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TOPOLOGICAL METHODS IN THE STUDY OF POSITIVE SOLUTIONS FOR OPERATOR EQUATIONS IN ORDERED BANACH SPACES

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Abstract. We study completely continuous maps in ordered Banach spaces having an invariant cone. We give conditions on minorants and majorants which yield the existence of at least one non-zero fixed point.

1. Introduction

In the study non-linear operators in ordered Banach spaces having an invariant cone it is often convenient to make use of minorants and majorants in order to establish the existence of non-zero fixed points (see [8, Chapters 3 and 4]). We present results which generalize those in Amann [1] and Krasnosel'skii [8], where the maps are supposed to possess more restrictive monotonicity properties. The proofs are based on the theory of topological degree in cones, more precisely, on the fixed point index. Applications to the existence of solutions to nonlinear boundary value problems are presented.

2. Main results

Let $(E, \|\cdot\|_E)$ be a real Banach space and P be a nonempty closed convex set in E. A set P is called a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$
(ii) $x \in P, -x \in P \Rightarrow x = \theta$, where θ denotes the zero element in E.

The cone P defines a linear ordering in E by $x \leq y$ if and only if $y - x \in P$. The cone P is said to be normal if there exists a constant $N > 0$ such that

$$\theta \leq x \leq y \Rightarrow \|x\| \leq N\|y\|, \quad x, y \in P.$$

If $L : E \to E$ is a bounded linear operator we define, $r(L)$, its spectral radius by

$$r(L) = \lim_{n\to+\infty} \|L^n\|^{1/n}.$$

Let $u_0 \in P$. Following [8], we say that the linear operator A is u_0-bounded below if for every non-zero $x \in P$ a natural number n and a positive number α can be found such that

$$\alpha u_0 \leq A^n x.$$

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Similarly, we say that the operator A is u_0-bounded above if for every non-zero $x \in P$ a natural number m and a positive number β can be found such that

$$A^m x \leq \beta u_0.$$

Finally, if for every $x \in P \setminus \{0\}$

$$\alpha u_0 \leq A^n x \leq \beta u_0.$$

for some n, then we call the operator A u_0-positive.

If the cone P is solid and for every non-zero x of P an n can be found such that $A^n x$ is an interior element of the cone, then the operator A is called strongly positive.

Let $v* \in P\setminus\{0\}$, furthermore, let the set $P(v*) = \{x \in P, x \geq \|x\| v*\}$. Therefore (see [9]) $P(v*)$ is a cone which allow plastering. In particular, $P(v*)$ is a normal, fully regular cone. The cone $P(v*)$ is solid (reproducing) if the cone P is solid (reproducing) and if the set $P(v*)$ contains at least one element whose norm is less than one.

For every open subset U of P (from now on, the topological notions of subsets of P refer to the relative topology of P as a topological subspace of E) and every compact map $F : \bar{U} \to P$ (F is continuous and $F(\bar{U})$ is relatively compact), which has no fixed points on ∂U, there exists an integer, $i_p(F, U)$, called the fixed point index of F on U with respect to P. This index satisfies the existence, homotopy, and excision properties of the Leray-Schauder degree.

If $r > 0$, we denote

$$P_r = \{x \in P : \|x\|_E < r\}, \quad P(v*)_r = \{x \in P(v*) : \|x\|_E < r\}.$$

Let X be a nonempty set and let Y be an ordered set. Following Amann [1], a map $g : X \to Y$ is said to be a majorant of the map $f : X \to Y$ if $f(x) \leq g(x)$ for all $x \in X$. Minorant are defined by reversing the above inequality sign.

Note that, throughout the remainder of this section the derivatives are considered to be with respect to the cone P. After these preparations we are ready for the statement of our main results.

Theorem 2.1. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, asymptotically linear minorant $g : P \to P$ such that $g'(\infty)$ is u_0-bounded below and satisfying the condition $g'(\infty)u_0 \geq \lambda_\infty u_0$ where $\lambda_\infty > 1$. Then there exists a positive number σ_∞ such that for every $\sigma \geq \sigma_\infty$, $i(f, P(v*)_\sigma) = 0$.*

Proof. Let $p \in P(v*) \setminus \{0\}$, we claim that it is sufficient to prove that there exists $\sigma_\infty > 0$ such that if $x \in P(v*)$ and $\lambda > 0$ satisfy $x = f(x) + \lambda p$, then $\|x\| < \sigma_\infty$. In fact, if the claim is true, then $i(f, P(v*)_\sigma) = 0$ for every $\sigma \geq \sigma_\infty$. Suppose that is not true, then we can find sequences $\lambda_n \in \mathbb{R}^+$ and
$x_n \in P(v*)$ such that $\|x_n\| \to \infty$ as $n \to \infty$ and

$$x_n = f(x_n) + \lambda_n p \geq g(x_n),$$

where the partial ordering is that induced by the cone P. Dividing the last inequality By $\|x_n\|$ we obtain

$$\frac{x_n}{\|x_n\|} \geq \frac{g(x_n)}{\|x_n\|} = \frac{g(x_n) - g'(\infty)x_n}{\|x_n\|} + \frac{g'(\infty)x_n}{\|x_n\|}.$$

