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EXISTENCE AND UNIQUENESS OF PSEUDO ALMOST PERIODIC SOLUTIONS FOR LIÉNARD-TYPE SYSTEMS WITH DELAYS

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Abstract. This article concerns Liénard-type systems with time-varying delays. By applying the theory of exponential dichotomies, the properties of pseudo almost periodic function, inequality analysis techniques, and contraction mapping principle, new criteria for the existence and uniqueness of pseudo almost periodic solutions are established. An example is given to illustrate the theoretical findings.

1. Introduction

It is well known that the Liénard-type systems have potential applications in many fields such as mechanics, physics and engineering and so on [3, 5, 6, 7, 8, 9]. Thus the problem on almost periodic solutions of the Liénard-type systems has attracted numerous scholars. In recent years, Gao and Liu [5] investigated the following Liénard-type equation with time-varying delays

$$\ddot{x}+g(x(t))\dot{x}(t)+h_0(x(t))+\sum_{l=1}^{m}h_l(x(t-\sigma_l(t)))=\rho(t), \tag{1.1}$$

where g and h_l $(l=0,1,2,\dots,m)$ are continuous functions on $\mathbb{R}$, $\sigma_l(t)\geq 0$ $(l=1,2,\dots,m)$ and $\rho(t)$ are almost periodic functions on $\mathbb{R}$. Form the viewpoint of mechanics, g usually stands for a damping or friction term, h_l denotes the restoring force, and $\rho_l(l=1,2,\dots,m)$ represents an externally force, σ_i denotes a time delay of the restoring force. For more details, we refer the reader to [1, 5]. Letting c^* be a positive constant and defining

$$f(x)=\int_0^x[g(u)-c^*]du, y=\frac{dx}{dt}+f(x),$$

Gao and Liu [5] obtained the following equivalent form of system (1.1),

$$\begin{gathered}\dot{x}(t)=-f(x(t))+y(t),\\ \dot{y}(t)=-c^*y(t)-[h_0(x(t))-c^*f(x(t))]-\sum_{l=1}^{m}h_l(x(t-\sigma_l(t)))+\rho(t).\end{gathered} \tag{1.2}$$

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By applying some analysis techniques and constructing a suitable Lyapunov function, they establish some sufficient conditions which guarantee the existence and exponential stability of the almost periodic solutions for system (1.2).

As we know, since the existence of pseudo almost periodic solutions has wide applications in various fields, especially in the economic, physics and biology [10, 11], it is a hot topic in qualitative theory of differential equations [10, 11]. In many cases, the nature of the almost periodic functions do not always hold in the set of pseudo almost periodic functions (see, e.g. [10, 11]). Based on the analysis above, we think that it is worthwhile to investigate the pseudo almost periodic solutions of Liénard-type systems. To the best of our knowledge, there is no paper that deal with this aspect for Liénard-type systems. Let

$$\begin{gathered} y=\dot{x}+ax-\theta_1(t), \\ \theta_2(t)=\rho(t)+a\theta_1(t)-\dot{\theta}_1(t). \end{gathered} \tag{1.3}$$

Then (1.2) becomes

$$\begin{gathered} \dot{x}(t)=-ax(t)+y(t)+\theta_1(t), \\ \dot{y}(t)=ay(t)-a^2x(t)-g(x(t))[y-ax+\theta_1(t)]-h_0(x(t)) \\ -\sum_{l=1}^{m}h_l(x(t-\sigma_l(t)))+\theta_2(t). \end{gathered} \tag{1.4}$$

Considering that the more reasonable models require the inclusion of the effect of changing environment, we modify system (1.4) as follows

$$\begin{gathered} \dot{x}(t)=-a(t)x(t)+y(t)+\theta_1(t), \\ \dot{y}(t)=a(t)y(t)-a^2(t)x(t)-g(x(t))[y-a(t)x+\theta_1(t)]-h_0(x(t)) \\ -\sum_{l=1}^{m}h_l(x(t-\sigma_l(t)))+\theta_2(t). \end{gathered} \tag{1.5}$$

The main aim of this article is to establish some sufficient conditions for the existence and uniqueness of pseudo almost periodic solutions of (1.5). The obtained results of this article are completely new and complement some previous studies.

