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CAUCHY PROBLEM FOR SOME FRACTIONAL NONLINEAR ULTRA-PARABOLIC EQUATIONS

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Abstract. Blowing-up solutions to nonlocal nonlinear ultra-parabolic equations is presented. The obtained results will contribute in the development of ultra-parabolic equations and enrich the existing non-extensive literature on fractional nonlinear ultra-parabolic problems. Our method of proof relies on a suitable choice of a test function and the weak formulation approach of the sought for solutions.

1. Introduction

This article aims to extend recent results by Kerbal and Kirane [10] by considering fractional in time and space nonlinear ultra-parabolic equations instead of classical ones. Indeed, we will present a blow-up result for the nonlocal nonlinear ultra-parabolic 2-times equation

$$\mathcal{L}u := u_{t_1} + D^{\alpha}_{0|t_2}(|u|^q - |u_1|^q)) + (-\Delta)^{\beta/2}(|u|^m) = |u|^p \tag{1.1}$$

posed for $(t_1, t_2, x) \in Q = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^N$, $N \in \mathbb{N}$ and supplemented with the initial conditions

$$u(t_1, 0; x) = u_1(t_1; x), \quad u(0, t_2; x) = u_2(t_2; x). \tag{1.2}$$

Here $p > m > 1$, $p > q > 1$ are real numbers and where for $0 < \alpha < 1$ and D^α is the fractional derivative in the sense of Riemann-Liouville. Then, we extend our results to the system of two equations

$$u_{t_1} + D^{\alpha_1}_{0|t_2}(|u|^s - |u_1|^s) + (-\Delta)^{\beta_1/2}(|u|^m) = |v|^q, \tag{1.3}$$

$$v_{t_1} + D^{\alpha_2}_{0|t_2}(|v|^r - |v_1|^r) + (-\Delta)^{\beta_2/2}(|v|^n) = |u|^p, \tag{1.4}$$

posed for $(t_1, t_2, x) \in Q = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N$, $N \in \mathbb{N}$, and supplemented with the initial conditions

$$u(t_1, 0; x) = u_1(t_1; x), \quad u(0, t_2; x) = u_2(t_2; x), \tag{1.5}$$

$$v(t_1, 0; x) = v_1(t_1; x), \quad v(0, t_2; x) = v_2(t_2; x). \tag{1.6}$$

Here p, q, r, s, are positive real numbers and $0 < \alpha_1, \alpha_2 < 1$, $0 < \beta_1, \beta_2 \leq 2$.

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The nonlocal operator $D^{\alpha}_{0|t}$ is defined, for a an absolutely continuous function $f:\mathbb{R}_+\to\mathbb{R}$, by

$$(D^{\alpha}_{0|t})f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(\sigma)}{(t-\sigma)^{\alpha}}d\sigma$$

and $\Gamma(\alpha)=\int_0^{\infty}r^{\alpha-1}e^{-r}dr$ is the Euler gamma function. The fractional power of the Laplacian $(-\Delta)^{\beta/2}$ $(0<\beta\leq 2)$ stands for diffusion in media with impurities and is defined as

$$(-\Delta)^{\beta/2}v(x)=\mathcal{F}^{-1}\Big(|\xi|^{\beta}\mathcal{F}(v)(\xi)\Big)(x),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ denotes its inverse and the operator $D^{\alpha}_{0|t}$ counts for the anomalous diffusion, a recently very much studied topic in probability, physics, chemistry, biology, image processing, etc, see for instance [1, 2, 3, 4, 5, 6, 7, 8, 11, 13, 14, 16] and their references. Classical multi-time or ultra-parabolic problems have received a special interest and attention by authors due to their application in real life problems, see for example [9, 10, 12, 17, 19], while the fractional analog are in their preliminary steps.

2. Preliminaries

Here, we need the right-hand fractional derivative in the sense of Riemann-Liouville

$$(D^{\alpha}_{t|T})f(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T\frac{f(\sigma)}{(\sigma-t)^{\alpha}}d\sigma,$$

for an absolutely continuous function $f:\mathbb{R}_+\to\mathbb{R}$. Note that for a differentiable function f, we have the so-called Caputo's fractional derivative

$$D^{\alpha}_{0|t}(f-f(0))(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f'(\sigma)}{(\sigma-t)^{\alpha}}d\sigma.$$

It is shown in [16, Corollary 2, p.46] that for f,g possessing appropriate regularity, the formula of integration by parts holds true

$$\int_0^T f(t)D^{\alpha}_{0|t}g(t)dt=\int_0^T g(t)D^{\alpha}_{t|T}f(t)dt.$$

We also need some preparatory lemmas based on the function ϕ defined by

$$\phi(t)=\begin{cases}\left(1-\frac{t}{T}\right)^{\lambda}, & 0\leq t\leq T,\\ 0, & t>T,\end{cases}\tag{2.1}$$

where $\lambda\geq 2$.

