, we study the singularities of the solution to the following Cauchy problem with analytic data in the complex domain

$$\begin{gathered} LU = \partial_t^2 U - x\partial_x^2 U + C\partial_x U - B(t^2 - 4x)^{-1}U = 0, \\ U(0,x) = U_0x), \\ U_t(0,x) = U_1(x), \end{gathered} \tag{1.1}$$

where C and B are real numbers. The variables t, x and the unknown U are complex numbers. Our aim is to give an explicit representation of the solution in terms of Gauss hypergeometric functions and study its singularities.

Equation (1.1) arises naturally by linearizing the nonlinear partial differential equation with double characteristics in involution

$$\partial_t^2 U - x\partial_x^2 U + C\partial_x U = aU^p,$$

around its self-similar solution $U = (t^2 - 4x)^{\frac{-1}{p-1}}$, where a is real number, $p > 1$ and $B = ap$.

In the complex domain, the study of the singularities of the solutions of the nonlinear Cauchy problem is very complicated. Especially if the roots of the characteristic polynomial are double and not holomorphic at the origin (as is in our case). Indeed the technical difficulties are such that even in the linear case there are no methods for obtaining general theorems.

However in the case of second order equations, many authors showed that the solutions of certain evolution or degenerate linear equations can be expressed in terms of hypergeometric functions. Since these hypergeometric functions have intrinsic singularities, they permit the analysis of the structure of the solutions and therefore to describe their singularities; see [2, 3, 4, 5, 7].

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We will show that the solution, depending on various parameters, might have singularities on the characteristic surfaces:

$$K_1 : x = 0, \quad K_2 : 4x - t^2 = 0. \tag{1.2}$$

Before developing the theory and to get some insight to the problem, we present four concrete examples.

Example 1.1. In $\mathbb{C}^2$ consider the Cauchy problem

$$\begin{gathered}\partial_t^2 U - x\partial_x^2 U + 2\partial_x U - 28(t^2 - 4x)^{-1}U = 0,\\ U(0, x) = x^4,\\ U_t(0, x) = 0.\end{gathered}$$

The solution is

$$U(t, x) = \frac{1}{4^2}(4x - t^2)^2(x^2 - t^2x + \frac{t^4}{6}).$$

We observe that U is polynomial.

Example 1.2. The Cauchy problem

$$\begin{gathered}\partial_t^2 U - x\partial_x^2 U - 2\partial_x U - 12(t^2 - 4x)^{-1}U = 0,\\ U(0, x) = x,\\ U_t(0, x) = 0,\end{gathered}$$

has solution

$$U(t, x) = \frac{1}{16}x^{-1}(4x - t^2)^2.$$

Thus it is singular only on $K_1 : x = 0$.

Example 1.3. In $\mathbb{C}^2$ consider the Cauchy problem

$$\begin{gathered}\partial_t^2 U - x\partial_x^2 U + \partial_x U + 6(t^2 - 4x)^{-1}U = 0,\\ U(0, x) = x,\\ U_t(0, x) = 0.\end{gathered}$$

Its solution is

$$U(t, x) = 4x^2(4x - t^2)^{-1}.$$

Thus, $U(t, x)$ is singular only on $K_2 : 4x - t^2 = 0$.

Example 1.4. Consider the Cauchy problem

$$\begin{gathered}\partial_t^2 U - x\partial_x^2 U - \partial_x U - 4(t^2 - 4x)^{-1}U = 0,\\ U(0, x) = x,\\ U_t(0, x) = 0.\end{gathered}$$

Its solution is

$$U(t, x) = \frac{1}{4}\big[t\sqrt{(4x - t^2)}\arcsin(\frac{t}{2\sqrt{x}}) + (4x - t^2)\big].$$

Thus, $U(t, x)$ is singular both on $K_1 : x = 0$ and on $K_2 : 4x - t^2 = 0$.

2. Statement of the problem

In a neighborhood Ω of the origin of $\mathbb{C}^2$, we consider the Cauchy problem with analytic data:

$$\begin{gathered} LU=\partial_t^2U-x\partial_x^2U+C\partial_xU-B(t^2-4x)^{-1}U=0,\\ U(0,x)=u_0(x),\\ U_t(0,x)=u_1(x), \end{gathered} \tag{2.1}$$

where

$$u_0(x)=\sum_{l=0}^{\infty}a_lx^l,\quad u_1(x)=\sum_{l=0}^{\infty}b_lx^l$$

are analytic and C, B are real numbers.

