Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 175, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

POSITIVITY AND NONEXISTENCE OF SOLUTIONS FOR QUASILINEAR INEQUALITIES

XIAOHONG LI

ABSTRACT. We prove a positivity property and a nonexistence theorem for weak solutions of quasilinear differential equalities in $\mathbb{R}^N$. To obtain our results, we use a comparison principle. Also we establish a criterium for the existence of positive radial solutions.

1. Introduction

In this article, we consider the positivity nonexistence for weak solutions of the anisotropic divergence structure quasilinear differential inequality

$$L(u) = -\operatorname{div}_L(a(x)A(|\nabla_L u|)\nabla_L u) \geq h(x)f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where A and f satisfy

- (A1) $A \in C(0,\infty)$ and $a, h \in C(\mathbb{R},\mathbb{R})$;
- (A2) $f : \mathbb{R} \to \mathbb{R}$ is continuous function and nondecreasing on $(-\infty, 0]$ satisfying $f(t) > 0$ for $t < 0$.

and L belongs to a wide class of anisotropic quasilinear operator including

$$\operatorname{div}_L(|\nabla_L u|^{p-2}\nabla_L u), \quad \operatorname{div}_L\Big(\frac{\nabla_L u}{\sqrt{1+|\nabla_L u|^2}}\Big), \quad \operatorname{div}_L\Big(\frac{|\nabla_L u|^{p-2}\nabla_L u}{(1+|\nabla_L u|^s)^k}\Big),$$

with $p > 1$, $s > 0$, $k \geq 0$.

Recently, this kind of problems have received a great attention. Tools based different forms of the maximum principle like the moving planes method or moving spheres method, nonlinear capacitary estimates and Pohozaev type identities, energy methods and Harnack inequality type argument, have been proved to be very successful for solving interesting problems related to be applications and to the general theory of partial differential equations. We refer to [1]-[29] and the references therein for some recent contributions.

In the special case of (1.1), given by $a(x) = h(x) \equiv 1$, the positivity results and nonexistence theorems are of great interest. For the Euclidean gradient case of (1.1), that is $\nabla_L u = \nabla u$, D'Ambrosio and Mitidieri [9] proved the positivity results and nonexistence theorems of weak solution $W^{1,p}_{\rm loc}$ in $\mathbb{R}^N$. Both in [6] and [8], the authors obtained the positivity results of solution C^1 solution of (1.1) in

2010 *Mathematics Subject Classification.* 35J60, 35J70.
Key words and phrases. Quasilinear inequality; positivity property; nonexistence theorem; comparison principle.

Submitted October 30, 2015. Published July 6, 2016.

the Heisenberg setting and Carnot Groups. An interesting discussion on the nonexistence of solutions of (1.1) for anisotropic case in $\mathbb{R}^N$ is given by L. D'Ambrosio in [7], who used the test function method developed by Mitidieri and Pohozaev (see [10]–[12]). In particular, both in [2] and [5], the authors proved positivity results by integral representation formulae, when L is the Laplacian operator or the polyharmonic operator in the Euclidean setting or, more generally L is a sub elliptic Laplacian on a Carnot group and f is nonnegative.

Motivated by the above works, in the present paper, we obtain the positivity property and the nonexistence Theorem for $W^{1,p}_{L,\mathrm{loc}}$ solutions of anisotropic quasilinear differential equality in $\mathbb{R}^N$. General results on the $W^{1,p}_{L,\mathrm{loc}}$ solutions for (1.1) are considered, which is based on a comparison principle and the analysis of an ordinary differential equation.

For the main results of the paper we shall present some preliminaries (see [4]-[9]). In this paper ∇ and $|\cdot|$ stands respectively for usual gradient in $\mathbb{R}^N$ and the Euclidean norm.