Lemma 2.1. *Let ϕ be as in (2.1). We have*

$$\int_0^T D^{\alpha}_{t,T}\phi(t)dt=C_{\alpha,\lambda}T^{1-\alpha},\tag{2.2}$$

where

$$C_{\alpha,\lambda}=\frac{\lambda\Gamma(\lambda-\alpha)}{(\lambda-\alpha+1)\Gamma(\lambda-2\alpha+1)}.$$

For a proof of the above lemma, see [11, 5].

Lemma 2.2. *Let ϕ be as in* (2.1) *and $p > 1$. Then for $p < \lambda + 1$,*

$$\int_0^T \phi^{1-p}(t)|\phi^{'}(t)|^p = C_p T^{1-p},$$

where

$$C_p = \frac{\lambda^p}{1+\lambda-p}.$$

For $\lambda > \alpha p - 1$,

$$\int_0^T \phi(t)^{1-p}|D^{\alpha}_{t,T}\phi(t)|^p dt = C_{p,\alpha} T^{1-\alpha p},$$

where

$$C_{p,\alpha} = \frac{\lambda^p}{(\lambda+1-p\alpha)}\Big\{\frac{\Gamma(\lambda-\alpha)}{\Gamma(\lambda-2\alpha+1)}\Big\}^p.$$

For a proof of the above lemma, see [11, 5]. We define the regular function ψ:

$$\psi(\xi) = \begin{cases} 1, & \text{if } 0 \le \xi \le 1,\\ \text{decreasing}, & \text{if } 1 \le \xi \le 2,\\ 0, & \text{if } \xi \ge 2, \end{cases} \tag{2.3}$$

which will be used hereafter.

3. Results

Solutions to (1.1) subject to conditions (1.2) are meant in the following weak sense.

Definition 3.1. A function $u \in L^m(Q)\bigcap L^p(Q)$ is called a weak solution to (1.1) if

$$\begin{aligned} &\int_Q |u|^p \varphi\, dP + \int_S u(0,t_2;x)\varphi(0,t_2;x)\, dP_2 + \int_Q |u(t_1,0;x)|^q D^{\alpha}_{t_2|T}\varphi\, dP \\ &= -\int_Q u\varphi_{t_1}\, dP + \int_Q |u|^q\, D^{\alpha}_{t_2|T}\varphi\, dP + \int_Q |u|^m\, (-\Delta)^{\beta/2}\varphi\, dP \end{aligned} \tag{3.1}$$

for any test function $\varphi \in C_0^\infty(Q)$; $S = \mathbb{R}_+ \times \mathbb{R}^N$, $P = (t_1,t_2,x)$ and $P_2 = (t_2,x)$, such that $\varphi(T,t_2;x) = \varphi(t_1,T;x) = 0$.

Note that every weak solution is a classical solution near the points (t_1,t_2,x) where $u(t_1,t_2,x)$ is positive.

Our main result dealing with equation (1.1) subject to (1.2) is given by the following theorem.

Theorem 3.2. *Assume that*

$$\int_S u(0,t_2;x)\varphi(0,t_2;x)dP_2 > 0, \quad \int_Q |u(t_1,0;x)|^q D^{\alpha}_{t_2|T}\varphi\, dP > 0.$$

If $1 < p \le \min\big(1+\frac{1}{N+1}, q(1+\frac{\alpha}{N+2-\alpha}), m(1+\frac{\beta}{N+2-\beta})\big)$, then Problem (1.1)-(1.2) *does not admit global weak solutions.*

For the proof, we need to recall the following proposition from [8, proposition 3.3].

Proposition 3.3 ([8]). *Suppose that $\delta \in [0,2]$, $\beta+1 \geq 0$, and $\theta \in \mathcal{C}_0^\infty(R^N)$. Then, the following point-wise inequality holds:*

$$|\theta(x)|^\beta \theta(x)(-\Delta)^{\delta/2}\theta(x) \geq \frac{1}{\beta+2}(-\Delta)^{\delta/2}|\theta(x)|^{\beta+2}.$$

Proof of Theorem 3.2. Our strategy of proof is to use the weak formulation of the solution with a suitable choice of the test function (see for example [15]). We assume that the solution is nontrivial and global. We choose the test function $\varphi(t_1, t_2, x)$ in the form

$$\varphi(t_1, t_2; x) = \varphi_1(t_1)\varphi_2(t_2)\varphi_3(x) \tag{3.2}$$

where $\varphi_1(t_1) = \psi(t_1/T)$, $\varphi_2(t_2) = (1 - t_2/T)^\lambda$ and $\varphi_3(x) = \psi(|x|^2/T^2)$.

