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Q-INTEGRAL EQUATIONS OF FRACTIONAL ORDERS

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Abstract. The aim of this paper is to study the existence of solutions for a class of q-integral equations of fractional orders. The techniques in this work are based on the measure of non-compactness argument and a generalized version of Darbo's theorem. An example is presented to illustrate the obtained result.

1. Introduction

In this paper, we are concerned with the following functional equation

$$x(t) = F\Big(t, x(a(t)), \frac{f(t, x(b(t)))}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big), \quad t \in I, \tag{1.1}$$

where $\alpha > 1$, $q \in (0,1)$, $I = [0,1]$, $f, u : [0,1] \times \mathbb{R} \to \mathbb{R}$, $a, b : I \to I$ and $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Equation (1.1) can be written as

$$x(t) = F\left(t, x(a(t)), f(t, x(b(t))) I_q^\alpha u(\cdot, x(\cdot))(t)\right), \quad t \in I,$$

where I_q^α is the q-fractional integral of order α defined by (see [1])

$$I_q^\alpha h(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s, \quad t \in [0,1].$$

Using a measure of non-compactness argument combined with a generalized version of Darbo's theorem, we provide sufficient conditions for the existence of at least one solution to (1.1). We present also some examples to illustrate our obtained result.

The measure of non-compactness argument was used by several authors to study the existence of solutions for various classes of integral equations. As examples, we refer the reader to Aghajani et al [2, 4, 5], Banaś et al [10, 14, 15], Banaś and Goebel [11], Banaś and Rzepka [16], Banaś and Martinon [12], Caballero et al [20, 21, 22], Darwish [25], Çakan and Ozdemir [23], Dhage and Bellale [28], Mursaleen and Mohiuddine [39], Mursaleen and Alotaibi [37], and the references therein. For other applications of the measure of non-compactness concept, we refer to [13, 38].

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In [24], via a measure of non-compactness concept, Darwish obtained an existence result for the singular integral equation

$$y(t)=f(t)+\frac{y(t)}{\Gamma(\alpha)}\int_0^t \frac{u(s,y(s))}{(t-s)^{1-\alpha}}\,ds, \quad t\in[0,1],\ \alpha>0.$$

The above equation can be written in the form

$$y(t)=f(t)+y(t)I^\alpha u(\cdot,y(\cdot))(t), \quad t\in[0,1],$$

where I^α is the Riemann-Liouville fractional integral of order α defined by (see [41])

$$I^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{h(s)}{(t-s)^{1-\alpha}}\,ds, \quad t\in[0,1].$$

Following the above work, there has been a great interest in the study of functional equations involving fractional integrals. In this direction, we refer the reader to [3, 19, 26, 27, 17, 18, 29] and the references therein.

The concept of q-calculus (quantum calculus) was introduced by Jackson (see [33, 34]). This subject is rich in history and has several applications (see [30, 35]). Fractional q-difference concept was initiated by Agarwal and by Al-Salam (see [1, 7]). Because of the considerable progress in the study of fractional differential equations, a great interest appeared from many authors in studying fractional q-difference equations (see for examples [6, 7, 31, 32, 36] and the references therein).

The paper is organized as follows. In Section 2, we fix some notations that will be used through this paper, we recall some basic tools on q-calculus and we collect some basic definitions and facts on the measure of non-compactness concept. In Section 3, we state and prove our main result. Next, we present an illustrative example.

2. Preliminaries

At first, we recall some concepts on fractional q-calculus and present additional properties that will be used later. For more details, we refer to [1, 8, 40].

Let q be a positive real number such that $q\neq 1$. For $x\in\mathbb{R}$, the q-real number $[x]_q$ is defined by

$$[x]_q=\frac{1-q^x}{1-q}.$$

The q-shifted factorial of real number x is defined by

$$(x,q)_0=1,\quad (x,q)_k=\prod_{i=0}^{k-1}(1-xq^i), \quad k=1,2,\dots,\infty.$$

For $(x,y)\in\mathbb{R}^2$, the q-analog of $(x-y)^k$ is defined by

$$(x-y)^{(0)}=1,\ (x-y)^{(k)}=\prod_{i=0}^{k-1}(x-q^iy),\ k=1,2,\dots$$

For $\beta\in\mathbb{R}$, $(x,y)\in\mathbb{R}^2$ and $x\geq 0$,

$$(x-y)^{(\beta)}=x^\beta\prod_{i=0}^{\infty}\frac{x-yq^i}{x-yq^{\beta+i}}.$$

Note that if $y=0$, then $x^{(\beta)}=x^\beta$.

The following inequality (see [31]) will be used later.

Lemma 2.1. *If $\beta > 0$ and $0 \le a \le b \le t$, then*
$$(t-b)^{(\beta)} \le (t-a)^{(\beta)}.$$

The q-gamma function is given by
$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \notin \{0,-1,-2,\dots\}.$$
We have the following property
$$\Gamma_q(x+1) = [x]_q \Gamma_q(x).$$
Let $f : [0,b] \to \mathbb{R}$ $(b > 0)$ be a given function. The q-integral of the function f is given by
$$I_q f(t) = \int_0^t f(s)\, d_q s = t(1-q) \sum_{n=0}^{\infty} f(tq^n) q^n, \quad t \in [0,b].$$
If $c \in [0,b]$, we have
$$\int_c^b f(s)\, d_q s = \int_0^b f(s)\, d_q s - \int_0^c f(s)\, d_q s.$$

Lemma 2.2. *If $f : [0,1] \to \mathbb{R}$ is a continuous function, then*
$$\Big|\int_0^t f(s)\, d_q s\Big| \le \int_0^t |f(s)|\, d_q s, \quad t \in [0,1].$$

Note that in general, if $0 \le t_1 \le t_2 \le 1$ and $f : [0,1] \to \mathbb{R}$ is a continuous function, the inequality
$$\Big|\int_{t_1}^{t_2} f(s)\, d_q s\Big| \le \int_{t_1}^{t_2} |f(s)|\, d_q s$$
is not satisfied. We remark that in many papers dealing with q-difference boundary value problems, the use of such inequality yields wrong results. As a counter-example, we refer the reader to [8, Page.12].