Let $\mu \in C(\mathbb{R}^N, \mathbb{R}^m)$ be a matrix $\mu := (\mu_{ij})$, $i = 1, \dots, m$, $j = 1, \dots, N$. For $i = 1, \dots, m$, let X_i and its formal adjoint X_i^* be defined as

$$X_i := \sum_{j=1}^{N} \mu_{ij}(\xi) \frac{\partial}{\partial \xi_j}, \quad X_i^* := -\sum_{j=1}^{N} \frac{\partial}{\partial \xi_j}(\mu_{ij}(\xi)\cdot), \tag{1.2}$$

where ∇_L and ∇_L^* are the vector field defined by

$$\nabla_L := (X_1, \dots, X_m)^T = \mu\nabla, \quad \nabla_L^* := (X_1^*, \dots, X_m^*)^T. \tag{1.3}$$

For any vector field $h = (h_1, \dots, h_m)^T \in C^1(\Omega, \mathbb{R}^m)$, we shall use the notation $\mathrm{div}_L(h) := \mathrm{div}(\mu^T h)$; that is

$$\mathrm{div}_L(h) = -\sum_{j=1}^{l} X_j^* h = -\nabla_L^* \cdot h. \tag{1.4}$$

Let $\delta := (\delta_1, \dots, \delta_N)$ be an N-uple of positive real numbers. Let $R > 0$, we shall denote by δ_R the anisotropic dilation $\delta_R : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\delta_R(x) = \delta_R(x_1, \dots, x_N) := (R^{\delta_1} x_1, \dots, R^{\delta_N} x_N). \tag{1.5}$$

The Jacobian of such a transformation is $J(\delta_R) = R^Q$ where $Q = \delta_1 + \delta_2 + \ldots + \delta_N$.

A nonnegative continuous function $H : \mathbb{R}^N \to \mathbb{R}^+$ is called a homogeneous norm if (i) $H(\xi) = 0$ if and only if $\xi = 0$, and (ii) it is homogeneous of degree 1 with respect to δ_R, i.e., $H(\delta_R(\xi)) = RH(\xi)$.

Notice that if H is a homogeneous norm differentiable a.e, then $|\nabla_L H|$ is homogeneous of degree 0 with respect to δ_R; hence $|\nabla_L H|$ is bounded.

For the rest of this article, we shall fix a homogeneous norm H differentiable away from 0. We set

$$\psi := |\nabla_L H| \tag{1.6}$$

and for $R > 0$, we define B_R as the ball of radius $R > 0$ generated by the norm H; that is, $B_R := \{x : H(x) < R\}$. Therefore

$$|B_R| = \int_{B_R} dx = R^Q \int_{H(x)<1} dx = C_H R^Q. \tag{1.7}$$

We shall assume that if $\nabla_L u = 0$ on a connected region Ω, then $u \equiv$ *constant* in such region.

Example 1.1. A simple canonical framework is the the Euclidean space $(\mathbb{R}^N, |\cdot|)$ with the Euclidean norm $|\cdot|$. In this case, $\mu = I_N$ is the identity matrix in N dimension, $\nabla_L = \nabla$ is the isotropic gradient and div_L is the divergence operator. The dilation δ_R defined by

$$\delta_R(x) = \delta_R(x_1, \dots, x_N) := (Rx_1, \dots, Rx_N)$$

is isotropic. Here, $Q = N$ is the dimension of the space. In this case, $\psi \equiv 1$ and B_R is the the Euclidean open ball of radius R centered at the origin.

Example 1.2 (Baouendi-Grushin type operator)**.** Let $\xi = (x, y) \in \mathbb{R}^n \times \mathbb{R}^k (= \mathbb{R}^N)$. Let $\gamma \geq 0$ and let μ be the following matrix

$$\begin{pmatrix} I_n & 0 \\ 0 & |x|^\gamma I_k \end{pmatrix}.$$

The corresponding vector field is $\nabla_\gamma = (\nabla_x, \quad |x|^\gamma \nabla_y)^T$ and the linear operator $L = \operatorname{div}_L(\nabla_L \cdot) = \Delta_x + |x|^{2\gamma} \Delta_y$ is the so-called Baouendi-Grushin operator. Notice that if $k = 0$ or $\gamma = 0$, the L coincides with the usual Laplacian operator. The vector field ∇_γ is homogeneous with respect to the dilation $\delta_R(x) = (Rx_1, \dots, Rx_n, R^{1+\gamma} y_1, \dots, R^{1+\gamma} y_k)$ and $Q = N + k\gamma$.