Now, replacing φ by φ^μ in (3.1), we estimate $\int_{Q_T} u\varphi^\mu_{t_1} dP$ using the ε-Young inequality as follows

$$\int_Q |u|\,|\varphi^\mu_{t_1}|\,dP \leq \varepsilon \int_Q |u|^p\varphi^\mu\,dP + C_\varepsilon \int_Q \varphi^{\mu-\frac{p}{p-1}}|\varphi_{t_1}|^{\frac{p}{p-1}}\,dP. \tag{3.3}$$

Similarly, we have

$$\int_Q |u|^q D^\alpha_{t_2|T}\varphi^\mu\,dP \leq \varepsilon \int_Q |u|^p\varphi^\mu\,dP + C_\varepsilon \int_Q |D^\alpha_{t_2|T}\varphi^\mu|^{\frac{p}{p-q}}\varphi^{-\frac{\mu q}{p-q}}\,dP, \tag{3.4}$$

where $p > q$. Observe that

$$\begin{aligned}
&\int_Q |u(t_1,0;x)|^q D^\alpha_{t_2|T}\varphi^\mu\,dP \\
&= \Big(\int_0^T D^\alpha_{t_2|T}\varphi_2^\mu(t_2)dt_2\Big)\int_S |u(t_1,0;x)|^q\varphi_3^\mu(x)\varphi_1^\mu(t_1)\,dP_1
\end{aligned} \tag{3.5}$$

with the help of Lemma 2.1 one can rewrite the equation (3.5) as

$$\int_Q |u(t_1,0;x)|^q D^\alpha_{t_2|T}\varphi^\mu\,dP = C_{\alpha,\lambda\mu}\,T^{1-\alpha}\int_S |u(t_1,0;x)|^q\varphi_3^\mu(x)\varphi_1^\mu(t_1)\,dP_1, \tag{3.6}$$

where $P_1 = (t_1, x)$. Using the convexity inequality in proposition 3.3 and the ε-Young inequality, the last term in the right hand side of equation (3.1) can be estimated by

$$\begin{aligned}
&\int_Q |u|^m\,(-\Delta)^{\beta/2}\varphi^\mu\,dP \\
&\leq \int_Q \mu\varphi^{\mu-1}|u|^m(-\Delta)^{\beta/2}\varphi\,dP \\
&\leq \varepsilon\int_Q \varphi^\mu|u|^p\,dP + C(\varepsilon)\int_Q |(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-m}}\varphi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}}\,dP.
\end{aligned} \tag{3.7}$$

Now, using (3.3), (3.4), (3.5), and (3.7), we obtain

$$\begin{aligned}
&\int_Q |u|^p\varphi^\mu\,dP + \int_S u(0,t_2;x)\varphi^\mu(0,t_2;x)\,dP_2\\
&+ C_{\alpha,\lambda\mu}T^{1-\alpha}\int_S |u(t_1,0;x)|^q\varphi_3^\mu(x)\varphi_1^\mu(t_1)\,dP_1\\
&\le 3\varepsilon\int_Q |u|^p\,\varphi^\mu\,dP + C_\varepsilon\Big(\int_{Q_T}\varphi^{\mu-\frac{p}{p-1}}|\varphi_{t_1}|^{\frac{p}{p-1}}\,dP\\
&\quad+\int_Q |D^\alpha_{t_2|T}\varphi^\mu|^{\frac{p}{p-q}}\varphi^{-\frac{\mu q}{p-q}}\,dP\\
&\quad+\int_Q |(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-m}}\varphi^{(p(\mu-1)-m\mu)\frac{1}{p-m}}\,dP\Big).
\end{aligned}\tag{3.8}$$