By letting $\delta_n = \frac{x_n}{\|x_n\|}$ we have

$$\delta_n - \frac{g(x_n) - g'(\infty)x_n}{\|x_n\|} - g'(\infty)\delta_n \in P. \tag{2.1}$$

Since $g'(\infty)$ is a positive linear map (see the proof of [1, Theorem 7.3]), we obtain

$$g'(\infty)\delta_n - g'(\infty)\Big(\frac{g(x_n) - g'(\infty)x_n}{\|x_n\|}\Big) - g'(\infty)(g'(\infty)\delta_n) \in P. \tag{2.2}$$

On the other hand, since $g'(\infty)\big|P$ is completely continuous (see [1, Theorem 7.3]), we may as well assume that $g'(\infty)\delta_n \to y \in P$. Then passing to the limit in (2.2) we obtain

$$y - g'(\infty)y \in P,$$

whence

$$y \geq g'(\infty)y.$$

Next, we prove that $y \in P \setminus \{0\}$. Indeed, since $x_n \in P(v*)$, we have $\delta_n \geq v*$. Whence, it follows that $g'(\infty)\delta_n - g'(\infty)v* \in P$. Then passing to the limit we find that $y \geq g'(\infty)v*$, and therefore $y \in P \setminus \{0\}$, because $g'(\infty)$ is u_0-bounded below. The inequality obtained $y \geq g'(\infty)y$ contradicts [8, Theorem 2.17 p. 91] and this completes the proof. □

Remark 2.2. If the cone P is solid and $g'(\infty)$ is strongly positive then for every non-zero x of P an n can be found such that $g'(\infty)^n x$ is an interior element of the cone, then for a sufficiently small $\alpha > 0$ the element $g'(\infty)^n x - \alpha u_0$ also will be an element of the cone, i.e. $\alpha u_0 \leq g'(\infty)^n x$ where u_0 is an arbitrary non zero element of P, that is $g'(\infty)$ is u_0-bounded below. On the other hand, it is well known that the spectral radius of $g'(\infty)$ is an eigenvalue to a positive eigenvector, and in fact the only eigenvalue with this property.

As a consequence of this we have this result.

Theorem 2.3. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, asymptotically linear minorant $g : P \to P$ such that $g'(\infty)$ is strongly positive and satisfying the condition $r(g'(\infty)) > 1$. Then there exists a positive number σ_∞ such that for every $\sigma \geq \sigma_\infty$, $i(f, P(v*)_\sigma) = 0$.*

Proof. There exists an $u_0 \in P \setminus \{0\}$ such that $g'(\infty)u_0 = \lambda_\infty u_0$ where $\lambda_\infty = r(g'(\infty)) > 1$. Therefore, all conditions of the preceding theorem are satisfied. □

Theorem 2.4. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, asymptotically linear majorant $h : P \to P$ such that $h'(\infty)$ is u_0-bounded above and satisfying the condition $h'(\infty)u_0 \leq \lambda_\infty u_0$ where $\lambda_\infty < 1$. Let $h'(\infty)v* \in P \setminus \{0\}$, then there exists a positive number σ_∞ such that for every $\sigma \geq \sigma_\infty$, $i(f, P(v*)_\sigma) = 1$.*

Proof. We claim that it is sufficient to prove that there exists $\sigma_\infty > 0$ such that if $x \in P(v*)$ and $t \in [0, 1]$ satisfy $x = tf(x)$, then $\|x\| < \sigma_\infty$. In fact, if the claim is true, then $i(f, P(v*)_\sigma) = 1$ for every $\sigma \geq \sigma_\infty$. In fact, Assuming the contrary, then we can find sequences $\lambda_n \in [0, 1]$ and $x_n \in P(v*)$ such that $\|x_n\| \to \infty$ as $n \to \infty$ and

$$x_n = \lambda_n f(x_n) \leq h(x_n),$$

By letting $\delta_n = x_n/\|x_n\|$ we rewrite the last inequality in the form

$$h'(\infty)\Big(\frac{h(x_n) - h'(\infty)x_n}{\|x_n\|}\Big) + h'(\infty)(h'(\infty)\delta_n) - h'(\infty)\delta_n \in P. \tag{2.3}$$

Without loss of generality, it can be assumed that $h'(\infty)\delta_n \to y \in P \setminus \{0\}$ (which is different from zero by the condition $y \geq h'(\infty)v*$), then passing to the limit in (2.3), we obtain

$$h'(\infty)y - y \in P.$$

Then

$$y \leq h'(\infty)y.$$

This inequality obtained contradicts [8, Theorem 2.18]. This completes the proof. □

Remark 2.5. Under the hypotheses of the last theorem and by the solution property of the fixed point index, f has a fixed point in $P(v*)_\sigma$.