Let $f : [0,1] \to \mathbb{R}$ be a given function. The fractional q-integral of order $\alpha \ge 0$ of the function f is given by $I_q^0 f(t) = f(t)$ and
$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s)\, d_q s, \quad t \in [0,1],\ \alpha > 0.$$
Note that for $\alpha = 1$, we have
$$I_q^1 f(t) = I_q f(t), \quad t \in [0,1].$$
If $f \equiv 1$, then
$$I_q^\alpha 1(t) = \frac{1}{\Gamma_q(\alpha+1)} t^\alpha, \quad t \in [0,1].$$

Now, we recall the notion of measure of non-compactness, which is the main tool used in this paper.

Let $\mathbb{E}$ be a Banach space over $\mathbb{R}$ with respect to a certain norm $\|\cdot\|$. We denote by $B_{\mathbb{E}}$ the set of all nonempty and bounded subsets of $\mathbb{E}$.

A mapping $\sigma : B_{\mathbb{E}} \to [0,\infty)$ is a measure of non-compactness in $\mathbb{E}$ (see [13]) if the following conditions are satisfied:

(i) for all $M \in B_{\mathbb{E}}$, we have
$$\sigma(M) = 0 \text{ implies } M \text{ is precompact;}$$

(ii) for every pair $(M_1, M_2) \in B_{\mathbb{E}} \times B_{\mathbb{E}}$, we have

$$M_1 \subseteq M_2 \Longrightarrow \sigma(M_1) \leq \sigma(M_2);$$

(iii) for every $M \in B_{\mathbb{E}}$,

$$\sigma(\overline{M}) = \sigma(M) = \sigma(\overline{\mathrm{co}} M),$$

where $\overline{\mathrm{co}} M$ denotes the closed convex hull of M;

(iv) for every pair $(M_1, M_2) \in B_{\mathbb{E}} \times B_{\mathbb{E}}$ and $\lambda \in (0,1)$, we have

$$\sigma(\lambda M_1 + (1-\lambda) M_2) \leq \lambda \sigma(M_1) + (1-\lambda)\sigma(M_2);$$

(v) if $\{M_n\}$ is a sequence of closed and decreasing (w.r.t $\subseteq$) sets in $B_{\mathbb{E}}$ such that $\sigma(M_n) \to 0$ as $n \to \infty$, then $M_\infty := \cap_{n=1}^{\infty} M_n$ is nonempty.

Let Λ be the set of functions $\eta : [0,\infty) \to [0,\infty)$ such that

(1) η is a non-decreasing function;
(2) η is an upper semi-continuous function;
(3) $\eta(s) < s$, for all $s > 0$.

For our purpose, we need the following generalized version of Darbo's theorem (see [2]).

Lemma 2.3. *Let D be a nonempty, bounded, closed and convex subset of a Banach space $\mathbb{E}$. Let $T : D \to D$ be a continuous mapping such that*

$$\sigma(TM) \leq \eta(\sigma(M)), \quad M \subseteq D,$$

where $\eta \in \Lambda$ and σ is a measure of non-compactness in $\mathbb{E}$. Then T has at least one fixed point.

Lemma 2.4. *Let $\eta_1, \eta_2 \in \Lambda$ and $\tau \in (0,1)$. Then the function $\gamma : [0,\infty) \to [0,\infty)$ defined by*

$$\gamma(t) = \max\{\eta_1(t), \eta_2(t), \tau\, t\}, \quad t \geq 0$$

belongs to the set Λ.

Proof. Let $(t,s) \in \mathbb{R}^2$ be such that $0 \leq t \leq s$. Since η_1, η_2 are non-decreasing and $\tau \in (0,1)$, we have

$$\begin{gathered} \eta_i(t) \leq \eta_i(s) \leq \gamma(s), \quad i = 1, 2\,, \\ \tau t \leq \tau s \leq \gamma(s), \end{gathered}$$

which yield $\gamma(t) \leq \gamma(s)$. Therefore, γ is a non-decreasing function. Now, for all $s > 0$, we have $\eta_i(s) < s$ (for $i = 1, 2$) and $\tau\, s < s$. Since $\gamma(s) \in \{\eta_1(s), \eta_2(s), \tau\, s\}$, we obtain

$$\gamma(s) < s, \quad s > 0.$$

On the other hand, it is well-known that the maximum of finitely many upper semi-continuous functions is upper semi-continuous. As consequence, the function γ belongs to the set Λ. $\square$

In what follows let $\mathbb{E} = C(I;\mathbb{R})$ be the set of real and continuous functions in the compact set I. It is well-known that $\mathbb{E}$ is a Banach space with respect to the norm

$$\|x\| = \max\{|x(t)| : t \in I\}, \quad x \in \mathbb{E}.$$

Now, let $M \in B_{\mathbb{E}}$ be fixed. For $(x,\rho) \in M \times (0,\infty)$, we denote by $\omega(x,\rho)$ the modulus of continuity of x; that is,

$$\omega(x,\rho) = \sup\{|x(t_1) - x(t_2)| : (t_1,t_2) \in I \times I,\ |t_1 - t_2| \le \rho\}.$$

Further on let us define

$$\omega(M,\rho) = \sup\{\omega(x,\rho) : x \in M\}.$$

Define the mapping $\sigma : B_{\mathbb{E}} \to [0,\infty)$ by

$$\sigma(M) = \lim_{\rho \to 0^+} \omega(M,\rho), \quad M \in B_{\mathbb{E}}.$$

Then σ is a measure of non-compactness in $\mathbb{E}$ (see [11]).