If we choose $\varepsilon=1/6$ (for example), then we obtain the estimate

$$\begin{aligned}
&\int_Q |u|^p\,\varphi^\mu\,dP + 2\int_S u(0,t_2;x)\varphi^\mu(0,t_2;x)\,dP_2\\
&+ C_{\alpha,\lambda\mu}T^{1-\alpha}\int_S |u(t_1,0;x)|^q\varphi_3^\mu(x)\varphi_1^\mu(t_1)\,dP_1\\
&\le C\Big(\int_Q \varphi^{\mu-\frac{p}{p-1}}|\varphi_{t_1}|^{\frac{p}{p-1}}\,dP + \int_Q |D^\alpha_{t_2|T}\varphi^\mu|^{\frac{p}{p-q}}\varphi^{-\frac{\mu q}{p-q}}\,dP\\
&\quad+\int_Q |(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-m}}\varphi^{(p(\mu-1)-m\mu)\frac{1}{p-m}}\,dP\Big)
\end{aligned}\tag{3.9}$$

for some positive constant C. The right hand side of (3.9) is now free of the unknown function u. Let us now pass to the new variables

$$\tau_1 = T^{-1}t_1,\quad \tau_2 = T^{-1}t_2,\quad y = T^{-1}x. \tag{3.10}$$

We have

$$\begin{aligned}
\int_Q \varphi^{\mu-\frac{p}{p-1}}\,|\varphi_{t_1}|^{\frac{p}{p-1}}\,dP &= \Big(\int_S \varphi_2^\mu\varphi_3^\mu dP_2\Big)\Big(\int_0^T \varphi_1^{\mu-\frac{p}{p-1}}\,|\varphi_{1,t_1}|^{\frac{p}{p-1}}dt_1\Big)\\
&= C_1T^{2+N-\frac{p}{p-1}}
\end{aligned}\tag{3.11}$$

where

$$C_1 = \Big(\int_{\Omega_2}\varphi_2^\mu\varphi_3^\mu dP_{\tau_2}\Big)\Big(\int_0^1 \psi^{\mu-\frac{p}{p-1}}|\psi_{\tau_1}|^{\frac{p}{p-1}}d\tau_1\Big)<\infty$$

with $\mu>\frac{p}{p-1}$ and $P_{\tau_2}=(\tau_2,y)$, $\Omega_2=\{1\le\tau_2+|y|\le 2\}$. Similarly, we obtain

$$\begin{aligned}
&\int_Q |D^\alpha_{t_2|T}\varphi^\mu|^{\frac{p}{p-q}}\varphi^{\frac{\mu q}{q-p}}\,dP\\
&=\Big(\int_S \varphi_1^\mu\varphi_3^\mu dP_1\Big)\Big(\int_0^T \varphi_2^{-\frac{\mu q}{p-q}}|D^\alpha_{t_2|T}\varphi_2^\mu|^{\frac{p}{p-q}}dt_2\Big)\\
&= C_2T^{2+N-\frac{\alpha p}{p-q}}
\end{aligned}\tag{3.12}$$

where

$$C_2 = \Big(\int_{\Omega_1}\varphi_1^\mu\varphi_3^\mu dP_{\tau_1}\Big)\Big(\int_0^1 \varphi_2^{-\frac{\mu q}{p-q}}|D^\alpha_{\tau_2}\varphi_2^\mu|^{\frac{p}{p-q}}d\tau_2\Big)<\infty$$

and $P_{\tau_1}=(\tau_1,y)$, $\Omega_1=\{1\le\tau_1+|y|\le 2\}$, and

$$\begin{aligned}
&\int_Q |(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-m}}\varphi^{(p(\mu-1)-m\mu)\frac{1}{p-m}}\,dP\\
&=\Big(\int_{\mathbb{R}^N}|(-\Delta)^{\beta/2}\varphi_3|^{\frac{p}{p-m}}\varphi_3^{(p(\mu-1)-m\mu)\frac{1}{p-m}}\,dx\Big)\Big(\int_{Q_T}\varphi_1^\mu\varphi_2^\mu\,dt_1\,dt_2\Big)\\
&=C_3T^{2+N-\frac{\beta p}{p-m}}
\end{aligned}\tag{3.13}$$

where

$$C_3=\int_{\text{support}\,\psi)}|(-\Delta_y)^{\beta/2}\psi|^{\frac{p}{p-m}}\psi^{(p(\mu-1)-m\mu)\frac{1}{p-m}}\,dy\int_{Q_T}\varphi_1^\mu\varphi_2^\mu\,d\tau_1\,d\tau_2<\infty$$

with $\mu>\frac{p}{p-m}$ and $Q_T=[0,T]\times[0,T]$. By (3.11)-(3.13), we obtain for (3.9) the following estimate