Note that the last theorem contains an essential restriction that f maps the cone $P(v*)$ into the cone $P(v*)$. In the following statement this assumption can be replaced by $f : P \to P$ where the cone P is supposed to be normal.

Theorem 2.6. *Let $f : P \to P$ be a completely continuous map. Suppose that f has a completely continuous, asymptotically linear majorant $h : P \to P$ such that $h'(\infty)$ is u_0-bounded above and satisfying the condition $h'(\infty)u_0 \leq \lambda_\infty u_0$ where $\lambda_\infty < 1$. Then there exists a positive number σ_∞ such that for every $\sigma \geq \sigma_\infty$, $i(f, P_\sigma) = 1$.*

Proof. It is sufficient to prove that there exists $\sigma_\infty > 0$ such that if $x \in P$ and $t \in [0, 1]$ satisfy $x = tf(x)$, then $\|x\| < \sigma_\infty$. Assuming the contrary, similarly, by letting $\delta_n = \frac{x_n}{\|x_n\|}$ (where $x_n \in P$, $\|x_n\| \to \infty$) we obtain

$$\frac{h(x_n) - h'(\infty)x_n}{\|x_n\|} + h'(\infty)\delta_n - \delta_n \in P. \tag{2.4}$$

Applying $h'(\infty)$ to this relation, we have

$$h'(\infty)\Big(\frac{h(x_n) - h'(\infty)x_n}{\|x_n\|}\Big) + h'(\infty)(h'(\infty)\delta_n) - h'(\infty)\delta_n \in P. \tag{2.5}$$

Without loss of generality, we can assume that $h'(\infty)\delta_n \to y \in P$, then passing to the limit in (2.5), we obtain

$$y \leq h'(\infty)y. \tag{2.6}$$

Next, we prove that $y \in P \setminus \{0\}$. Indeed, suppose that $y = 0$. By (2.4) we obtain

$$0 \leq \delta_n \leq \frac{h(x_n) - h'(\infty)x_n}{\|x_n\|} + h'(\infty)\delta_n.$$

Since P is normal we obtain

$$1 \leq N\Big\|\frac{h(x_n) - h'(\infty)x_n}{\|x_n\|} + h'(\infty)\delta_n\Big\|$$

where N is the normal constant of P. From the limits

$$\frac{h(x_n) - h'(\infty)x_n}{\|x_n\|} + h'(\infty)\delta_n \to 0 \quad \text{as} \quad n \to \infty,$$

we obtain a contradiction. The inequality (2.6) obtained contradicts [8, Theorem 2.18]. This completes the proof. □

In what follows, we shall use the following theorem of Krein and Rutman [10, Theorem 6.2].

Theorem 2.7. *Let A be a completely continuous linear operator satisfying the following two conditions:*

(1) $A(P) \subset P$,
(2) *there exists an element $u \in P$, $\|u\| = 1$, a scalar $c > 0$, and a natural number p such that $A^p u \geq cu$.*

Then A has nonzero eigenvalues; among those of maximal modulus there is a positive one not less than $c^{\frac{1}{p}}$, to which corresponds a characteristic vector $v \in P$ of the operator A:

$$Av = \rho v \quad (\rho \geq c^{\frac{1}{p}}, v \in P, v \neq 0).$$

This theorem is also found in Krasnoselskii's book [8, p.67]. It should be noted that the requirement $\|u\| = 1$ in the above theorem is not essential. Now, we can prove the following result.

Theorem 2.8. *Let (E, P) be an ordered Banach space with normal cone, and let $f : P \to P$ be a completely continuous map. Suppose that f has a completely continuous, asymptotically linear majorant $h : P \to P$, such that $h'(\infty)$ does not possess a positive eigenvector to an eigenvalue greater than or equal to 1. Then f has a fixed point.*

Proof. The proof is carried out analogously to the proof of the preceding theorem with the following modifications: from (2.6), we obtain by using Theorem 2.6 that $h'(\infty)$ has a positive eigenvector to an eigenvalue greater than or equal to 1, which contradicts the hypothesis of the theorem. □

Remark 2.9. The last theorem is a generalization of [1, Theorems 7.4, 13.7] by Amann, and it is a generalization of [8, Theorem 4.9'] by Krasnosel'skii, where the maps are supposed to be monotone, which is a restrictive hypothesis. And also it is a generalization of [8, Theorems 4.8] where the norm is also supposed to be monotone.