Example 1.3 (Heisenberg-Kohn operator)**.** Let $\xi = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{H}^n (= \mathbb{R}^N)$ and let μ be the matrix

$$\begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}.$$

The corresponding vector field ∇_H is the Heisenberg gradient on the Heisenberg group $\mathbb{H}^n$. The vector field ∇_H is homogeneous with respect to the dilation $\delta_R(x) = (Rx, Ry, , R^2 t)$ and $Q = 2n + 2$. In $\mathbb{H}^1$ the corresponding vector fields are $X = \partial_x + 2y\partial_t$, $Y = \partial_y - 2x\partial_t$. In this case $Q = 4$. This is the simplest case of more general setting: the Carnot group. More details are given in [4]-[9].

Example 1.4 (Heisenberg-Greiner operator)**.** Let $\xi = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ $(= \mathbb{R}^N)$, $r := |(x, y)|$, $\gamma \geq 1$ and let μ be the following matrix

$$\begin{pmatrix} I_n & 0 & 2\gamma y r^{2\gamma - 2} \\ 0 & I_n & -2\gamma x r^{2\gamma - 2} \end{pmatrix}.$$

The corresponding vector fields are $X_i = \partial_{x_i} + 2\gamma y_i r^{2\gamma-2} \partial_t$, $Y_i = \partial_{y_i} - 2\gamma x_i r^{2\gamma - 2} \partial_t$ for $i = 1, \dots, n$.

For $\gamma \geq 1$, $L = \operatorname{div}_L(\nabla_L \cdot)$ is the sub-Laplacian Δ_H on the Heisenberg group $\mathbb{R}^n$. If $\gamma = 2, 3, \dots, L$ is a Greiner operator. The vector field associated to μ is homogeneous with respect to the dilation $\delta_R(x) = (Rx, Ry, R^{2\gamma} t)$ and $Q = 2n + 2\gamma$.

Let $\Omega \subset \mathbb{R}^N$ be an open set and $p > 1$. Throughout this paper we shall denote by

$$W_L^{1,p}(\Omega) = \{u \in L^p(\Omega) : \ |\nabla_L u| \in L^p(\Omega)\},$$
$$W_{L,\mathrm{loc}}^{1,p}(\Omega) = \{u \in L_{\mathrm{loc}}^p(\Omega) : \ |\nabla_L u| \in L_{\mathrm{loc}}^p(\Omega)\}.$$

Notice that when $\mu = I_n$, where I_n is the identity matrix, then $W_L^{1,p}(\Omega) = W^{1,p}(\Omega)$ and $W_{L,\mathrm{loc}}^{1,p}(\Omega) = W_{\mathrm{loc}}^{1,p}(\Omega)$.

Definition 1.5. We shall say that $u \in W^{1,p}_{L,\mathrm{loc}}(\Omega)$ satisfies (1.1) in the weak sense and for any nonnegative test function $\varphi \in C^1_0(\Omega)$ such that

$$\int_\Omega a(x)A(|\nabla_L u|)\nabla_L u \cdot \nabla_L \varphi \, dx \geq \int_\Omega h(x)f(u)\varphi \, dx \tag{1.8}$$

holds.

Definition 1.6. Let $d : \Omega \to \mathbb{R}$ be a nonnegative non constant measurable function. For $\alpha \neq 0$, d^α is called an L_p–harmonic function if

$$L_p d^\alpha = \mathrm{div}_L(|\nabla_L d^\alpha|^{p-2}\nabla_L d^\alpha) = 0$$

in the weak sense; that is, for every nonnegative $\varphi \in C^1_0(\Omega)$, we have

$$\int_\Omega |\nabla_L d^\alpha|^{p-2}\nabla_L d^\alpha \cdot \nabla_L \varphi = \alpha|\alpha|^{p-2}\int_\Omega d^{(\alpha-1)(p-2)}|\nabla_L d|^{p-2}\nabla_L d \cdot \nabla_L \varphi = 0, \tag{1.9}$$

where $d^{(\alpha-1)(p-1)}|\nabla_L d|^{p-2} \in L^1_{\mathrm{loc}}(\Omega)$.

The main result in this paper is the following theorems.