$$\begin{aligned}
&\int_Q|u|^p\,\varphi^\mu\,dP+2\int_S u(0,t_2;x)\varphi^\mu(0,t_2;x)\,dP_2\\
&+C_{\alpha,\lambda\mu}T^{1-\alpha}\int_S|u(t_1,0;x)|^q\varphi_3^\mu(x)\varphi_1^\mu(t_1)\,dP_1\\
&\le C_1T^{2+N-\frac{p}{p-1}}+C_2T^{2+N-\frac{\alpha p}{p-q}}+C_3T^{2+N-\frac{\beta p}{p-m}},
\end{aligned}\tag{3.14}$$

then

$$\begin{aligned}
&\int_Q|u|^p\varphi^\mu\,dP+2\int_S u(0,t_2;x)\varphi^\mu(0,t_2;x)\,dP_2\\
&+C_{\alpha,\lambda\mu}T^{1-\alpha}\int_S|u(t_1,0;x)|^q\varphi_3^\mu(x)\varphi_1^\mu(t_1)\,dP_1\\
&\le\tilde{C}\Big(T^{2+N-\frac{p}{p-1}}+T^{2+N-\frac{\alpha p}{p-q}}+T^{2+N-\frac{\beta p}{p-m}}\Big)
\end{aligned}\tag{3.15}$$

where $\tilde{C}=\max\{C_1,C_2,C_3\}$. Now, for the first case, we require:

(a) $2+N-\frac{p}{p-1}<0$ or $1<p\le 1+\frac{1}{N+1}$, for $p>q$ and $m>1$.
(b) $2+N-\frac{\alpha p}{p-q}<0$ or $1<p\le q(1+\frac{\alpha}{N+2-\alpha})$, for $p>m>1$.
(c) $2+N-\frac{\beta p}{p-m}<0$ or $1<p\le m\big(1+\frac{\beta}{N+2-\beta}\big)$.

Letting T aproach infinity in (3.15), we obtain a contradiction as the left hand side is positive while the right hand side goes to zero.

For the second case, we assume the exponents of T in (3.15) are zeros. Applying Hölder's inequality to the right hand side of inequality (3.9), we obtain

$$\begin{aligned}
&\int_Q|u|^p\,\varphi^\mu\,dP+2\int_S u(0,t_2;x)\varphi^\mu(0,t_2;x)\,dP_2\\
&+C_{\alpha,\lambda\mu}T^{1-\alpha}\int_S|u(t_1,0;x)|^q\varphi_3^\mu(x)\varphi_1^\mu(t_1)\,dP_1\\
&\le\Big(\int_{C_T}|u|^p\varphi^\mu\,dP\Big)^{1/p}C(\varphi)
\end{aligned}\tag{3.16}$$

where

$$\begin{aligned}
C(\varphi)=C\Big(&\int_Q\varphi^{-\frac{\mu}{p-1}}|\varphi_{t_1}^\mu|^{\frac{p}{p-1}}\,dP+\int_Q|D^\alpha_{t_2|T}\varphi^\mu|^{\frac{p}{p-q}}\varphi^{-\frac{\mu q}{p-q}}\,dP\\
&+\int_Q|(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-m}}\varphi^{(p(\mu-1)-m\mu)\frac{1}{p-m}}\,dP\Big).
\end{aligned}$$

Whereupon, using Lebesgue's dominated convergence theorem we have

$$\int_Q |u|^p \varphi \, dP \le \tilde{C} \implies \lim_{T\to\infty} \int_{C_T} |u|^p dP = 0,$$

where $C_T = \{(t_1, t_2, x) | \; T \le t_1 + t_2 + |x| \le 2T\}$.

Then, letting T appraoch infinity in (3.16), the right-hand side approaches zero, which is again contradiction. □

4. A 2 × 2 system with a 2-dimensional fractional time

We consider

$$u_{t_1} + D^{\alpha_1}_{0|t_2}(|u|^s - |u_1|^s) + (-\Delta)^{\beta_1/2}(|u|^m) = |v|^q, \tag{4.1}$$

$$v_{t_1} + D^{\alpha_2}_{0|t_2}(|v|^r - |v_1|^r) + (-\Delta)^{\beta_2/2}(|v|^n) = |u|^p, \tag{4.2}$$

posed for $(t_1, t_2, x) \in Q = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N$, $N \in \mathbb{N}$, and supplemented with the initial conditions