Remark 2.10. Suppose, in addition, that P has nonempty interior and that $h'(\infty)$ is strongly positive. Then the conditions of the above theorem can be replaced by $r(h'(\infty)) < 1$.

Remark 2.11. We observe that the statement of the previous theorems remains valid if the map f is supposed to have a minorant or a majorant only on the set $\Xi = \{x \in P(v*), \|x\| \geq R_0\}$, where R_0 is a large number.

Analogously, in the determination of the values of the fixed point index of the map f on the set P_r, where r is a small number, it is convenient to make use of the right derivative at zero. Corresponding statements follows from the same arguments which were applied above. Therefore, we shall not repeat these arguments. We observe only that instead of subsequences of elements the norm of which increases indefinitely subsequences converging to zero must be considered.

Theorem 2.12. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, right differentiable at zero minorant $g : P \to P$ such that $g(0) = 0$, $g'_+(0)$ is u_0-bounded below and satisfying the condition $g'_+(0)u_0 \geq \lambda_0 u_0$ where $\lambda_0 > 1$. Then there exists a positive number σ_0 such that for every $0 < \sigma \leq \sigma_0$, $i(f, P(v*)_\sigma) = 0$.*

As a consequence of the last theorem if the cone P is solid, we have the following statement.

Theorem 2.13. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, right differentiable at zero minorant $g : P \to P$ such that $g(0) = 0$, $g'_+(0)$ is strongly positive and satisfying the condition $r(g'_+(0)) > 1$. Then there exists a positive number σ_0 such that for every $0 < \sigma \le \sigma_0$, $i(f, P(v*)_\sigma) = 0$.*

The following statement is analogous to Theorem 2.4.

Theorem 2.14. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, right differentiable at zero majorant $h : P \to P$ such that $h(0) = 0$, $h'_+(0)$ is u_0-bounded above and satisfying the condition $h'_+(0)u_0 \le \lambda_0 u_0$ where $\lambda_0 < 1$, Let $h'_+(0)v* \in P \setminus \{0\}$, then there exists a positive number σ_0 such that for every $0 < \sigma \le \sigma_0$, $i(f, P(v*)_\sigma) = 1$.*

Remark 2.15. Note that the statement of the last three theorems remains valid if the map f is supposed to have a minorant or a majorant only on the set $\Xi = \{x \in P(v*), \|x\| \le r_0\}$, where r_0 is a small number.

We assume that $f(\theta) = \theta$ and raise the question concerning the existence in the cone of other (different from θ) fixed points for the positive map f. In this connection, we first mention some conditions following from Theorems 2.1 and 2.14.

Theorem 2.16. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, asymptotically linear minorant $g : P \to P$ such that $g'(\infty)$ is u_0-bounded below and satisfying the condition $g'(\infty)u_0 \ge \lambda_\infty u_0$ where $\lambda_\infty > 1$. Suppose that f has a completely continuous, right differentiable at zero majorant $h : P \to P$ such that $h(0) = 0$, $h'_+(0)$ is u_0-bounded above and satisfying the condition $h'_+(0)u_0 \le \lambda_0 u_0$ where $\lambda_0 < 1$. Let $h'_+(0)v* \in P \setminus \{0\}$, then f has at least one positive fixed point $x \in P(v*) \setminus \{0\}$.*

Proof. Theorems 2.1 and 2.14, imply the existence of a real numbers σ_0, σ_∞ with $0 < \sigma_0 < \sigma_\infty$ such that $i(f, P(v*)_{\sigma_\infty}) = 0$ and $i(f, P(v*)_{\sigma_0}) = 1$, hence by the additivity property $i(f, P(v*)_{\sigma_\infty} \setminus \overline{P(v*)_{\sigma_0}}) = -1$. Consequently, the solution property of the fixed point index implies the existence of at least one fixed point x with $\sigma_0 < \|x\|_E < \sigma_\infty$. □

Remark 2.17. We remark that under the hypotheses of the last theorem, if it is assumed that P has nonempty interior and that $v* \in \mathring{P}$, then it follows from the fact that the fixed point obtained $x \in P(v*)$ that it belongs to the interior of the cone P ($x \in \mathring{P}$).

Similarly, from Theorems 2.4 and 2.12 we have the following result.