Theorem 1.7. *Let d^α be defined as Definition 1.6 and $h(x) \geq M\big|\nabla_L d\big|^p$, where $p > 1$ and M is a positive constant. Assume* (A1), (A2), *$a(x) > M$ and*

$$\int_{-\infty}^{+\infty}\Big(\int_t^{+\infty} f(s)ds\Big)^{-1/p}dt = +\infty \tag{1.10}$$

hold. Let A satisfy (i) $A(t) \geq t^{p-2}$ or (ii) $A(t) \leq t^{p-2}$, and let $tA(t)$ be strictly increasing for $t > 0$. If u is a $W^{1,p}_{L,\mathrm{loc}}$ solution of (1.1)*, then $u \geq 0$.*

Remark 1.8. The condition that $t \longmapsto tA(t)$ is strictly increasing is a minimal requirement for ellipticity of (1.1). Furthermore, it allows singular and degenerate behavior of the operator A at $t = 0$.

Remark 1.9. Similar results have been proved in [6]–[9], when $a(x) = h(x) \equiv 1$, under different conditions. In particular, if we set $d(x) = |x|$, then the condition $h(x) \geq M\big|\nabla_L d\big|^p$ reduces to $h(x) \geq M$ in the Euclidean case.

Theorem 1.10. *Assume that d^α is defined as Definition 1.6 and $h(x) \geq M\big|\nabla_L d\big|^p$, where $p > 1$ and M is a positive constant. Let $f : \mathbb{R} \to \mathbb{R}$ be a positive, non-increasing and continuous function satisfying* (1.10)*. Then* (1.1) *has no solutions.*

2. Proofs of Theorems 1.7 and 1.10

To prove theorem 1.7, we establish some preliminary results in this section. Now, we shall prove a comparison lemma that it is useful when considering solutions of inequalities of the form

$$\mathrm{div}_L\,(a(x)A_1(|\nabla_L u|)\nabla_L u) \geq g_1(x,u) \quad \text{in } \Omega, \tag{2.1}$$

$$\mathrm{div}_L(MA_2(|\nabla_L v|)\nabla_L v) \leq g_2(x,v) \quad \text{in } \Omega, \tag{2.2}$$

where M is a positive constant. Here, for $i = 1, 2$, A_i is a continuous function such that $A_i > 0$ for $t > 0$ and $g_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

In a similar manner to Definition 1.5, we can define the solution of equalities (2.1) and (2.2).

Definition 2.1. We shall say that $u \in W^{1,p}_{L,\mathrm{loc}}(\Omega)$ satisfies (2.1) (resp. (2.2)) in the weak sense, and for any nonnegative test function $\varphi \in C^1_0(\Omega)$ such that

$$-\int_\Omega a(x)A_1(|\nabla_L u|)\nabla_L u \cdot \nabla_L \varphi \, dx \geq \int_\Omega g_1(x,u)\varphi \; dx \tag{2.3}$$

(resp.

$$-\int_\Omega a(x)A_2(|\nabla_L u|)\nabla_L u \cdot \nabla_L \varphi \, dx \leq \int_\Omega g_2(x,u)\varphi \, dx\Big) \tag{2.4}$$

holds.

Lemma 2.2 (Comparison principle)**.** *Let Ω be a bounded open set. Let u and v be respectively solutions of* (2.1) *and* (2.2) *of class $W^{1,p}_{L,\mathrm{loc}}(\Omega)$. Assume that $a(x) \geq M$ and*

- (i) *for any $x \in \Omega$, $t \geq s$ there holds $g_1(x,t) \geq g_2(x,s)$, $g_1(x,\cdot)$ is not decreasing.*
- (ii) *(1) $A_1(t) \geq A_2(t)$ for $t > 0$ and the function $tA_2(t)$ is strictly increasing for $t > 0$; or (2) $A_1(t) \leq A_2(t)$ for $t > 0$ and the function $tA_1(t)$ is strictly increasing for $t > 0$;*
- (iii) *$u \leq v$ on $\partial\Omega$.*

Then $u \leq v$ in Ω.