The remainder of the paper is organized as follows. In Section 2, we introduce some notations, lemmas and definitions. In Section 3, we present some new sufficient conditions for the existence and uniqueness of pseudo almost periodic solutions for the Liénard-type system with delays. An example is given to illustrate the effectiveness of the obtained results in Section 4. In Section 5, we give a brief conclusion.

2. Preliminary results

In this section, we would like to recall some notation, basic definitions and lemmas which are used in what follows. For convenience, we denote by $\mathbb{R}^q(\mathbb{R}=\mathbb{R}^1)$ the set of all q-dimensional real vectors (real numbers). Let

$$\{x_i(t)\}=(x_1(t),x_2(t),\dots,x_n(t))\in\mathbb{R}^n.$$

For any $x(t)=\{x_i(t)\}\in\mathbb{R}^n$, we let $|x|$ denote the absolute value vector given by $|x|=\{|x_i(t)|\}$ and define $\|x(t)\|=\max_{1\le i\le n}\{|x_i(t)|\}$. A matrix or vector $U\ge 0$ means that all entries of U are greater than or equal to zero. $U>0$ can be defined similarly. For matrices or vectors U and V, $U \geq V$ (resp. $U > V$) means that $U - V \geq 0$ (resp. $U - V > 0$). Throughout this paper, we use the notation

$$v^+ = \sup_{t\in\mathbb{R}} |v(t)|, \quad v^- = \inf_{t\in\mathbb{R}} |v(t)|,$$

where $v(t)$ is a bounded continuous function. Let $\mathrm{BC}(\mathbb{R},\mathbb{R})$ denote the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}$, and $\mathrm{BUC}(\mathbb{R},\mathbb{R})$ be the set of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Obviously, $(\mathrm{BC}(\mathbb{R},\mathbb{R}), \|\cdot\|)$ is a Banach space where $\|\cdot\|$ denotes the sup norm $\|v\|_\infty := \sup_{t\in\mathbb{R}} \|v(t)\|$.

Definition 2.1 ([4, 11])**.** Let $v(t) \in \mathrm{BC}(\mathbb{R},\mathbb{R}), v(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(u,\varepsilon) = \{\varrho : \|v(t+\varrho) - v(t)\| < \varepsilon \text{ for all } t \in \mathbb{R}\}$ is relatively dense; that is, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$; for any interval with length $l(\varepsilon)$, there exists a number $\varrho = \varrho(\varepsilon)$ in this interval such that $\|v(t+\varrho) - v(t)\| < \varepsilon$, for all $t \in \mathbb{R}$.

We denote by $\mathrm{AP}(\mathbb{R},\mathbb{R})$ the set of the almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$. Besides, the concept of pseudo almost periodicity (PAP) was introduced by Zhang [11] in the early nineties. It is a natural generalization of the classical almost periodicity. Precisely, define the class of functions $\mathrm{PAP}_0(\mathbb{R},\mathbb{R})$ as follows:

$$\big\{v \in \mathrm{BC}(\mathbb{R},\mathbb{R}) : \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |v(t)|dt = 0\big\}.$$

A function $v \in \mathrm{BC}(\mathbb{R},\mathbb{R})$ is called pseudo almost periodic if it can be expressed as $v = v_1 + v_2$, where $v_1 \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $v_2 \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R})$. The collection of such functions will be denoted by $\mathrm{PAP}(\mathbb{R},\mathbb{R})$. The functions v_1 and v_2 in the above definition are, respectively, called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function v. The decomposition given in definition above is unique. Observe that $(\mathrm{PAP}(\mathbb{R},\mathbb{R}), \|\cdot\|)$ is a Banach space and $\mathrm{AP}(\mathbb{R},\mathbb{R})$ is a proper subspace of $(\mathrm{PAP}(\mathbb{R},\mathbb{R})$ since the function $v_2(t) = \sin^2 t + \sin^4 \sqrt{11}t + \exp(-t^6 \sin^4 t)$ is pseudo almost periodic function but not almost periodic.