Our purpose is to construct an exact solution in terms of hypergeometric functions and to show that the solution might have singularities on the characteristic surfaces (1.2). We denote

$$\begin{gathered} \Omega_R=\{(t,x)\in\mathbb{C}^2:|t|<R,\ |x|<R\},\\ K_1=\{(t,x)\in\Omega_R:x=0\},\\ K_2=\{(t,x)\in\Omega_R:4x-t^2=0\}. \end{gathered}$$

Using these notation, we have the following theorems.

Theorem 2.1. *The Cauchy problem* (2.1) *has a unique solution of the form :*

$$U(t,x)=\sum_{l\geq 0}(a_lV_{l,\beta}+b_lW_{l,\beta'}),$$

with

$$\begin{aligned} V(t,x)&=V_{l,\beta}(t,x)\\ &=(4)^{-\beta}(x)^{l-\beta}(4x-t^2)^{\beta}F\big(\beta-l,C+\beta-l+1,\frac{1}{2};\frac{t^2}{4x}\big), \end{aligned} \tag{2.2}$$

where β satisfies

$$\beta(2l-\beta-C-\frac{1}{2})=\frac{1}{4}B,$$

$$\begin{aligned} W(t,x)&=W_{l,\beta'}(t,x)=\\ &=(4)^{-\beta'}(x)^{l-\beta'}(4x-t^2)^{\beta'}tF\big(\beta'-l,C+\beta'-l+1,\frac{3}{2};\frac{t^2}{4x}\big), \end{aligned} \tag{2.3}$$

where

$$\beta'(2l-\beta'-C+\frac{1}{2})=\frac{1}{4}B,$$

and F denotes the Gauss hypergeometric function. This solution might have singularities on $K=K_1\cup K_2$.

2.1. Construction of the solutions.

According to the principle of superposition, it is sufficient to study the following two Cauchy problems:

$$\begin{gathered} LV=0,\\ V(0,x)=x^l,\\ V_t(0,x)=0, \end{gathered} \tag{2.4}$$

and

$$
\begin{gathered}
LW = 0, \\
W(0,x) = 0, \\
W_t(0,x) = x^l.
\end{gathered} \tag{2.5}
$$

First we solve (2.4). Setting the Characteristics equations $x = \xi_1$ and $4x - t^2 = \xi_2$. Let $V(t,x) = V_l(t,x) = \xi_1^l v(z)$ with $z = 1 - \frac{\xi_2}{4\xi_1}$. Substituting $\xi_1^l v(z)$ for V , $LV = 0$ becomes

$$
\begin{aligned}
LV = {}& \frac{1}{2}\xi_1^{l-1}V' + z\xi_1^{l-1}V'' - z^2\xi_1^{l-1}V'' - 2z\xi_1^{l-1}V' + lz\xi_1^{l-1}V' \\
& + zl\xi_1^{l-1}V' - l(l-1)\xi_1^{l-1}V + C(l\xi_1^{l-1}V - z\xi_1^{l-1}V') + \frac{B}{\xi_2}\xi_1^l V = 0.
\end{aligned}
$$

Simplifying this equation and replacing $\frac{\xi_1}{\xi_2}$ by $\frac{1}{4(1-z)}$, we have

$$
\begin{aligned}
& z(1-z)v'' + \big(\frac{1}{2} - (C + 2(1-l))z\big)v' \\
& + (l(C - l + 1) - \frac{1}{4}B(\frac{1}{z-1}))v = 0.
\end{aligned} \tag{2.6}
$$

The substitution $v = (1-z)^\beta y$ leads to

$$
\begin{aligned}
& z(1-z)y'' + \big(\frac{1}{2} - (C + 2(1 - l + \beta))z\big)y' \\
& + \big(l(C - l + 1) + \beta(\beta + C + 1 - 2l)\big)y = 0.
\end{aligned} \tag{2.7}
$$

Therefore (2.7) is equivalent to a Gauss differential equation with parameters

$$
\big(\beta - l, C + \beta - l + 1, \frac{1}{2}\big),
$$

if and only if

$$
\beta\big(2l - \beta - C - \frac{1}{2}\big) = \frac{1}{4}B. \tag{2.8}
$$

According to hypergeometric equation theory, we have: A first solution of Gauss equation for $|z| < 1$ is

$$
y_1(z) = F(a,b;c;z) = F\big(\beta - l, C + \beta - l + 1; \frac{1}{2}; z\big).
$$