$$u(t_1, 0; x) = u_1(t_1; x), \quad u(0, t_2; x) = u_2(t_2; x), \tag{4.3}$$

$$v(t_1, 0; x) = v_1(t_1; x), \quad v(0, t_2; x) = v_2(t_2; x). \tag{4.4}$$

Here p, q, r, s, are positive real numbers and $0 < \alpha_1, \alpha_2 < 1$, $0 < \beta_1, \beta_2 \le 2$. Let us set

$$I_0 = \int_S u_2(0, t_2, x)\varphi(0, t_2, x) dP_2 + \int_Q |u_1|^s D^{\alpha_1}_{t_2|T} \varphi \, dP$$

$$J_0 = \int_S v_2(0, t_2, x)\varphi(0, t_2, x) dP_2 + \int_Q |v_1|^r D^{\alpha_2}_{t_2|T} \varphi \, dP$$

Definition 4.1. We say that $(u, v) \in (L^p \cap L^m) \times (L^q \cap L^n)$ is a weak formulation to system (4.1)-(4.2) if

$$\begin{gathered}
\int_Q |v|^q \varphi \, dP + I_0 = -\int_Q u \, \varphi_{t_1} \, dP + \int_Q |u|^s \, D^{\alpha_1}_{t_2|T}\varphi \, dP + \int_Q |u|^m \, (-\Delta)^{\beta_1/2}\varphi \, dP \\
\int_Q |u|^p \varphi \, dP + J_0 = -\int_Q v \, \varphi_{t_1} \, dP + \int_Q |v|^r \, D^{\alpha_2}_{t_2|T}\varphi \, dP + \int_Q |v|^n \, (-\Delta)^{\beta_2/2}\varphi \, dP
\end{gathered} \tag{4.5}$$

for any test function $\varphi \in C_0^\infty$. Now, set

$$\begin{gathered}
\sigma_1 = -\frac{q[1 - p(N+1)] + N + 2}{pq - 1}, \\
\sigma_2 = -\frac{q[\alpha_1 - p(N+1)] + r(N+2)}{pq - r}, \\
\sigma_3 = -\frac{q[\beta_1 - p(N+1)] + n(N+2)}{pq - n}, \\
\sigma_4 = -\frac{q[s - p(N+2-\alpha_1)] + s(N+2)}{pq - s}, \\
\sigma_5 = -\frac{q[s\alpha_2 - p(N+2-\alpha_1)] + sr(N+2)}{pq - sr}, \\
\sigma_6 = -\frac{q[s\beta_2 - p(N+2-\alpha_1)] + sn(N+2)}{pq - sn},
\end{gathered}$$

$$\sigma_7 = -\frac{q[m - p(N+2-\beta_1)] + m(N+2)}{pq - m},$$
$$\sigma_8 = -\frac{q[m\alpha_2 - p(N+2-\beta_1)] + rm(N+2)}{pq - rm}$$
$$\sigma_9 = -\frac{q[m\beta_2 - p(N+2-\beta_1)] + nm(N+2)}{pq - nm}.$$

Theorem 4.2. *Let $p > 1$, $q > 1$, $p > m$, $p > s$, $q > n$, $q > r$ and assume that*

$$\int_S u_2(0,t_2,x)\varphi^\mu(0,t_2,x)dP_2 > 0, \quad \int_Q |u_1|^s D^{\alpha_1}_{t_2|T}\varphi^\mu\, dP > 0,$$
$$\int_S v_2(0,t_2,x)\varphi^\mu(0,t_2,x)dP_2 > 0, \quad \int_Q |v_1|^r D^{\alpha_2}_{t_2|T}\varphi^\mu\, dP > 0,$$

then solutions to system (4.1)-(4.2) *blow-up whenever*

$$\max\{\sigma_1,\dots,\sigma_9;\, \delta_1,\dots,\delta_9\} \le 0.$$

Proof of theorem 4.2. Assume that the solution is nontrivial and global. Next, replacing φ by φ^μ in (4.5) and then using Hölder's inequality to estimate the RHS, we obtain the following estimates:

- For $p > 1$,
$$-\int_Q u\varphi^\mu_{t_1}\, dP \le \mu\Big(\int_Q |u|^p\varphi^\mu\, dP\Big)^{1/p}\Big(\int_Q \varphi^{\mu-\frac{p}{p-1}}\,|\varphi_{t_1}|^{\frac{p}{p-1}}\, dP\Big)^{\frac{p-1}{p}}. \tag{4.6}$$
- For $p > s$,
$$\int_Q |u|^s D^{\alpha_1}_{t_2|T}\varphi^\mu\, dP \le \Big(\int_Q |u|^p\varphi^\mu dP\Big)^{s/p}\Big(\int_Q \varphi^{-\frac{s\mu}{p-s}}\,|D^{\alpha_1}_{t_2|T}\varphi^\mu|^{\frac{p}{p-s}} dP\Big)^{\frac{p-s}{p}}. \tag{4.7}$$
- For $p > m$,
$$\int_Q |u|^m(-\Delta)^{\frac{\beta_1}{2}}\varphi^\mu \le \mu\Big(\int_Q |u|^p\varphi^\mu\Big)^{\frac{m}{p}}\Big(\int_Q \varphi^{\mu-\frac{p}{p-m}}|(-\Delta)^{\frac{\beta_1}{2}}\varphi|^{\frac{p}{p-m}}\Big)^{\frac{p-m}{p}}. \tag{4.8}$$

Similarly, we have

- For $q > 1$,
$$-\int_Q v\varphi^\mu_{t_1}\, dP \le \mu\Big(\int_Q |v|^q\varphi^\mu\, dP\Big)^{\frac{1}{q}}\Big(\int_Q \varphi^{\mu-\frac{q}{q-1}}\,|\varphi_{t_1}|^{\frac{q}{q-1}}\, dP\Big)^{\frac{q-1}{q}}. \tag{4.9}$$
- For $q > r$,
$$\int_Q |v|^r\, D^{\alpha_2}_{t_2|T}\varphi^\mu\, dP \le \Big(\int_Q |v|^q\varphi^\mu dP\Big)^{\frac{r}{q}}\Big(\int_Q \varphi^{-\frac{r\mu}{q-r}}\,|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{q}{q-r}} dP\Big)^{\frac{q-r}{q}}. \tag{4.10}$$
- For $q > n$
$$\int_Q |v|^n(-\Delta)^{\frac{\beta_2}{2}}\varphi^\mu \le \mu\Big(\int_Q |v|^q\varphi^\mu\Big)^{\frac{n}{q}}\Big(\int_Q \varphi^{\mu-\frac{q}{q-n}}|(-\Delta)^{\frac{\beta_2}{2}}\varphi|^{\frac{q}{q-n}}\Big)^{\frac{q-n}{q}}. \tag{4.11}$$

If we set

$$I_u := \int_Q |u|^p\varphi^\mu dP, \quad I_v := \int_Q |v|^q\varphi^\mu\, dP,$$
$$A(p) = \mu\Big(\int_Q \varphi^{\mu-\frac{p}{p-1}}|\varphi_{t_1}|^{\frac{p}{p-1}}\, dP\Big)^{\frac{p-1}{p}},$$

$$\begin{aligned}
A(q) &= \mu\Big(\int_Q \varphi^{\mu-\frac{q}{q-1}}|\varphi_{t_1}|^{\frac{q}{q-1}}\,dP\Big)^{\frac{q-1}{q}},\\
B(p,s) &= \Big(\int_Q \varphi^{-\frac{s\mu}{p-s}}|D^{\alpha_1}_{t_2|T}\varphi^{\mu}|^{\frac{p}{p-s}}dP\Big)^{\frac{p-s}{p}},\\
B(q,r) &= \Big(\int_Q \varphi^{-\frac{r\mu}{q-r}}|D^{\alpha_2}_{t_2|T}\varphi^{\mu}|^{\frac{q}{q-r}}dP\Big)^{\frac{q-r}{q}},\\
C(p,m) &= \mu\Big(\int_Q \varphi^{\mu-\frac{p}{p-m}}|(-\Delta)^{\frac{\beta_1}{2}}\varphi|^{\frac{p}{p-m}}dP\Big)^{\frac{p-m}{p}},\\
C(q,n) &= \mu\Big(\int_Q \varphi^{\mu-\frac{q}{q-n}}|(-\Delta)^{\frac{\beta_2}{2}}\varphi|^{\frac{q}{q-n}}\,dP\Big)^{\frac{q-n}{q}},\\
I_0^{\mu} &= \int_S u_2(0,t_2,x)\varphi^{\mu}(0,t_2,x)dP_2 + \int_Q |u_1|^s D^{\alpha_1}_{t_2|T}\varphi^{\mu}\,dP,\\
J_0^{\mu} &= \int_S v_2(0,t_2,x)\varphi^{\mu}(0,t_2,x)dP_2 + \int_Q |v_1|^r D^{\alpha_2}_{t_2|T}\varphi^{\mu}\,dP,
\end{aligned}$$

then, using estimates (4.6)-(4.11), we can write (4.5) as

$$\begin{gathered}
I_v + I_0^{\mu} \le I_u^{1/p}A(p) + I_u^{s/p}B(p,s) + I_u^{\frac{m}{p}}C(p,m),\\
I_u + J_0^{\mu} \le I_v^{\frac{1}{q}}A(q) + I_v^{\frac{r}{q}}B(q,r) + I_v^{\frac{n}{q}}C(q,n).
\end{gathered}$$