Definition 2.2 ([2, 12])**.** *Let $x \in \mathbb{R}^n$ and $Q(t)$ be a $n\times n$ continuous matrix defined on $\mathbb{R}$. The linear system*

$$\dot{x}(t) = Q(t)x(t) \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α and projection P and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$\begin{gathered}\|X(t)PX^{-1}(s)\| \leq ke^{-\alpha(t-s)}, \quad \text{for } t \geq s,\\ \|X(t)(I-P)X^{-1}(s)\| \leq ke^{-\alpha(s-t)}, \quad \text{for } t \leq s,\end{gathered}$$

where I is the identity matrix.

Lemma 2.3 ([6])**.** *Suppose that $Q(t)$ is an almost periodic matrix function and $g(t) \in \mathrm{PAP}(\mathbb{R},\mathbb{R}^p)$. If the linear system (2.1) admits an exponential dichotomy, then pseudo almost periodic system*

$$\dot{x}(t) = Q(t)x(t) + g(t) \tag{2.2}$$

has a unique pseudo almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)ds. \tag{2.3}$$

Lemma 2.4 ([6]). *Let $Q(t) = (q_{ij})_{n\times n}$ be an almost matrix defined on $\mathbb{R}$, and there exists a positive constant ς such that*

$$|q_{ii}| - \sum_{j=1,j\neq i} |q_{ij}| \geq \varsigma, \quad \forall t \in \mathbb{R}, i = 1, 2, \dots, n.$$

Then the linear system (2.1) *admits an exponential dichotomy on $\mathbb{R}$.*

Let

$$\begin{gathered}
\varrho = \max\big\{\frac{\sup_{t\in\mathbb{R}} |\theta_1(t)|}{a^-}, \frac{\sup_{t\in\mathbb{R}} |\theta_2(t)|}{a^-}\big\}, \\
\kappa = \max\big\{\frac{1}{a^-}, \frac{\sup_{t\in\mathbb{R}}[a^2(t) + G(1 + a(t) + \theta_1(t)) + \sum_{l=0}^m H_l]}{a^-}\big\}, \\
\delta = \max\Big\{\frac{1}{a^-}, \frac{\sup_{t\in\mathbb{R}}\big[a^2(s) + G|a(t)| + G|\theta_1(t)| + 2G + \sum_{l=0}^m H_l\big]}{a^-}\Big\}
\end{gathered} \tag{2.4}$$

and

$$\Omega^* = \Big\{\varphi \| \|\varphi - \varphi_0\|_\infty \leq \frac{\kappa\varrho}{1-\kappa}, \varphi = (\varphi_1, \varphi_2)^T \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^2) \cap \mathrm{BUC}(\mathbb{R}, \mathbb{R}^2)\Big\},$$

where

$$\varphi_0 = \Big(\int_{-\infty}^t e^{-\int_s^t a(\theta)d\theta}\theta_1(s)ds, \int_t^{+\infty} e^{-\int_t^s a(\theta)d\theta}|\theta_2(s)|ds\Big)^T.$$

Lemma 2.5. *Ω^* is a closed subset $\mathrm{PAP}(\mathbb{R}, \mathbb{R}^2)$.*

The proof of the above lemma is similar to that of [10, Lemma 2.1]. Here we omit it.

Throughout this paper, we make the following assumptions for system (1.5):

- (H1) For $l = 0, 1, 2, \dots, m$, there exist nonnegative constants $H_0, H_1, H_2, \dots, H_m$ such that
$$|h_l(x) - h_l(y)| \leq H_l|x - y| \quad \text{for all } x, y \in \mathbb{R}, h_l(0) = 0.$$
- (H2) There exists a positive constant G such that $g(x) \leq G$ for all $x \in \mathbb{R}$.
- (H3) $a(t), \theta_1(t), \theta_2(t), \sigma_l(t) \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}), a(t) > 0$, for all $t \in \mathbb{R}$, where $i = 1, 2, \dots, m$.
- (H4) $\kappa < 1, \frac{\varrho}{1-\kappa} < 1, \delta < 1$.

3. Existence and uniqueness of pseudo almost periodic solutions

In this section, we establish sufficient conditions on the existence and uniqueness of pseudo almost periodic solutions of (1.5).