3. Main result

Define the operator T on $\mathbb{E} = C(I;\mathbb{R})$ by

$$(Tx)(t) = F\Big(t, x(a(t)), \frac{f(t,x(b(t)))}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s,x(s))\, d_q s\Big), \quad (x,t) \in \mathbb{E} \times I.$$

We consider the assumption

(A1) The functions $f, u : [0,1] \times \mathbb{R} \to \mathbb{R}$, $a, b : I \to I$ and $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

Proposition 3.1. *Under assumption* (A1), the operator T maps $\mathbb{E}$ into itself.

Proof. From assumption (A1), we have just to show that the operator H defined on $\mathbb{E}$ by

$$(Hx)(t) = \int_0^t (t-qs)^{(\alpha-1)} u(s,x(s))\, d_q s, \quad (x,t) \in \mathbb{E} \times I \tag{3.1}$$

maps $\mathbb{E}$ into itself. To do this, let us fix $x \in \mathbb{E}$. For all $t \in I$, we have

$$\begin{aligned} (Hx)(t) &= \int_0^t (t-qs)^{(\alpha-1)} u(s,x(s))\, d_q s \\ &= t(1-q) \sum_{n=0}^{\infty} q^n (t - q^{n+1} t)^{(\alpha-1)} u(tq^n, x(tq^n)) \\ &= t^{\alpha}(1-q) \sum_{n=0}^{\infty} q^n (1 - q^{n+1})^{(\alpha-1)} u(tq^n, x(tq^n)). \end{aligned}$$

On the other hand, since $0 < q^{n+1} < 1$, using Lemma 2.1, we have

$$(1-q^{n+1})^{(\alpha-1)} \le (1-0)^{(\alpha-1)} = 1.$$

Then by the continuity of u and using the Weierstrass convergence theorem, we obtain the desired result. □

We consider also the following assumptions.

(A2) There exist a constant $C_F > 0$ and a non-decreasing function $\varphi_F : [0,\infty) \to [0,\infty)$ such that

$$|F(t,x,y) - F(t,z,w)| \le \varphi_F(|x-z|) + C_F|y-w|, \quad (t,x,y,z,w) \in I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

(A3) There exists a constant $C_f > 0$ such that

$$|f(t,x) - f(t,y)| \le C_f|x-y|, \quad (t,x,y) \in I \times \mathbb{R} \times \mathbb{R}.$$

(A4) There exists a non-decreasing and continuous function $\varphi_u : [0,\infty) \to [0,\infty)$ such that

$$|u(t,x) - u(t,y)| \le \varphi_u(|x-y|), \quad (t,x,y) \in I \times \mathbb{R} \times \mathbb{R},\ \varphi_u(t) < t,\ t > 0,$$
$$u(t,0) = 0, \quad t \in I.$$

(A5) There exists $r_0 > 0$ such that

$$\varphi_F(r_0) + F^* + \frac{C_F(C_f r_0 + f^*)\varphi_u(r_0)}{\Gamma_q(\alpha+1)} \le r_0,$$

where

$$F^* = \max\{|F(t,0,0)| : t \in I\} \quad \text{and} \quad f^* = \max\{|f(t,0)| : t \in I\}.$$

Let the closed ball of center 0 and radius r_0 be denote by

$$\overline{B(0,r_0)} = \{x \in \mathbb{E} : \|x\| \le r_0\}.$$

Proposition 3.2. *Under assumptions* (A1)–(A5), *the operator* T *maps* $\overline{B(0,r_0)}$ *into itself.*

Proof. Let $x \in \overline{B(0,r_0)}$. Using the considered assumptions, for all $t \in I$, we have

$$\begin{aligned}
&|(Tx)(t)| \\
&\le \Big|F\Big(t, x(a(t)), \frac{f(t,x(b(t)))}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s,x(s))\, d_q s\Big) - F(t,0,0)\Big| \\
&\quad + |F(t,0,0)| \\
&\le \varphi_F(|x(a(t))|) + C_F \frac{|f(t,x(b(t)))|}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} |u(s,x(s))|\, d_q s + F^* \\
&\le \varphi_F(\|x\|) + C_F \frac{|f(t,x(b(t))) - f(t,0)| + |f(t,0)|}{\Gamma_q(\alpha)} \\
&\quad \times \int_0^t (t-qs)^{(\alpha-1)} |u(s,x(s))|\, d_q s + F^* \\
&\le \varphi_F(\|x\|) + C_F \frac{(C_f |x(b(t))| + f^*)\, \varphi_u(\|x\|)}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)}\, d_q s + F^* \\
&\le \varphi_F(\|x\|) + C_F \frac{(C_f \|x\| + f^*)\, \varphi_u(\|x\|)}{\Gamma_q(\alpha+1)} t^\alpha + F^* \\
&\le \varphi_F(r_0) + C_F \frac{(C_f r_0 + f^*)\, \varphi_u(r_0)}{\Gamma_q(\alpha+1)} + F^*.
\end{aligned}$$

Therefore,

$$\|Tx\| \le \varphi_F(r_0) + C_F \frac{(C_f r_0 + f^*)\, \varphi_u(r_0)}{\Gamma_q(\alpha+1)} + F^*, \quad x \in \overline{B(0,r_0)}.$$