Theorem 2.18. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, asymptotically linear majorant $h : P \to P$ such that $h'(\infty)$ is u_0-bounded above and satisfying the condition $h'(\infty)u_0 \le \lambda_\infty u_0$ where $\lambda_\infty < 1$. Suppose that f has a completely continuous, right differentiable at zero minorant $g : P \to P$ such that $g(0) = 0$, $g'_+(0)$ is u_0-bounded below and satisfying the condition $g'_+(0)u_0 \ge \lambda_0 u_0$ where $\lambda_0 > 1$. Let $h'(\infty)v* \in P \setminus \{0\}$, then f has at least one positive fixed point $x \in P(v*) \setminus \{0\}$.*

If P is solid and normal, then Theorems 2.8 and 2.13 imply the following statement.

Theorem 2.19. *Let $f : P(v*) \to P(v*)$ be a completely continuous map. Suppose that f has a completely continuous, right differentiable at zero minorant $g : P \to P$ such that $g(0) = 0$, $g'_+(0)$ is strongly positive and satisfying the condition $r(g'_+(0)) > 1$. Suppose that f has a completely continuous, asymptotically linear majorant $h : P \to P$, such that $h'(\infty)$ does not possess a positive eigenvector to an eigenvalue greater than or equal to 1. Then f has at least one positive fixed point $x \in P \setminus \{0\}$.*

Remark 2.20. It is not difficult to prove another statements following from the previous results concerning the existence of a non-trivial solution $x \in P$.

3. Applications to nonlinear differential equations

Consider the second-order two-point boundary problem for the ordinary differential equation

$$\begin{gathered} -x" = f(t,x), \quad 0 \le t \le 1, \\ x(0) = x(1) = 0, \end{gathered} \tag{3.1}$$

where $f(t,x)$ is continuous on $0 \le t \le 1, x \ge 0$, $f(t,x) \ge 0$ for all $(t,x) \in [0,1] \times \mathbb{R}^+$ and $f(t,0) \equiv 0$. Then the boundary problem (3.1) has a trivial solution $x(t) \equiv 0$. Problems of the form (3.1) arise in many applications (see [6]), where usually the existence of positive solutions is of interest.

In this section, we are interested in producing sufficient conditions for the existence of positive solution to (3.1) in the special case when

$$g(s,y) \le f(s,y) \le h(s,y), \quad \forall (s,y) \in [0,1] \times [0,\infty),$$

where $h, g : [0,1) \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions and $h(t,0) \equiv 0$.

Let $E = C[0,1]$ and $P = \{x \in C[0,1] : x(t) \ge 0\}$. Then P is a normal cone in E. Also define the norm

$$\|x\| = \max_{0 \le t \le 1} |x(t)|.$$

It is well-known that $x \in C^2[0,1]$ is a solution of (3.1) if and only if $x \in C[0,1]$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s) f(s,x(s)) ds,$$

where $G(t,s)$ is the Green function of the differential operator $-x"$ with the boundary condition $x(0) = x(1) = 0$; that is,

$$G(t,s) = \begin{cases} t(1-s), & t \le s \\ s(1-t), & t \ge s. \end{cases}$$

Now, Consider the operator

$$Fx(t) = \int_0^1 G(t,s) f(s,x(s)) ds.$$

Then equation (3.1) has a continuous, non-negative and non-trivial solution if and only if there exists $x \in P \setminus \{0\}$ such that $x = Fx$. Define the operators:

$$Gx(t) = \int_0^1 G(t,s) g(s,x(s)) ds, \quad Hx(t) = \int_0^1 G(t,s) h(s,x(s)) ds.$$

Obviously, $Gx(t) \le Fx(t) \le Hx(t)$ for every $t \in [0,1]$ and $x \in C[0,1]$. The following theorem gives sufficient condition so that (3.1) has a solution in $P \setminus \{0\}$.

Theorem 3.1. *Suppose that*

(A1) *there exists a continuous function* $a : [0,1] \to \mathbb{R} \quad (a(s) \neq 0)$ *such that*
$$\lim_{y\to 0^+} \frac{h(t,y)}{y} = a(t), \quad \textit{uniformly for } \ t \in [0,1]$$
where
$$a(t) \le m < \frac{48}{5}, \quad \textit{for all } t \in [0,1] \tag{3.2}$$

(A2) *there exists a continuous function* $b : [0,1] \to \mathbb{R}$ *such that*
$$\lim_{y\to +\infty} \frac{g(t,y)}{y} = b(t), \quad \textit{uniformly for } t \in [0,1]$$
where
$$b(t) \ge M > 12, \quad \textit{for all } t \in [0,1] \tag{3.3}$$

Then equation (3.1) *has a non-trivial solution* $x \in C^2[0,1]$ *satisfying* $x(t) \ge \|x\| t(1-t)$ *for all* $0 \le t \le 1$.