A second solution is

$$
\begin{aligned}
y_2(z) &= z^{1/2}F\big(a - c + 1, b - c + 1; 2 - c; z\big) \\
&= z^{1/2}F\big(\beta - l + \frac{1}{2}, C + \beta - l + \frac{3}{2}, \frac{3}{2}; z\big).
\end{aligned}
$$

A complete solution of the Gauss equation is

$$
\begin{aligned}
y = {}& DF\big(\beta - l, C + \beta - l + 1, \frac{1}{2}; z\big) \\
& + Ez^{1/2}F\big(\beta - l + \frac{1}{2}, C + \beta - l + \frac{3}{2}, \frac{3}{2}; z\big),
\end{aligned} \tag{2.9}
$$

with $z = \frac{t^2}{4x}$, for $|z| < 1$, where D and E are constants.

It follows that $V = x^l(1-z)^\beta y$ is a solution of $LV = 0$. Taking into account the Cauchy data

$$
V(0,x) = x^l, \quad V_t(0,x) = 0.
$$

we have to choose $D = 1$ and $E = 0$. Hence $V(t,x)$ reduces to

$$V(t,x) = V_{l\beta}(t,x) = x^l\big(1 - \frac{t^2}{4x}\big)^{\beta} F\big(\beta - l, C + \beta - l + 1, \frac{1}{2}; \frac{t^2}{4x}\big). \tag{2.10}$$

In a similar way, we solve the second Cauchy problem (2.4) by setting

$$W(t,x) = tx^l\big(1 - \frac{t^2}{4x}\big)^{\beta'} y;$$

we obtain

$$\begin{aligned} W(t,x) &= W_{l,\beta'}(t,x) \\ &= t(4)^{-\beta'} x^{l-\beta'}(4x - t^2)^{\beta'} F\big(\beta' - l, C + \beta' - l + 1, \frac{3}{2}; \frac{t^2}{4x}\big), \end{aligned} \tag{2.11}$$

where β' and l satisfy

$$\beta'\big(\frac{1}{2} + 2l - C - \beta'\big) = \frac{B}{4}.$$

Remark 2.2. When $\beta - l = -n$ or $C + \beta - l + 1 = -n$, $n \in \mathbb{N}$, the solution V of (2.4) is

$$\begin{aligned} V(t,x) &= (4)^{l-n}(x)^n(4x - t^2)^{l-n}\Big(\sum_{i=0}^{n} \alpha_i\big(\frac{t^2}{4x}\big)^i\Big), \\ &= (4)^{l-n}(4x - t^2)^{l-n}\Big(\sum_{i=0}^{n} \alpha_i x^{n-i}\big(\frac{t^2}{4}\big)^i\Big); \end{aligned}$$

its last term is

$$c_n t^{2n}(4x - t^2)^{l-n},$$

where c_n depends on parameters C, l and β.

Therefore, we have some results for V.

1. When $l - n \geq 0$, the solution $V(t,x)$ is a polynomial.
2. When $l - n < 0$, the solution is singular on the surface $K_2 : 4x - t^2 = 0$.
3. When $C + \beta - l + 1 = -n$, we have the following results:
   - (i) For $C < -n - 1$, the solution is singular on the surface K_1.
   - (ii) For $C > -n - 1$ and $l - (C + n + 1) < 0$, the solution is singular on the surface K_2.
   - (iii) For $C > -n - 1$ and $l - (C + n + 1) > 0$, the solution is polynomial.

2.2. **Singularities.** In this section, we study the singularities of the solution. The mapping

$$z = \frac{t^2}{4x},$$

transforms

$$\begin{gathered} t = 0, \text{ into } z = 0, \\ K_2 : t^2 - 4x = 0, \text{ into } z = 1, \\ K_1 : x = 0, \text{ into } z = \infty. \end{gathered}$$

By construction, the solution U is composed of a hypergeometric function which is holomorphic on $D - (0, 1, \infty)$, where D is the Riemann sphere. So, the study of the singularities of the solution is reduced to those corresponding well known properties of Gauss functions. It follows that U is ramified around $K_1 \cup K_2$.