Using the above inequality and assumption (A5), we obtain the desired result. $\square$

Proposition 3.3. *Under assumptions* (A1)–(A5), *the operator* T *maps continuously* $\overline{B(0,r_0)}$ *into itself.*

Proof. Define the operators γ_1, γ_2 and γ_3 on $\mathbb{E}$ by

$$(\gamma_1 x)(t) = t, \quad (x,t) \in \mathbb{E} \times I,$$
$$(\gamma_2 x)(t) = x(a(t)), \quad (x,t) \in \mathbb{E} \times I,$$

$$(\gamma_3 x)(t) = f(t, x(b(t))), \quad (x,t) \in \mathbb{E} \times I.$$

Obviously, $\gamma_1 : \mathbb{E} \to \mathbb{E}$ is continuous. Moreover, for all $x, y \in \mathbb{E}$, we have

$$|(\gamma_2 x)(t) - (\gamma_2 y)(t)| = |x(a(t)) - y(a(t))| \le \|x - y\|, \quad t \in I,$$

which implies that

$$\|\gamma_2 x - \gamma_2 y\| \le \|x - y\|, \quad (x,y) \in \mathbb{E} \times \mathbb{E}.$$

Therefore, γ_2 is uniformly continuous on $\mathbb{E}$. Similarly, for all $x, y \in \mathbb{E}$, for all $t \in I$, we have

$$\begin{aligned} |(\gamma_3 x)(t) - (\gamma_3 y)(t)| &= |f(t, x(b(t))) - f(t, y(b(t)))| \\ &\le C_f |x(b(t)) - y(b(t))\| \le C_f \|x - y\|, \end{aligned}$$

which implies

$$\|\gamma_3 x - \gamma_3 y\| \le C_f \|x - y\|, \quad (x,y) \in \mathbb{E} \times \mathbb{E}.$$

Then γ_3 is also uniformly continuous on $\mathbb{E}$. So, in order to prove that T is continuous on $\overline{B(0, r_0)}$, we only need to show that the operator H defined by (3.1) is continuous on $\overline{B(0, r_0)}$. To do this, let us consider $\varepsilon > 0$ and $(x, y) \in \overline{B(0, r_0)} \times \overline{B(0, r_0)}$ such that $\|x - y\| \le \varepsilon$. For all $t \in I$, we have

$$\begin{aligned} (Hx)(t) - (Hy)(t) &= \int_0^t (t - qs)^{(\alpha-1)} u(s, x(s))\, d_q s - \int_0^t (t - qs)^{(\alpha-1)} u(s, y(s))\, d_q s \\ &= \int_0^t (t - qs)^{(\alpha-1)} \left(u(s, x(s)) - u(s, y(s)) \right) d_q s. \end{aligned}$$

Set

$$u_{r_0}(\varepsilon) = \sup\{|u(t,x) - u(t,y)| : t \in I,\ (x,y) \in [-r_0, r_0] \times [-r_0, r_0],\ |x - y| \le \varepsilon\},$$

we obtain

$$(Hx)(t) - (Hy)(t) \le \frac{t^\alpha}{[\alpha]_q} u_{r_0}(\varepsilon) \le \frac{u_{r_0}(\varepsilon)}{[\alpha]_q},$$

for all $t \in I$. Therefore,

$$\|Hx - Hy\| \le \frac{u_{r_0}(\varepsilon)}{[\alpha]_q}.$$

Passing to the limit as $\varepsilon \to 0^+$ and using the uniform continuity of u on the compact set $I \times [-r_0, r_0]$, we obtain

$$\lim_{\varepsilon \to 0^+} \frac{u_{r_0}(\varepsilon)}{[\alpha]_q} = 0,$$

which completes the proof. □

To prove our main result, the following additional assumptions are needed.

(A6) The function $\varphi_F : [0, \infty) \to [0, \infty)$ is continuous and it satisfies $\varphi_F(s) < s$ for $s > 0$.

(A7) The function $a : I \to I$ satisfies

$$|a(t) - a(s)| \le \varphi_a(|t - s|), \quad (t,s) \in I \times I,$$

where $\varphi_a : [0, \infty) \to [0, \infty)$ is non-decreasing and $\lim_{t \to 0^+} \varphi_a(t) = 0$.

(A8) The function $b : I \to I$ satisfies

$$|b(t) - b(s)| \le \varphi_b(|t - s|), \quad (t,s) \in I \times I,$$

where $\varphi_b : [0, \infty) \to [0, \infty)$ is non-decreasing and $\lim_{t \to 0^+} \varphi_b(t) = 0$.

(A9) We suppose that
$$0<\varphi_u(r_0)<\frac{\Gamma_q(\alpha+1)}{C_F C_f}\quad\text{and}\quad \frac{C_F}{\Gamma_q(\alpha)}(C_f r_0+f^*)<1.$$

Our main result is the following.

Theorem 3.4. *Under assumptions* (A1)–(A9), Equation (1.1) has at least one solution $x^*\in C(I;\mathbb{R})$ satisfying $\|x^*\|\le r_0$.