Remark 2.3. If $a(x) \equiv 1$, $M = 1$ and $A_1 = A_2$ in (2.1) and (2.2), similar results to Lemma 2.2 have been proved under different conditions. [4, 6, 8, 21, 22, 27]. In Lemma 2.2, we extend some results of [6, 8], which hold for C^1 solutions, to the large class of $W^{1,p}_{L,\mathrm{loc}}$ solutions.

Proof of Lemma 2.2. Let $\epsilon > 0$ be fixed and set $v_\epsilon = v + \epsilon$. It is a simple to check that the function v_ϵ satisfies the inequality

$$\mathrm{div}_L(MA_2(|\nabla_L v|)\nabla_L v) \leq g_2(x, v_\epsilon) \quad \text{in } \Omega,$$

Therefore, for any nonnegative test function $\varphi \in C^1_0(\Omega)$ we have

$$-\int_\Omega MA_2(|\nabla_L v|)\nabla_L v \cdot \nabla_L \varphi \, dx \leq \int_\Omega g_2(x, v_\epsilon)\varphi \, dx$$

By subtraction we obtain

$$\begin{aligned} &-\int_\Omega (a(x)A_1(|\nabla_L u|)\nabla_L u) - MA_2(|\nabla_L v|)\nabla_L v) \cdot \nabla_L \varphi \, dx \\ &\geq \int_\Omega (g_1(x,u) - g_2(x,v_\epsilon))\varphi \, dx \end{aligned} \tag{2.5}$$

We choose the nonnegative $\varphi = ((u - v_\epsilon)^+)^2$ as test function in (2.5). Obviously, $\varphi \in W^{1,p}_L(\Omega)$ and φ has compact support since $u - v_\epsilon < 0$ on $\partial\Omega$. Then, we obtain

$$\begin{aligned} &-2\int_\Omega (a(x)A_1(|\nabla_L u|)\nabla_L u) - MA_2(|\nabla_L v|)\nabla_L v) \cdot (\nabla_L u - \nabla_L v)(u - v_\epsilon)^+ \, dx \\ &\geq \int_\Omega (g_1(x,u) - g_2(x,v_\epsilon))((u - v_\epsilon)^+)^2 \, dx \end{aligned} \tag{2.6}$$

Since $a(x) \geq M > 0$, we have

$$
\begin{aligned}
&(a(x)A_1(|\nabla_L u|))\nabla_L u) - MA_2(|\nabla_L v|)\nabla_L v) \cdot (\nabla_L u - \nabla_L v) \\
&= a(x)A_1(|\nabla_L u|)|\nabla_L u|^2 + MA_2(|\nabla_L v|)|\nabla_L v|^2 \\
&\quad - (a(x)A_1(|\nabla_L u|) + MA_2(|\nabla_L v|))(\nabla_L u \cdot \nabla_L v) \\
&= (a(x)A_1(|\nabla_L u|)|\nabla_L u| - MA_2(|\nabla_L v|)|\nabla_L v|)(|\nabla_L u| - |\nabla_L v|) \\
&\quad + (a(x)A_1(|\nabla_L u|) + MA_2(|\nabla_L v|))(|\nabla_L u||\nabla_L v| - \nabla_L u \cdot \nabla_L v) \\
&\geq M((A_1(|\nabla_L u|)|\nabla_L u| - A_2(|\nabla_L v|)|\nabla_L v|)(|\nabla_L u| - |\nabla_L v|) \\
&\quad + (A_1(|\nabla_L u|) + A_2(|\nabla_L v|))(|\nabla_L u||\nabla_L v| - \nabla_L u \cdot \nabla_L v)) \\
&=: M(I_1 + I_2).
\end{aligned}
\tag{2.7}
$$

Since $A_i(t) > 0$ for $t > 0$, where $i = 1, 2$, we have $I_2 \geq 0$.