Proof. We are going to prove that all conditions of Theorem 2.16 are satisfied. For it, we must observe that $F : P \to P$ is a completely continuous map. Moreover:

(a) To prove that $F(P) \subset P(u_0)$, where $(u_0(t) = t(1-t))$, it suffices to observe that (which is not difficult to prove) for every $t, \tau, s \in [0,1]$, the inequality
$$G(t,s) \ge u_0(t) G(\tau, s) \tag{3.4}$$
is valid. In fact, if $x(t) \ge 0, t, \tau \in [0,1]$, then from (3.4) we have
$$\begin{aligned} Fx(t) &= \int_0^1 G(t,s) f(s, x(s)) ds \\ &\ge \int_0^1 u_0(t) G(\tau, s) f(s, x(s)) ds \\ &= u_0(t) Fx(\tau). \end{aligned}$$
for any $t, \tau \in [0,1]$. from which it follows that $Fx(t) \ge \|F(x)\| u_0(t)$, $(t \in [0,1])$, and therefore $F(P) \subset P(u_0)$.

(b) By using (A2), we shall prove the existence of $G'(\infty)$ the derivative of G along P at infinity. In fact, we are going to see that for all $x \in P$.
$$G'(\infty) x(t) = \int_0^1 G(t,s) b(s) x(s) ds;$$
that is,
$$\lim_{\substack{\|x\| \to \infty \\ x \in P}} \frac{G(x) - G'(\infty) x}{\|x\|} = 0.$$
For it, we must prove that for all $\varepsilon \in \mathbb{R}^+$ there exists $K_1(\varepsilon) \in \mathbb{R}^+$ such that
$$\|x\| \ge K_1(\varepsilon)(x \in P) \Rightarrow \frac{\|G(x) - G'(\infty)x\|}{\|x\|} \le \varepsilon.$$
Let $\varepsilon > 0$, then from (A2) there is $K(\varepsilon) \in \mathbb{R}^+$ such that
$$|g(s,y) - b(s)y| \le \varepsilon |y|, \quad \forall s \in [0,1] \quad \forall y \in \mathbb{R} : y \ge K(\varepsilon).$$

Let,

$$M(\varepsilon) = \sup\Big\{|g(s,y) - b(s)y|, s \in [0,1], y \in [0, K(\varepsilon]\Big\} + 1$$

and $K_1(\varepsilon) = \frac{M(\varepsilon)}{\varepsilon}$.
Then if $x \in P$ satisfies $\|x\| \geq K_1(\varepsilon)$, we obtain

$$|G(x)(t) - G'(\infty)x(t)| \leq \int_0^1 G(t,s)|g(s,x(s)) - b(s)x(s)|ds.$$

Take

$$B^1 = \{s \in [0,1] : x(s) \geq K(\varepsilon)\} \quad \text{and} \quad B^2 = \{s \in [0,1] : x(s) < K(\varepsilon)\},$$

then

$$\begin{aligned}
|G(x)(t) - G'(\infty)x(t)| &\leq \int_{[0,1]\cap B^1} |g(s,x(s)) - b(s)x(s)|ds \\
&\quad + \int_{[0,1]\cap B^2} |g(s,x(s)) - b(s)x(s)|ds \\
&\leq \int_0^1 \varepsilon|x(s)|ds + \int_0^1 M(\varepsilon)ds \\
&\leq \varepsilon\|x\| + M(\varepsilon) \\
&\leq \varepsilon\|x\| + K_1(\varepsilon)\varepsilon \\
&\leq 2\varepsilon\|x\|, \quad \forall t \in [0,1].
\end{aligned}$$

Therefore

$$\|G(x) - G'(\infty)x\| \leq 2\varepsilon\|x\|, \quad \forall x \in P; \|x\| \geq K_1(\varepsilon).$$

(c) Analogously, one may verify, by using (A1) that there exists $H'_+(0)$ the right derivative of H along P at 0, and that

$$H'_+(0)(x)(t) = \int_0^1 G(t,s)a(s)x(s)ds$$

(d) Now, let $y(t) \geq 0$ and $y(t) \neq 0$, then from [8, Lemma 7.6, p.283], positive numbers α and β can be found such that

$$\alpha t(1-t) \leq \int_0^1 G(t,s)y(s)ds \leq \beta t(1-t),$$

then by (3.2) and (3.3), positive numbers α' and β' can be found such that

$$\begin{aligned}
&\alpha' t(1-t) \leq G'(\infty)y(t), \\
&H'_+(0)y(t) \leq \beta' t(1-t),
\end{aligned}$$

that is $H'_+(0)$ is u_0-bounded above, and $G'(\infty)$ is u_0- bounded below.