Theorem 3.1. *Suppose that* (H1)–(H4) *hold. Then there exists a unique pseudo almost periodic solution of system* (1.5) *in the region Ω^*.*

Proof. Denote $x(t) = x(t; t_0, \varphi)$. Let $\varphi = (\varphi_1, \varphi_2)^T \in \Omega$ and $f(t, \mu) = \varphi_j(t - \mu)$. In view of [11, Theorem 5.3 p.58], and [9, Definition 5.7 p.59], we can conclude that the uniform continuity of φ_2 implies that $f \in \mathrm{PAP}(\mathbb{R} \times \Omega)$ and f is continuous in $\mu \in L$ and uniformly in $t \in \mathbb{R}$ for all compact subset L of $\Omega \subset \mathbb{R}$. According to $\sigma_l \in \mathrm{PAP}(\mathbb{R}, \mathbb{R})(l = 1, 2, \dots, m)$ and [11, Theorem 5.11], we know that $\varphi_1(t - \sigma_l(t)) \in \mathrm{PAP}(\mathbb{R}, \mathbb{R})$. It follows from [11, Corollary 5.4] and the composition theorem of pseudo almost periodic functions that

$$\varphi_2(t) + \theta_1(t) \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}),$$

$$
\begin{aligned}
&- a^2(t)\varphi_1(t) - g(\varphi_1(t))[\varphi_2(t) - a(t)\varphi_1(t) + \theta_1(t)] \\
&- h_0(\varphi_1(t)) - \sum_{l=1}^{m} h_l(\varphi_1(t - \sigma_l(t))) \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}).
\end{aligned}
$$

Now we consider the auxiliary system

$$
\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -a(t) & 0 \\ 0 & a(t) \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix}, \tag{3.1}
$$

where

$$
\begin{gathered}
\gamma_1(t) = \varphi_2(t) + \theta_1(t) \\
\gamma_2(t) = -a^2(t)\varphi_1(t) - g(\varphi_1(t))[\varphi_2(t) - a(t)\varphi_1(t) + \theta_1(t)] \\
- h_0(\varphi_1(t)) - \sum_{l=1}^{m} h_l(\varphi_1(t - \sigma_l(t))) + \theta_2(t).
\end{gathered}
$$

Then it follows from Lemma 2.4 that the following system

$$
\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -a(t) & 0 \\ 0 & a(t) \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{3.2}
$$

admits an exponential dichotomy on $\mathbb{R}$. Define a projection P as follows

$$
P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
$$

Applying Lemma 2.3, we can conclude that system (3.1) has exactly one pseudo almost periodic solution which takes the form

$$
\begin{bmatrix} x_\varphi(t) \\ y_\varphi(t) \end{bmatrix} = \begin{bmatrix} \int_{-\infty}^{t} e^{-\int_s^t a(\theta)d\theta} \gamma_1(s) ds \\ -\int_t^{+\infty} e^{-\int_t^s a(\theta)d\theta} \gamma_2(s) ds \end{bmatrix},
$$

where

$$
\begin{gathered}
\gamma_1(s) = \varphi_2(s) + \theta_1(s), \\
\gamma_2(s) = -a^2(s)\varphi_1(s) - g(\varphi_1(s))[\varphi_2(s) - a(s)\varphi_1(s) + \theta_1(s)] \\
- h_0(\varphi_1(s)) - \sum_{l=1}^{m} h_l(\varphi_1(s - \sigma_l(s))) + \theta_2(s).
\end{gathered}
$$

Define a mapping $\Gamma : \Omega^* \to \mathrm{PAP}(\mathbb{R}, \mathbb{R}^2)$ as follows

$$
(\Gamma\varphi)(t) = \begin{bmatrix} x_\varphi(t) \\ y_\varphi(t) \end{bmatrix}, \quad \forall \varphi \in \Omega^*.
$$