Proof. From Proposition 3.3, we know that $T:\overline{B(0,r_0)}\to\overline{B(0,r_0)}$ is a continuous operator. Now, let M be a nonempty subset of $\overline{B(0,r_0)}$. Let $\rho>0$, $x\in M$ and $(t_1,t_2)\in I\times I$ be such that $|t_1-t_2|\le\rho$. Without restriction of the generality, we may assume that $t_1\ge t_2$. We have

$$\begin{aligned}
&|(Tx)(t_1)-(Tx)(t_2)|\\
&=\Big|F\Big(t_1,x(a(t_1)),\frac{f(t_1,x(b(t_1)))}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big)\\
&\quad-F\Big(t_2,x(a(t_2)),\frac{f(t_2,x(b(t_2)))}{\Gamma_q(\alpha)}\int_0^{t_2}(t_2-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big)\Big|\\
&\le\Big|F\Big(t_1,x(a(t_1)),\frac{f(t_1,x(b(t_1)))}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big)\\
&\quad-F\Big(t_2,x(a(t_1)),\frac{f(t_1,x(b(t_1)))}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big)\Big|\\
&\quad+\Big|F\Big(t_2,x(a(t_1)),\frac{f(t_1,x(b(t_1)))}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big)\\
&\quad-F\Big(t_2,x(a(t_2)),\frac{f(t_2,x(b(t_2)))}{\Gamma_q(\alpha)}\int_0^{t_2}(t_2-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big)\Big|\\
&=(I)+(II).
\end{aligned}\tag{3.2}$$

• Estimate for (I). We have

$$\begin{aligned}
&\Big|\frac{f(t_1,x(b(t_1)))}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big|\\
&\le\frac{|f(t_1,x(b(t_1)))|}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}|u(s,x(s))|\,d_qs\\
&\le\frac{|f(t_1,x(b(t_1)))-f(t_1,0)|+|f(t_1,0)|}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}\varphi_u(|x(s)|)\,d_qs\\
&\le\frac{(C_f|x(b(t_1))|+f^*)\varphi_u(\|x\|)}{\Gamma_q(\alpha+1)}t_1^\alpha\\
&\le\frac{(C_f\|x\|+f^*)\varphi_u(\|x\|)}{\Gamma_q(\alpha+1)}\\
&\le\frac{(C_f r_0+f^*)\varphi_u(r_0)}{\Gamma_q(\alpha+1)}=D.
\end{aligned}$$

Set

$$\begin{aligned}
C(F,\delta)=\sup\big\{&|F(t,x,y)-F(s,x,y)|:(t,s)\in I\times I,\ |t-s|\le\rho,\\
&x\in[-r_0,r_0],\ y\in[-D,D]\big\},
\end{aligned}$$

we obtain

$$(I) \le C(F,\delta). \tag{3.3}$$

• Estimate for (II). We have

$$\begin{aligned}
(II) &\le \varphi_F(|x(a(t_1)) - x(a(t_2))|) \\
&\quad + \frac{C_F}{\Gamma_q(\alpha)}\Big| f(t_1, x(b(t_1))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \\
&\quad - f(t_2, x(b(t_2))) \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big|.
\end{aligned}$$

On the other hand,

$$|x(a(t_1)) - x(a(t_2))| \le \omega(x \circ a, \rho),$$

which yields

$$\varphi_F(|x(a(t_1)) - x(a(t_2))|) \le \varphi_F(\omega(x \circ a, \rho)).$$

Now, we have

$$\begin{aligned}
&\Big| f(t_1, x(b(t_1))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \\
&\quad - f(t_2, x(b(t_2))) \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big| \\
&\le \Big| f(t_1, x(b(t_1))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \\
&\quad - f(t_2, x(b(t_2))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big| \\
&\quad + \Big| f(t_2, x(b(t_2))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \\
&\quad - f(t_2, x(b(t_2))) \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big| \\
&\le \frac{|f(t_1, x(b(t_1))) - f(t_2, x(b(t_2)))| \varphi_u(\|x\|)}{[\alpha]_q} \\
&\quad + |f(t_2, x(b(t_2)))| \Big| \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \\
&\quad - \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big| \\
&= (III) + (IV).
\end{aligned}$$

Let us define

$$\omega_f(r_0, \rho) = \sup\{|f(t,x) - f(s,x)| : (t,s) \in I \times I,\ |t-s| \le \rho,\ x \in [-r_0, r_0]\}.$$

Then

$$\begin{aligned}
(III) &\le \frac{\varphi_u(\|x\|)}{[\alpha]_q} |f(t_1, x(b(t_1))) - f(t_1, x(b(t_2)))| \\
&\quad + \frac{\varphi_u(\|x\|)}{[\alpha]_q} |f(t_1, x(b(t_2))) - f(t_2, x(b(t_2)))| \\
&\le \frac{[C_f |x(b(t_1)) - x(b(t_2))| + \omega_f(r_0, \rho)]\, \varphi_u(r_0)}{[\alpha]_q}
\end{aligned}$$

$$\le \frac{[C_f\omega(x\circ b,\rho)+\omega_f(r_0,\rho)]\,\varphi_u(r_0)}{[\alpha]_q}.$$

Now, let us estimate (IV). At first, we have

$$\begin{aligned}|f(t_2,x(b(t_2)))| &\le |f(t_2,x(b(t_2)))-f(t_2,0)|+|f(t_2,0)|\\ &\le C_f|x(b(t_2))|+f^*\le C_f r_0+f^*.\end{aligned}$$

Next, we have

$$\begin{aligned}&\Big|\int_0^{t_1}(t_1-qs)^{(\alpha-1)}u(s,x(s))\,d_qs-\int_0^{t_2}(t_2-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big|\\ &=(1-q)\sum_{n=0}^{\infty}q^n(1-q^{n+1})^{(\alpha-1)}\left|t_1^\alpha u(q^nt_1,x(q^nt_1))-t_2^\alpha u(q^nt_2,x(q^nt_2))\right|.\end{aligned}$$