(e) Direct calculation shows that

$$\begin{aligned}
\int_0^1 G(t,s)s(1-s)ds &= (1-t)\int_0^t s^2(1-s)ds + t\int_t^1 s(1-s)^2 ds \\
&= (1-t)\int_0^t s^2(1-s)ds + t\int_0^{1-t} s^2(1-s)ds \\
&= (1-t)\big(\frac{t^3}{3} - \frac{t^4}{4}\big) + t\big(\frac{(1-t)^3}{3} - \frac{(1-t)^4}{4}\big)
\end{aligned}$$

$$= t(1-t)\Big(\frac{-t^2+t+1}{12}\Big).$$

We consider the function $\phi(t) = -t^2 + t + 1$. Obviously, $\phi'(t) = -2t + 1 \geq 0$ for $0 \leq t \leq \frac{1}{2}$ and $\phi'(t) \leq 0$ for $\frac{1}{2} \leq t \leq 1$. Therefore, the inequalities $\phi(1) = \phi(0) = 1 \leq \phi(t) \leq \phi(\frac{1}{2}) = \frac{5}{4}$ hold. From this, and by virtue of (3.2) and (3.3) we obtain that the operators $H'_+(0)$ and $G'(\infty)$ satisfy the condition

$$G'(\infty)u_0 \geq \lambda'_\infty u_0,$$
$$H'_+(0)u_0 \leq \lambda'_0 u_0$$

where $\lambda'_\infty = M/12 > 1$ and $\lambda'_0 = 5m/48 < 1$.

Finally, from (a), (b), (c), (d), and Theorem 2.16, we obtain the existence of a point $x \in P(u_0) \setminus \{0\}$ satisfying $Fx = x$. This completes the proof of the theorem. □

4. Applications to nonlinear integral equations

In this section we shall study the existence of positive solutions of integral equations of the form

$$x(t) = \int_0^{\tau(t)} f(t, s, x(t-s-l))\, ds, \tag{4.1}$$

which formulate a model to explain the evolution of certain infectious diseases and it may also be considered as a growth equation for single species populations when the birth rate varies seasonally. It include, on a particular case, different equations suggested by other authors (see [3, 5, 7, 12, 13, 11]). We are interested in producing sufficient conditions for the existence of positive periodic solution to (4.1) in the special case where

$$f(t,s,y) \leq g(t,s,y), \quad \forall (t,s,y) \in \mathbb{R}\times\mathbb{R}\times[0,+\infty[,$$

under the following assumptions (H) on functions f and g :
$f, g : \mathbb{R}\times\mathbb{R}\times[0,+\infty[\to \mathbb{R}$ are continuous functions with :

- (A3) $f(t,s,0) = 0$ for all $(t,s) \in \mathbb{R}\times\mathbb{R}$,
- (A4) $f(t,s,y) \geq 0$, $g(t,s,y) \geq 0$, $f(t,s,y) \leq g(t,s,y)$ for all $(t,s,y) \in \mathbb{R}\times\mathbb{R}\times[0,+\infty[$ and there exists a positive number $w, (w > 0)$ such that $f(t+w,s,y) = f(t,s,y)$ and $g(t+w,s,y) = g(t,s,y)$, for all $(t,s,y) \in \mathbb{R}\times\mathbb{R}\times[0,+\infty[$,
- (A5) l is a nonnegative constant and $\tau : \mathbb{R} \to \mathbb{R}^+$ is a continuous and λ-periodic function ($\lambda > 0$) such that $\frac{\omega}{\lambda} = \frac{p}{q}, p, q \in \mathbb{N}$.

Denote by P the cone of nonnegative functions in the real Banach space E, of all real and continuous $q\omega-$ periodic functions defined on $\mathbb{R}$, where if $x \in E$

$$\|x\| = \max_{0\leq t\leq q\omega} |x(t)|.$$

We are interested in the existence of solution of (4.1) in $P \setminus [0]$. Define the operator $F, G : E \to E$ by

$$Fx(t) = \int_0^{\tau(t)} f(t,s,x(t-s-l))\, ds,$$
$$Gx(t) = \int_0^{\tau(t)} g(t,s,x(t-s-l))\, ds.$$

Then equation (4.1) has a continuous, nonnegative, and nontrivial $q\omega$-periodic solution if and only if there exists $x \in P \setminus [0]$ satisfying $x = Fx$. Now, we present and prove our main results.