In view of the definition of the norm in Banach space $\mathrm{PAP}(\mathbb{R}, \mathbb{R}^2)$, we have

$$
\begin{aligned}
\|\varphi_0\| &\le \sup_{t\in\mathbb{R}} \max\Big\{ \int_{-\infty}^{t} e^{-\int_s^t a(\theta)d\theta} |\theta_1(s)| ds, \int_t^{+\infty} e^{-\int_t^s a(\theta)d\theta} |\theta_2(s)| ds \Big\} \\
&\le \max\Big\{ \frac{\sup_{t\in\mathbb{R}} |\theta_1(t)|}{a^-}, \frac{\sup_{t\in\mathbb{R}} |\theta_2(t)|}{a^-} \Big\} = \varrho.
\end{aligned} \tag{3.3}
$$

Thus

$$
\|\varphi\|_\infty \le \|\varphi - \varphi_0\| + \|\varphi_0\|_\infty \le \frac{\kappa\varrho}{1-\kappa} + \varrho = \frac{\varrho}{1-\kappa} < 1. \tag{3.4}
$$

By (3.4), we have

$$\begin{aligned}
&\|\Gamma\varphi-\varphi_0\|_\infty \\
&=\sup_{t\in\mathbb{R}}\max\Big\{\Big|\int_{-\infty}^{t} e^{-\int_s^t a(\theta)d\theta}|\varphi_2(s)|ds\Big|,\Big|\int_t^{+\infty} e^{-\int_t^s a(\theta)d\theta}|\gamma_2^*(s)|ds\Big|\Big\} \\
&\le \max\Big\{\frac{1}{a^-},\frac{\sup_{t\in\mathbb{R}}[a^2(t)+G(1+a(t)+\theta_1(t))+\sum_{l=0}^m H_l]}{a^-}\Big\}\|\varphi\|_\infty \\
&=\kappa\|\varphi\|_\infty\le\frac{\kappa\varrho}{1-\kappa},
\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}
\gamma_2^*(s)=&-a^2(s)\varphi_1(s)-g(\varphi_1(s))[\varphi_2(s)-a(s)\varphi_1(s)+\theta_1(s)] \\
&-h_0(\varphi_1(s))-\sum_{l=1}^m h_l(\varphi_1(s-\sigma_l(s))).
\end{aligned}$$

Then

$$\|\Gamma\varphi\|_\infty\le\|\Gamma\varphi-\varphi_0\|+\|\varphi_0\|_\infty\le\frac{\kappa\varrho}{1-\kappa}+\varrho=\frac{\varrho}{1-\kappa}<1. \tag{3.6}$$

In view of (3.1), we know that $(\Gamma(\varphi)(t))'$ is bounded on $\mathbb{R}$. Thus we can conclude that $\Gamma(\varphi)(t)$ is uniformly continuous on $\mathbb{R}$, and $\Gamma\varphi\in\Omega^*$ which implies that T is a self-mapping from Ω^* to Ω^*.