2.3. **Convergence of the solution.** The study of the convergence of this solution is reduced to estimate the Gauss functions. For this, we apply the following result.

Lemma 2.3. *If $a \geq b > c > 0$ and $d = a + b - c$, then for $|z| < 1$,*

$$F(a, b; c; z) \ll (1 - z)^{-d} \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}. \tag{2.12}$$

For the proof of this lemma, see [6].

Theorem 2.4. *The series $\sum_{l\geq 0} a_l V_l$ and $\sum_{l\geq 0} \sum b_l W_l$ converge for $|x| < \frac{R}{4}$, where R is the radius of convergence of u_0 and u_1.*

Proof. We have

$$V(t, x) = V_l(t, x) = (x)^l \big(1 - \frac{t^2}{4x}\big)^{\beta} F\big(\beta - l, C + \beta - l + 1; \frac{1}{2}; \frac{t^2}{4x}\big).$$

Let

$$\beta = \beta_1 = \frac{2l - C - \frac{1}{2} - \sqrt{\Delta}}{2}$$

be one of the roots of the equation (2.8), where

$$\Delta = \big(2l - C - \frac{1}{2}\big)^2 - B \tag{2.13}$$

Putting : $a = \beta - l$, $b = C + \beta - l + 1$, and $c = \frac{1}{2}$, we have

$$d = a + b - c = 2\beta - 2l + C + \frac{1}{2} = -\sqrt{\Delta} < 0.$$

In this case, we recall the Euler transformation

$$F(\alpha, \delta; \gamma; z) = (1 - z)^{\gamma-\alpha-\delta} F(\gamma - \alpha, \gamma - \delta; \gamma; z).$$

Applying this transformation to $F(\beta - l, C + \beta - l + 1; \frac{1}{2}; \frac{t^2}{4x})$, we obtain

$$\begin{aligned} &F\big(\beta - l, C + \beta - l + 1; \frac{1}{2}; \frac{t^2}{4x}\big) \\ &= \big(1 - \frac{t^2}{4x}\big)^{\sqrt{\Delta}} F\big(l - \beta + \frac{1}{2}, l - C - \beta - \frac{1}{2}; \frac{1}{2}; \frac{t^2}{4x}\big). \end{aligned}$$

For l large, $l - \beta = l + o(1)$, and so by applying Lemma 2.3 to $F\big(l - \beta + \frac{1}{2}, l - C - \beta - \frac{1}{2}; \frac{1}{2}; \frac{t^2}{4x}\big)$ we have

$$F\big(l - \beta + \frac{1}{2}, l - C - \beta - \frac{1}{2}; \frac{1}{2}; \frac{t^2}{4x}\big) \ll \big(1 - \frac{t^2}{4x}\big)^{-\sqrt{\Delta}} M,$$

where $y = \frac{\sqrt{\Delta}}{2}$ and

$$M = \frac{\Gamma(\frac{1}{2})\Gamma(2y)}{\Gamma(y - \frac{2C+1}{4})\Gamma(y + \frac{2C+3}{4})}.$$

Stirling's formula gives

$$M \sim 2^{2y-1} \frac{\sqrt{2\pi}(y)^{y-\frac{1}{2}} e^{-y} \sqrt{2\pi}(y)^y e^{-y}}{\Big(\sqrt{2\pi}(y)^{y-(\frac{2C+3}{4})} e^{-y}\Big)\Big(\sqrt{2\pi}(y)^{y+(\frac{2C+1}{4})} e^{-y}\Big)};$$

then $M \sim 2^{2y-1}$. Therefore,

$$|V(t, x)| \leqslant 2^{2y-1} |x|^l |1 - \frac{t^2}{x}|^{\beta_1} |1 - \frac{t^2}{x}|^{\sqrt{\Delta}} |1 - \frac{t^2}{x}|^{-\sqrt{\Delta}}$$

$$\leqslant 2^{2y-1}|x|^l|1-\frac{t^2}{x}|^{\beta_1}.$$

As $\beta_1 \to 0$ for l large, we deduce that

$$\limsup_{l\to\infty}|V_l|^{1/l} \le 4|x|.$$

It follows that $\sum_{l\geq 0} a_l V_l$ converges for

$$|x| < \frac{R}{4}.$$

In the similar way, we show that $\sum b_l W_l$ converges for

$$|x| < \frac{1}{4}R.$$

This completes the proof. □

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