Since $I_0^{\mu}, J_0^{\mu} > 0$, we have

$$I_v \le I_u^{1/p}A(p) + I_u^{s/p}B(p,s) + I_u^{\frac{m}{p}}C(p,m), \tag{4.12}$$

$$I_u \le I_v^{\frac{1}{q}}A(q) + I_v^{\frac{r}{q}}B(q,r) + I_v^{\frac{n}{q}}C(q,n). \tag{4.13}$$

Now, from (4.12) and (4.13), we have

$$\begin{aligned}
I_v + I_0^{\mu} \le{}& \Big(I_v^{\frac{1}{pq}}A^{1/p}(q) + I_v^{\frac{r}{pq}}B^{1/p}(q,r) + I_v^{\frac{n}{pq}}C^{1/p}(q,n)\Big)A(p)\\
&+ (I_v^{\frac{s}{pq}}A^{s/p}(q) + I_v^{\frac{rs}{pq}}B^{s/p}(q,r) + I_v^{\frac{ns}{pq}}C^{s/p}(q,n))B(p,s)\\
&+ \Big(I_v^{\frac{m}{pq}}A^{\frac{m}{p}}(q) + I_v^{\frac{rm}{pq}}B^{\frac{m}{p}}(q,r) + I_v^{\frac{nm}{pq}}C^{\frac{m}{p}}(q,n)\Big)C(p,m).
\end{aligned}$$

Then Young's inequality implies

$$\begin{aligned}
I_v + I_0^{\mu} \le K\Big\{&\Big(A^{1/p}(q)A(p)\Big)^{\frac{pq}{pq-1}} + \Big(B^{1/p}(q,r)A(p)\Big)^{\frac{pq}{pq-r}}\\
&+ \Big(C^{1/p}(q,n)A(p)\Big)^{\frac{pq}{pq-n}} + \Big(A^{s/p}(q)B(p,s)\Big)^{\frac{pq}{pq-s}}\\
&+ \Big(B^{s/p}(q,r)B(p,s)\Big)^{\frac{pq}{pq-rs}} + \Big(C^{s/p}(q,n)B(p,s)\Big)^{\frac{pq}{pq-ns}}\\
&+ \Big(A^{\frac{m}{p}}(q)C(p,m)\Big)^{\frac{pq}{pq-m}} + \Big(B^{\frac{m}{p}}(q,r)C(p,m)\Big)^{\frac{pq}{pq-rm}}\\
&+ \Big(C^{\frac{m}{p}}(q,n)C(p,m)\Big)^{\frac{pq}{pq-nm}}\Big\}
\end{aligned}$$

for some positive constant K. Using the scaled variables (3.2) we obtain

$$A(p) = CT^{-1+(N+2)(1-1/p)}, \quad A(q) = CT^{-1+(N+2)(1-1/q)},$$

$$B(p,s) = CT^{-\alpha_1+(N+2)(1-s/p)}, \quad B(q,r) = CT^{-\alpha_2+(N+2)(1-r/q)},$$
$$C(p,m) = CT^{-\beta_1+(N+2)(1-m/p)}, \quad C(q,n) = CT^{-\beta_2+(N+2)(1-n/q)},$$

for some positive constant C. Hence, we obtain

$$I_v + I_0^\mu \le K\{T^{\sigma_1} + T^{\sigma_2} + \dots + T^{\sigma_9}\}. \tag{4.14}$$

Similarly, we obtain for I_u the estimate

$$I_u + J_0^\mu \le K\{T^{\delta_1} + T^{\delta_2} + \dots + T^{\delta_9}\}. \tag{4.15}$$

Finally, passing to the limit as $T \to \infty$, we observe that:

Either $\max\{\sigma_1, \dots, \sigma_9; \delta_1, \dots, \delta_9\} < 0$ and in this case, the right hand side tends to zero while the left hand side is strictly positive. Hence, we obtain a contradiction.

Or $\max\{\sigma_1, \dots, \sigma_9; \delta_1, \dots, \delta_9\} = 0$ and in this case, following the analysis similar as in one equation, we prove a contradiction. □

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