First we consider the case (ii) (1). From $A_1(t) \geq A_2(t)$ for $t > 0$ and the function $tA_2(t)$ is increasing for $t > 0$, we obtain

$$
\begin{aligned}
I_1 &= (A_1(|\nabla_L u|)|\nabla_L u| - A_2(|\nabla_L v|)|\nabla_L v|)(|\nabla_L u| - |\nabla_L v|) \\
&\geq (A_2(|\nabla_L u|)|\nabla_L u| - A_2(|\nabla_L v|)|\nabla_L v|)(|\nabla_L u| - |\nabla_L v|) \geq 0.
\end{aligned}
\tag{2.8}
$$

Therefore, $\int_\Omega M(I_1 + I_2)(u - v_\epsilon)^+ dx \geq 0$. Since $g_1(x, \cdot)$ is not decreasing,

$$
\begin{aligned}
&\int_\Omega (g_1(x, u) - g_2(x, v))((u - v_\epsilon)^+)^2 \, dx \\
&\geq (g_1(x, u) - g_1(x, v_\epsilon))((u - v_\epsilon)^+)^2 \, dx \geq 0,
\end{aligned}
\tag{2.9}
$$

and then combining (2.6) with (2.7), we obtain

$$
\begin{aligned}
0 &\geq \int_\Omega M(I_1 + I_2)(u - v_\epsilon)^+ dx \\
&\geq \int_\Omega (a(x)A_1(|\nabla_L u|)\nabla_L u) - MA_2(|\nabla_L v|)\nabla_L v) \cdot (\nabla_L u - \nabla_L v)(u - v_\epsilon)^+ \, dx \\
&\geq \int_\Omega (g_1(x, u) - g_2(x, v_\epsilon))((u - v_\epsilon)^+)^2 \, dx \geq 0;
\end{aligned}
\tag{2.10}
$$

that is,

$$
\int_\Omega (I_1 + I_2)(u - v_\epsilon)^+ dx = 0. \tag{2.11}
$$

The following proof is by contradiction. Assume that $\varphi = u - v - \epsilon > 0$ for $x \in \Omega$, then we have $I_1 = I_2 = 0$ by (2.11). We claim that $\nabla_L u = \nabla_L v$. Indeed, If $\nabla_L u \neq \nabla_L v$, by $I_2 = 0$, we obtain

$$
|\nabla_L u||\nabla_L v| = \nabla_L u \cdot \nabla_L v \tag{2.12}
$$

and

$$
\begin{aligned}
(|\nabla_L u| - |\nabla_L v|)^2 &= |\nabla_L u|^2 - 2|\nabla_L u||\nabla_L v| + |\nabla_L v|^2 \\
&= |\nabla_L u|^2 - 2\nabla_L u \cdot \nabla_L v + |\nabla_L v|^2 \\
&= (\nabla_L u - \nabla_L v)^2,
\end{aligned}
\tag{2.13}
$$

which implies $|\nabla_L u| \neq |\nabla_L v|$. Moreover from $I_1 = 0$ and the monotonicity of tA_2, we obtain

$$\begin{aligned} 0 &= (A_1(|\nabla_L u|)|\nabla_L u| - A_2(|\nabla_L v|)|\nabla_L v|)(|\nabla_L u| - |\nabla_L v|) \\ &\geq (A_2(|\nabla_L u|)|\nabla_L u| - A_2(|\nabla_L v|)|\nabla_L v)(|\nabla_L u| - |\nabla_L v|)| > 0, \end{aligned} \tag{2.14}$$

which is a contradiction. Thus, we have $\nabla_L u = \nabla_L v$, which implies $\nabla_L \varphi = \nabla_L((u-v_\epsilon)^+)^2 = 0$; that is, $\varphi = ((u-v_\epsilon)^+)^2 \equiv$ *constant* in Ω. Since $\varphi \in W^{1,p}_{L,\mathrm{loc}}(\Omega)$, we have $\varphi = ((u-v_\epsilon)^+)^2 \equiv 0$ in Ω, that is $u - v_\epsilon \leq 0$, which is a contradiction of our assumption. Thus, $u \leq v + \epsilon$ in Ω. Letting $\epsilon \to 0$ completes the proof.