We can write

$$\begin{aligned}&|t_1^\alpha u(q^nt_1,x(q^nt_1))-t_2^\alpha u(q^nt_2,x(q^nt_2))|\\ &\le t_1^\alpha\,|u(q^nt_1,x(q^nt_1))-u(q^nt_1,x(q^nt_2))|\\ &\quad+|t_1^\alpha u(q^nt_1,x(q^nt_2))-t_2^\alpha u(q^nt_2,x(q^nt_2))|\\ &\le \varphi_u(|x(q^nt_1)-x(q^nt_2)|)+A_\rho\\ &\le \varphi_u(\omega(x,\rho))+A_\rho,\end{aligned}$$

where

$$\begin{aligned}A_\rho=\sup\big\{&|\mathcal{N}(\tau,s,x)-\mathcal{N}(\tau',s',x)|:(\tau,s,\tau',s')\in I^4,\ |\tau-\tau'|\le\rho,\\ &|s-s'|\le\rho,\ x\in[-r_0,r_0]\big\}\end{aligned}$$

and

$$\mathcal{N}(\tau,s,x)=\tau^\alpha u(s,x),\ (\tau,s,x)\in I\times I\times\mathbb{R}.$$

Then, we obtain

$$\begin{aligned}&\Big|\int_0^{t_1}(t_1-qs)^{(\alpha-1)}u(s,x(s))\,d_qs-\int_0^{t_2}(t_2-qs)^{(\alpha-1)}u(s,x(s))\,d_qs\Big|\\ &\le\varphi_u(\omega(x,\rho))+A_\rho.\end{aligned}$$

As consequence, we have

$$(IV)\le(C_fr_0+f^*)(\varphi_u(\omega(x,\rho))+A_\rho).$$

Using the above inequalities, we obtain

$$\begin{aligned}(II)\le\varphi_F(\omega(x\circ a,\rho))+\frac{C_F}{\Gamma_q(\alpha)}\Big(&\frac{[C_f\omega(x\circ b,\rho)+\omega_f(r_0,\rho)]\varphi_u(r_0)}{[\alpha]_q}\\ &+(C_fr_0+f^*)(\varphi_u(\omega(x,\rho))+A_\rho)\Big).\end{aligned}$$

Now, observe that from assumption (A7), we have

$$\begin{aligned}\omega(x\circ a,\rho)&=\sup\{|x(a(t))-x(a(s))|:(t,s)\in I\times I,\ |t-s|\le\rho\}\\ &\le\sup\{|x(\mu)-x(\nu)|:(\mu,\nu)\in I\times I,\ |\mu-\nu|\le\varphi_a(\rho)\}\\ &=\omega(x,\varphi_a(\rho)).\end{aligned}$$

Similarly, from assumption (A8), we have

$$\omega(x\circ b,\rho)\le\omega(x,\varphi_b(\rho)).$$

Then

$$
\begin{aligned}
(II) \leq \varphi_F(\omega(x,\varphi_a(\rho))) + \frac{C_F}{\Gamma_q(\alpha)}\Big(&\frac{[C_f\omega(x,\varphi_b(\rho)) + \omega_f(r_0,\rho)]\varphi_u(r_0)}{[\alpha]_q} \\
&+ (C_f r_0 + f^*)(\varphi_u(\omega(x,\rho)) + A_\rho)\Big).
\end{aligned}
\tag{3.4}
$$

Next, using (3.2), (3.3) and (3.4), we obtain

$$
\begin{aligned}
\omega(Tx,\rho) \leq C(F,\delta) + \varphi_F(\omega(x,\varphi_a(\rho))) + \frac{C_F}{\Gamma_q(\alpha)}\Big(&\frac{[C_f\omega(x,\varphi_b(\rho)) + \omega_f(r_0,\rho)]\varphi_u(r_0)}{[\alpha]_q} \\
&+ (C_f r_0 + f^*)(\varphi_u(\omega(x,\rho)) + A_\rho)\Big),
\end{aligned}
$$

which yields

$$
\begin{aligned}
\omega(TM,\rho) \leq C(F,\delta) &+ \varphi_F(\omega(M,\varphi_a(\rho))) \\
&+ \frac{C_F}{\Gamma_q(\alpha)}\Big(\frac{[C_f\omega(M,\varphi_b(\rho)) + \omega_f(r_0,\rho)]\varphi_u(r_0)}{[\alpha]_q} \\
&+ (C_f r_0 + f^*)(\varphi_u(\omega(M,\rho)) + A_\rho)\Big).
\end{aligned}
$$

Recall that from assumptions (A7)–(A8), we have

$$
\lim_{\rho\to 0^+} \varphi_a(t) = \lim_{\rho\to 0^+} \varphi_b(t) = 0.
$$

Then passing to the limit as $\rho \to 0^+$ in the above inequality, we obtain

$$
\sigma(TM) \leq \varphi_F(\sigma(M)) + \frac{C_F}{\Gamma_q(\alpha)}\Big(\frac{C_f\sigma(M)\varphi_u(r_0)}{[\alpha]_q} + (C_f r_0 + f^*)\varphi_u(\sigma(M))\Big).
$$

Therefore,

$$
\sigma(TM) \leq \eta(\sigma(M)),
$$

where

$$
\eta(t) = \max\{\varphi_F(t), L\varphi_u(t), Nt\}, \quad t \geq 0,
$$

with

$$
L = \frac{C_F}{\Gamma_q(\alpha)}(C_f r_0 + f^*), \quad N = \frac{C_F C_f}{\Gamma_q(\alpha+1)}\varphi_u(r_0).
$$

From assumption (A9) and Lemma 2.4, the function η belongs also to the set Λ. Finally, applying Lemma 2.3, we obtain the existence of at least one fixed point of the operator T in $\overline{B(0,r_0)}$, which is a solution to (1.1). □