Now we prove that Γ is a contraction mapping. According to (2.4), we obtain

$$\begin{aligned}
&\|(\Gamma\varphi)(t)-(\Gamma\psi)(t)\| \\
&=(|((\Gamma\varphi)(t)-(\Gamma\psi)(t))_1|,((\Gamma\varphi)(t)-(\Gamma\psi)(t))_2|)^T \\
&\le\Big(\Big|\int_{-\infty}^{t} e^{-\int_s^t a(\theta)d\theta}|\varphi_2(s)-\psi_2(s)|ds\Big|,\Big|\int_t^{+\infty} e^{-\int_t^s a(\theta)d\theta}|\gamma_2^*(s)-\bar{\gamma}_2^*(s)|ds\Big|\Big)^T \\
&\le\Big(\int_{-\infty}^{t} e^{-\int_s^t a(\theta)d\theta}ds\sup_{t\in\mathbb{R}}|\varphi_2(s)-\psi_2(s)|,\int_t^{+\infty} e^{-\int_t^s a(\theta)d\theta}ds \\
&\quad\times\Big\{a^2(s)\sup_{t\in\mathbb{R}}|\varphi_1(s)-\psi_1(s)|+G|a(t)|\sup_{t\in\mathbb{R}}|\varphi_1(s)-\psi_1(s)| \\
&\quad+G|\theta_1(t)|\sup_{t\in\mathbb{R}}|\varphi_1(s)-\psi_1(s)|+G\sup_{t\in\mathbb{R}}|\varphi_1(s)-\psi_1(s)|+G\sup_{t\in\mathbb{R}}|\varphi_2(s)-\psi_2(s)| \\
&\quad+H_0\sup_{t\in\mathbb{R}}|\varphi_1(s)-\psi_1(s)|+\sum_{l=1}^m H_l\sup_{t\in\mathbb{R}}|\varphi_1(s-\sigma_l(s))-\psi_1(s-\sigma_l(s))|\Big\}\Big)^T \\
&\le\Big(\int_{-\infty}^{t} e^{-\int_s^t a(\theta)d\theta}ds\|\varphi(s)-\psi(s)\|_\infty,\int_t^{+\infty} e^{-\int_t^s a(\theta)d\theta}ds \\
&\quad\sup_{t\in\mathbb{R}}\Big[a^2(s)+G|a(t)|+G|\theta_1(t)|+2G+\sum_{l=0}^m H_l\Big]\|\varphi(s)-\psi(s)\|_\infty\Big)^T,
\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}
\bar{\gamma}_2^*(s)=&-a^2(s)\psi_1(s)-g(\psi_1(s))[\psi_2(s)-a(s)\psi_1(s)+\theta_1(s)] \\
&-h_0(\psi_1(s))-\sum_{l=1}^m h_l(\psi_1(s-\sigma_l(s))).
\end{aligned}$$

It follows from (3.7) that

$$
\begin{aligned}
&\|\Gamma\varphi-\Gamma\psi\|_\infty \\
&\leq \max\Big\{\frac{1}{a^-}, \frac{\sup_{t\in\mathbb{R}}\big[a^2(s)+G|a(t)|+G|\theta_1(t)|+2G+\sum_{l=0}^m H_l\big]}{a^-}\Big\}\|\varphi(s)-\psi(s)\|_\infty \\
&= \delta\|\varphi(s)-\psi(s)\|_\infty.
\end{aligned}
$$

By (H4), we can conclude that the mapping Γ is a contraction. It follows from Lemma 2.5 that the mapping Γ has a unique fixed point $\bar{z}=(\bar{x},\bar{y})^T\in\Omega^*, \Gamma\bar{z}=\bar{z}$. Thus system (1.5) has a pseudo almost periodic solution in Ω^*. The proof is complete. □

4. Example

In this section, we will give an example to support our theoretical predictions. Considering the following Liénard-type system with time-varying delays

$$
\begin{aligned}
\dot{x}(t) &= -a(t)x(t)+y(t)+\theta_1(t), \\
\dot{y}(t) &= a(t)y(t)-a^2(t)x(t)-g(x(t))[y-a(t)x+\theta_1(t)] \\
&\quad -h_0(x(t))-\sum_{l=1}^m h_l(x(t-\sigma_l(t)))+\theta_2(t).
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{gathered}
a(t)=24+2\sin t, \beta_1(t)=-2-20\cos t, \theta_2(t)=\cos\sqrt{5}t+\sin\sqrt{7}t, \\
h_l(x)=\frac{1}{2}(|x+1|-|x-1|), g(x)=\arctan x, \sigma_l(t)=\frac{l}{2}\cos^2 t,
\end{gathered}
$$

where $l=0,1,2$. Then $H=G=1, a^-=22$, $\kappa=\frac{1}{22}<1$, $\delta=\frac{1}{22}<1$, $\varrho=\frac{11}{12}<1$. Then (H1)–(H4) in Theorem 3.1 hold, thus system (4.1) has an unique positive pseudo almost periodic solution.

Conclusions. In this article, a Liénard-type system with time-varying delays is studied. With the aid of the theory of exponential dichotomies, the properties of pseudo almost periodic function, inequality analysis technique and contraction mapping principle, some new sufficient conditions to ensure the existence and uniqueness of pseudo almost periodic solutions of the model are derived. These conditions are expressed in simple algebraic formulae which are very easily checked in practice. We give an example to support the theoretical predictions. The obtained results are essentially new and complement the previously known studies.

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