Now, we consider the case (ii)(2), which proof is the same as that of (ii)(1). By virtue of (2.8) and (2.10), we only to replace (2.8) and (2.10) by the following inequalities

$$\begin{aligned} I_1 &= (A_1(|\nabla_L u|)|\nabla_L u| - A_2(|\nabla_L v|)|\nabla_L v|)(|\nabla_L u| - |\nabla_L v|) \\ &\geq (A_1(|\nabla_L u|)|\nabla_L u| - A_1(|\nabla_L v|)|\nabla_L v|)(|\nabla_L u| - |\nabla_L v|) \geq 0 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} 0 &= (A_1(|\nabla_L u|)|\nabla_L u| - A_2(|\nabla_L v|)|\nabla_L v|)(|\nabla_L u| - |\nabla_L v|) \\ &\geq (A_1(|\nabla_L u|)|\nabla_L u| - A_1(|\nabla_L v|)|\nabla_L v)(|\nabla_L u| - |\nabla_L v|)| > 0 \end{aligned} \tag{2.16}$$

respectively in the proof of (1). The proof is complete. $\square$

Lemma 2.4 ([4, Lemma2.18]). *Let $p > 1$, $\alpha \in \mathbb{R}^N$, $\alpha \neq 0$. $d^\alpha \in C^2(\Omega)$ be a positive L_p-harmonic function; that is, $L_p d^\alpha = 0$ in the weak sense of* (1.9)*. Let $u(x) := \phi(d(x))$, we have*

$$\begin{aligned} L_p u &= (p-1)\big|\nabla_L d\big|^p \big|\nabla_L \phi'(d)\big|^{p-2}\big(\phi'(d) + \frac{1-\alpha}{d}\phi'(d)\big) \\ &= \big|\nabla_L d\big|^p d^{(p-1)(\alpha-1)}\Big(d^{(p-1)(1-\alpha)}\big|\phi'(d)\big|^{p-2}\phi'(d)\Big)' \end{aligned} \tag{2.17}$$

Remark 2.5. Some special cases of Lemma 2.4 are the following. In the Euclidean case, we set $d(x) = |x| := r$, then (2.17) reduces to

$$L_p u = (p-1)|\phi'(r)|^{p-2}\Big(\phi''(r) + \frac{N-1}{p-1}\frac{\phi'(r)}{r}\Big),$$

which is discussed in [18, 21, 22] for $p \neq 2$ and [23] for $p = 2$. In the Heisenberg setting studied in [6], we set

$$d(x) = |x|_H = \Big(\sum_{i=1}^n (\xi_i^2 + \eta_i^2)^2 + \tau^2\Big)^{1/4} := r,$$

then (2.17) reduces to

$$L_p u = (p-1)\psi^p|\phi'(r)|^{p-2}\Big(\phi''(r) + \frac{Q-1}{r}\phi'(r)\Big), \tag{2.18}$$

where $\psi = \big|\nabla_L |x|_H\big|$. In the Carnot group considered in [6, 8, 9], setting

$$d(x) := N_p = \begin{cases} \Gamma(x)^{\frac{p-1}{p-Q}}, & p > 1,\ p \neq Q, \\ \exp(-\Gamma(x)), & p = Q, \end{cases} \tag{2.19}$$

where Γ is the fundamental solution of the quasilinear operator

$$L_p u = \mathrm{div}_L(|\nabla_L u|^{p-1}\nabla_L u)$$

at the origin, we obtain (2.18) with $r = N_p$ and $\psi = |\nabla_L N_p|$.

Lemma 2.6. *Let $p > 1$ and let g be a continuous and non-decreasing function on $[0, +\infty)$ satisfying $g(t) > 0$ for $t > 0$. If*

$$\int_1^{+\infty} \Big(\int_1^t g(s)ds \Big)^{-1/p} dt = \infty, \tag{2.20}$$

then for any $c > 0$ and $\sigma > 0$, there exists $R > 0$ and a function ϕ satisfying

$$\big(r^\sigma |\phi'(r)|^{p-2}\phi'(r)\big)' = r^\sigma g(\phi(r)), \quad \phi(0) = c, \quad \phi'(0) = 0, \tag{2.21}$$

where ϕ is increasing on $[0, R]$ and $\phi \to \infty$ as $r \to R$.

Remark 2.7. For the proof of Lemma 2.6, we can see [18, 21, 23]. Osserman [23] proved the result in the case $p = 2$. On the other hand, Naito and Usami [21] obtained the result when $\sigma = N - 1$, and then Ghergu and Rădulescu [18] given a generalization proof of Lemma 2.6.