We end the paper with the following illustrative example. Consider the integral equation

$$
x(t) = \frac{t}{32} + \frac{x(t)}{4} + [\alpha]_q\Big(\frac{t}{2} + \frac{x(t)}{4}\Big)\int_0^t (t-qs)^{(\alpha-1)}\frac{x(s)}{(2+s^2)}\,d_qs, \tag{3.5}
$$

for $t \in I = [0,1]$, where $\alpha > 1$ and $q \in (0,1)$. Observe that (3.5) can be written in the form (1.1), where

$$
\begin{gathered}
a(t) = t, \quad t \in I, \quad b(t) = t, \quad t \in I, \\
F(t,x,y) = \frac{t}{32} + \frac{x}{4} + \Gamma_q(\alpha+1)y, \quad (t,x,y) \in I \times \mathbb{R} \times \mathbb{R}, \\
f(t,x) = \frac{t}{2} + \frac{x}{4}, \quad (t,x) \in I \times \mathbb{R},
\end{gathered}
$$

$$u(t,x)=\frac{x}{(2+t^2)},\quad (t,x)\in I\times\mathbb{R}.$$

Now, let us check that the required assumptions by Theorem 3.4 are satisfied.

• Assumption (A1). It is trivial.

• Assumption (A2). For all $(t,x,y,z,w)\in I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, we have

$$\begin{aligned}|F(t,x,y)-F(t,z,w)|&=\Big|\frac{x}{4}+\Gamma_q(\alpha+1)y-\frac{z}{4}-\Gamma_q(\alpha+1)w\Big|\\&\le\frac{|x-z|}{4}+\Gamma_q(\alpha+1)|y-w|.\end{aligned}$$

Then assumption (A2) is satisfied with

$$\varphi_F(t)=\frac{t}{4},\quad t\ge 0,$$
$$C_F=\Gamma_q(\alpha+1).$$

• Assumption (A3). For all $(t,x,y)\in I\times\mathbb{R}\times\mathbb{R}$, we have

$$|f(t,x)-f(t,y)|=\frac{|x-y|}{4}.$$

Then assumption (A3) is satisfied with $C_f=\frac{1}{4}$.

• Assumption (A4). For all $(t,x,y)\in I\times\mathbb{R}\times\mathbb{R}$, we have

$$|u(t,x)-u(t,y)|=\frac{|x-y|}{2+t^2}\le\frac{|x-y|}{2}.$$

Take $\varphi_u(t)=\frac{t}{2}$, $t\ge 0$, assumption (A4) holds.

• Assumption (A5). At first, in our case, we have $F^*=\frac{1}{32}$ and $f^*=\frac{1}{2}$. Now, the inequality

$$\varphi_F(r_0)+F^*+\frac{C_F(C_fr_0+f^*)\varphi_u(r_0)}{\Gamma_q(\alpha+1)}\le r_0$$

is equivalent to

$$r_0^2-4r_0+\frac{1}{4}\le 0.$$

The above inequality is satisfied for any $r_0\in[\frac{4-\sqrt{15}}{2},\frac{4+\sqrt{15}}{2}]$.

• Assumptions (A6)–(A8) are trivial.

• Assumption (A9). The inequality

$$0<\varphi_u(r_0)<\frac{\Gamma_q(\alpha+1)}{C_FC_f}$$

is equivalent to $0<r_0<8$. The inequality

$$\frac{C_F}{\Gamma_q(\alpha)}(C_fr_0+f^*)<1$$

is equivalent to

$$r_0<\frac{4}{[\alpha]_q}-2.$$

A simple computation gives us that

$$\Big[\frac{4-\sqrt{15}}{2},\frac{4+\sqrt{15}}{2}\Big]\cap\Big(0,\frac{4}{[\alpha]_q}-2\Big)\ne\emptyset$$

for $\alpha=3/2$ and $q=1/2$. Therefore, all the assumptions (A1)–(A9) are satisfied for $\alpha=3/2$ and $q=1/2$. By Theorem 3.4, we have the following result.

Theorem 3.5. *For $(\alpha, q) = (3/2, 1/2)$, Equation (3.5) has at least one solution $x^* \in C(I;\mathbb{R})$.*