Theorem 4.1. *Suppose that f and g satisfy assumptions (H) and:*

(A7) *there exists a continuous function $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*
$$\lim_{y \to 0^+} \frac{f(t,s,y)}{y} = a(t,s), \quad \textit{uniformly for } (t,s) \in \mathbb{R} \times \mathbb{R},$$

(A8) *there exists a continuous function $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*
$$\lim_{y \to +\infty} \frac{g(t,s,y)}{y} = b(t,s), \quad \textit{uniformly for } (t,s) \in \mathbb{R} \times \mathbb{R},$$

(A9) $\overset{\circ}{A}_t = \emptyset \quad \forall t \in \mathbb{R}$, *where* $A_t = [s \in \mathbb{R} : a(t, t-s) = 0]$.

Then if
$$r(L(\tau,a)) > 1, \quad \textit{and} \quad r(L(\tau,b)) < 1, \tag{4.2}$$
equation (4.1) has a solution in $P \setminus [0]$, where $r(L(\tau,a))$ means the spectral radius of the linear operator $L(\tau,a) : E \to E$ defined by
$$L(\tau,a)x(t) = \int_0^{\tau(t)} a(t,s)x(t-s-l)\, ds, \quad \forall x \in E,$$
(analogously for $r(L(\tau,b))$ and $L(\tau,b)$).

Proof. We must observe that (E,P) is an ordered Banach space with $\overset{\circ}{P} \neq \emptyset$ and P is normal. Also it is not difficult to see that $F, G : P \to P$ are completely continuous. Moreover:

(a) It is easy to prove (see[3, Theorem 2.1]) the existence of $G'(+\infty)$, the derivative of G along P at infinity. In fact
$$G'(+\infty)(x)(t) = L(\tau,b)x(t), \quad \forall x \in E$$

(b) Also, it's not difficult to prove the existence of $F'_+(0)$ the right derivative of F along P at 0, and
$$F'_+(0)(x)(t) = L(\tau,a)x(t), \quad \forall x \in E,$$

(c) It is easily seen (see[3, Theorem 2.1]) that $F'_+(0)$ is strongly positive. Now, from (b), (c) and (4.2) and by using [1, Lemma 13.1] there exists a number $\sigma > 0$ such that $i_P(F, P_\sigma) = 0$. On the other hand from (a) and (4.2) we can assure from the proof of Theorem 2.8 that there exists a number $R > \sigma > 0$ such that $i_P(F, P_R) = 1$. Hence by the additivity property $i(F, P_R \setminus \bar{P_\sigma}) = 1$. Consequently, the solution property of the fixed point index implies the existence of at least one fixed point x with $\sigma < \|x\|_E < R$. This completes the proof of the theorem. □

Now we present an example of Theorem 4.1 which cannot be studied from the results of [3, 5, 11].

Example 4.2. Let $h : [0, +\infty] \to \mathbb{R}^+$ be a continuous function satisfying
$$h(0) = 0 \quad h'(0) = \alpha > 0, \quad \lim_{y \to +\infty} \frac{h(y)}{y} = \beta > 0$$

and take $d : \mathbb{R} \to \mathbb{R}$ a continuous, positive and ω-periodic function ($\omega > 0$) and $l = 0$. If

$$f(t,s,y) = d(t-s)h(y)(1+\sin^2 y), \quad \forall (t,s,y) \in \mathbb{R} \times \mathbb{R} \times [0,+\infty]$$

and

$$g(t,s,y) = 2d(t-s)h(y), \quad \forall (t,s,y) \in \mathbb{R} \times \mathbb{R} \times [0,+\infty],$$

then hypotheses (A7)–(A9) of Theorem 4.1 are satisfied with $a(t,s) = \alpha d(t-s)$, and $b(t,s) = 2\beta d(t-s)$. Consequently if

$$1 < r(L(\tau,a)), \quad r(L(\tau,b)) < 1, \tag{4.3}$$

equation (4.1) has a solution in $P \setminus \{0\}$. Note that in the particular case where $d(t) \equiv d \in \mathbb{R}^+$ conditions (4.3) are satisfied if we take

$$\frac{1}{\alpha d} < \min_{t\in\mathbb{R}} \tau(t) \le \max_{t\in\mathbb{R}} \tau(t) < \frac{1}{2\beta d}.$$

Here we use that fact that (see [11] and [3]).

$$\min_{t\in\mathbb{R}} \int_0^{\tau_1(t)} \alpha(t,s)ds \le r(L(\tau_1,\alpha)) \le \max_{t\in\mathbb{R}} \int_0^{\tau_1(t)} \alpha(t,s)ds.$$

for every continuous function $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is ω-periodic in t.

Note that this example cannot be studied by [3, Theorem 2.1] because the condition (A8) is not satisfied ($\lim_{y\to+\infty} f(t,s,y)/y$ does not exist!).

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