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References

[1] R. P. Agarwal; *Certain fractional q-integrals and q-derivatives*, Proc. Cambridge Philos. Soc., **66** (1969), 365–370.
[2] A. Aghajani, R. Allahyari, M. Mursaleen; *A generalization of Darbo's theorem with application to the solvability of systems of integral equations*, J. Comput. Appl. Math., **260** (2014), 68–77.
[3] A. Aghajani, J. Banaś, Y. Jalilian; *Existence of solutions for a class of nonlinear Volterra singular integral equations*, Comput. Math. Appl., **62** (2011), 1215–1227.
[4] A. Aghajani, M. Mursaleen, A. Shole Haghighi; *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta Math. Sci., **35** B (3) (2015), 552–566.
[5] A. Aghajani, E. Pourhadi, J.J. Trujillo; *Application of measure of noncompactness to a Cauchy problem for fractional differential equations in Banach spaces*, Fract. Calc. Appl. Anal., **16** (4) (2013), 962–977.
[6] B. Ahmad, J. Nieto; *Basic theory of nonlinear third-order q-difference equations and inclusions*, Math. Model. Anal., **18** (1) (2013), 122–135.
[7] W. Al-Salam; *Some fractional q-integrals and q-derivatives*, Proc. Edinb. Math. Soc., **15** (2) (1966/1967), 135–140 .
[8] M. H. Annaby, Z. S. Mansour; *q-fractional calculus and equations*, Lecture Notes in Mathematics 2056, Springer-Verlag, Berlin, 2012.
[9] F. Atici, P. Eloe; *Fractional q-calculus on a time scale*, J. Nonlinear Math. Phys., **14** (3) (2007), 341–352.
[10] J. Banaś, J. Caballero, J. Rocha, K. Sadarangani; *Monotonic solutions of a class of quadratic integral equations of Volterra type*, Comput. Math. Appl., **49** (2005), 943–952.
[11] J. Banaś, K. Goebel; *Measures of Noncompactness in Banach Spaces*, in: Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, New York and Basel, 1980.
[12] J. Banaś, A. Martinon; *Monotonic solutions of a quadratic integral equation of Volterra type*, Comput. Math. Appl., **47** (2004), 271–279.
[13] J. Banaś, M. Mursaleen; *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Dehli, 2014.
[14] J. Banaś, M. Lecko, W.G. El-Sayed; *Existence theorems of some quadratic integral equations*, J. Math. Anal. Appl., **222** (1998), 276–285.
[15] J. Banaś, J. Rocha, K. Sadarangani; *Solvability of a nonlinear integral equation of Volterra type*, J. Comput. Appl. Math., **157** (2003), 31–48.
[16] J. Banaś, B. Rzepka; *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl., **284** (2003), 165–173.
[17] J. Banaś, B. Rzepka; *Monotonic solutions of a quadratic integral equation of fractional order*, J. Math. Anal. Appl., **332** (2007), 1370–1378.
[18] J. Banaś, D. O'Regan; *On existence and local attractivity of solutions of a quadratic integral equation of fractional order*, J. Math. Anal. Appl., **345** (2008), 573-582.
[19] M. Benchohra, D. Seba; *Integral equations of fractional order with multiple time delays in banach spaces*, Electron. J. Differential Equations, **56** (2012), 1–8.
[20] J. Caballero, B. López, K. Sadarangani; *On monotonic solutions of an integral equation of Volterra type with supremum*, J. Math. Anal. Appl., **305** (2005), 304–315.
[21] J. Caballero, B. López, K. Sadarangani; *Existence of nondecreasing and continuous solutions for a nonlinear integral equation with supremum in the kernel*, Z. Anal. Anwend., **26** (2007), 195–205.
[22] J. Caballero, J. Rocha, K. Sadarangani; *On monotonic solutions of an integral equation of Volterra type*, J. Comput. Appl. Math., **174** (2005), 119–133.

[23] U. Çakan, I. Ozdemir; *An application of Darbo fixed-point theorem to a class of functional integral equations*, Numer. Funct. Anal. Optim. **36** (2015), 29–40.
[24] M. A. Darwish; *On quadratic integral equation of fractional orders*, J. Math. Anal. Appl., **311** (2005), 112–119.
[25] M. A. Darwish; *On solvability of some quadratic functional-integral equation in Banach algebra*, Commun. Appl. Anal. **11** (2007), 441–450.
[26] M. A. Darwish; *On monotonic solutions of a singular quadratic integral equation with supremum*, Dynam. Systems Appl., **17** (2008) 539–550.
[27] M. A. Darwish, K. Sadarangani; *On a quadratic integral equation with supremum involving Erdélyi-Kober fractional order*, Mathematische Nachrichten, **288** (5-6) (2015) 566–576.
[28] B. C. Dhage, S. S. Bellale; *Local asymptotic stability for nonlinear quadratic functional integral equations*, Electron. J. Qual. Theory Differ. Equ., **10** (2008), 1–13.
[29] A. M. A. El-Sayed, H. H. G. Hashem; *Existence results for coupled systems of quadratic integral equations of fractional orders*, Optimization Letters, **7** (2013), 1251—260.
[30] T. Ernst; *The History of q-Calculus and a New Method*, U. U. D. M. Report 2000:16, ISSN 1101–3591, Department of Mathematics, Uppsala University, 2000.
[31] R. Ferreira,; *Nontrivial solutions for fractional q-difference boundary value problems*, Electron. J. Qual. Theory Differ. Equ., **70** (2010), 1–10.
[32] R. Ferreira; *Positive solutions for a class of boundary value problems with fractional q-differences*, Comput. Math. Appl., **61** (2011), 367–373.
[33] F. H. Jackson; *On q-functions and a certain difference operator*, Trans. Roy Soc. Edin., **46** (1908), 253–281.
[34] F. H. Jackson; *On q-definite integrals*, Quart. J. Pure and Appl. Math., **41** (1910), 193–203.
[35] V. Kac, P. Cheung; *Quantum calculus*, Springer Science & Business Media, 2002.
[36] S. Liang, J. Zhang; *Existence and uniqueness of positive solutions for three-point boundary value problem with fractional q-differences*, J. Appl. Math. Comput., **40** (2012), 277–288.
[37] M. Mursaleen, A. Alotaibi; *Infinite system of differential equations in some BK spaces*, Abstract Appl. Anal., Volume 2012, Article ID 863483, 20 pages.
[38] M. Mursaleen, A. K. Noman; *Compactness by the Hausdorff measure of noncompactness*, Nonlinear Anal., **73** (2010), 2541–2557.
[39] M. Mursaleen, S. A. Mohiuddine; *Applications of measures of noncompactness to the infinite system of differential equations in l^p spaces*, Nonlinear Anal. **75** (2012), 2111–2115.
[40] P. M. Rajković, S. D. Marinković, M. S. Stanković; *On q-analogues of Caputo derivative and Mittag-Leffler function*, Fract. Calc. Appl. Anal., **10** (2007), 359–373.
[41] S.G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon et alibi, 1993.

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