Proof of Theorem 1.7. Let $\sigma = (p-1)(1-\alpha) > 0$, and let ϕ be a solution of (2.21) such that $\phi(r) \to +\infty$ as $r \to R$. We set $v(x) := \phi(d(x))$, where $d(x)$ satisfy the condition of Lemma 2.4. By Lemma 2.6 then v satisfies

$$\begin{aligned} &\operatorname{div}_L(M|\nabla_L v|^{p-1}\nabla_L v) \\ &= M\big|\nabla_L d\big|^p d^{(p-1)(\alpha-1)}\Big(d^{(p-1)(1-\alpha)}\big|\phi'(d)\big|^{p-2}\phi'(d)\Big)' \\ &= M\big|\nabla_L d\big|^p g(v) := g_2(x, v), \end{aligned} \tag{2.22}$$

$\phi(d(x)) \to +\infty$ as $d(x) \to R$, and $v(0) = c$ in $\Omega_R = \{x : d(x) < R\}$.

On the other hand, let $g(t) := f(-t)$ and $u = -U$ in (1.1). Then the function g satisfies the assumptions of Theorem 1.7; therefore we obtain

$$\begin{aligned} \operatorname{div}_L(a(x)A(|\nabla_L U|)\nabla_L U) &\geq h(x)f(-u) = h(x)g(U) \\ &\geq M\big|\nabla_L d\big|^p g(U) := g_1(x, U) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{2.23}$$

Since $U(x) \leq v(x)$ for $d(x)$ close to R, combining (2.22) and (2.23), we can apply the comparison Lemma 2.2. Thus $U(x) \leq v(x)$ in Ω_R. In particular in the neighborhood of origin we have $U(x) \leq c$, i.e., $U(0) \leq c$. Letting $c \to 0$, it follows that $U(0) \leq 0$. Hence $u(0) \geq 0$. Since the inequality (1.1) is invariant under translations in the weak sense (1.8) in $\mathbb{R}^N$, we obtain $u(x) \geq 0$ in $\mathbb{R}^N$. □

Proof of Theorem 1.10. By contradiction, assume that u is a solution of (1.1). Fix $\beta \in \mathbb{R}$ and set $v = u - \beta$, then v solves the inequality $-\Delta v \geq f(u) = f(v + \beta)$. Since the function $f(\cdot + \beta)$ satisfies the hypothesis of Theorem 1.7, we have $v \geq 0$, that is $u \geq \beta$. Since the inequality $u \geq \beta$ holds for any β, we obtain $u = +\infty$, which is impossible. Hence, we obtain the conclusion. □

Acknowledgements. This work was supported by the National Natural Science Foundation of China (No. 11301301, No. 11571295 and No. 11401347) and the Foundation for Outstanding Middle-Aged and Young Scientists of Shandong Province (No. BS2013SF027). The author thanks the anonymous reviewer for his/her careful review and helpful suggestions for improvement.

References

[1] M. F. Bidaut-Véron, S. Pohozaev; *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math., 84 (2001), 1–49.

[2] G. Caristi, L. D'Ambrosio, E. Mitidieri; *Representation formulae for solutions to some classes of higher order systems and related Liouville theorems*, Milan J. Math., 76 (2008), 27–67.

[3] G. Molica Bisci, M. Ferrara; *Subelliptic and parametric equations on Carnot groups*, Proc. Amer. Math. Soc., 144 (2016), 3035-3045.

[4] L. D'Ambrosio; *Hardy-type inequalities related to degenerate elliptic differential operators*, Ann. Sc. Norm. Super. Pisa Cl. Sci., (5) IV (2005), no. 3, 451–486.

[5] L. D'Ambrosio, E. Mitidieri, S. I. Pohozaev; *Representation formulae and inequalities for solutions of a class of second order partial differential equations*. Trans. Amer. Math. Soc., 358 (2006), no. 2, 893–910.

[6] L. D'Ambrosio, E. Mitidieri; *Positivity property of solutions of some quasilinear elliptic inequalities*, Functional analysis and evolution equations, 2008, 147–155.

[7] L. D'Ambrosio; *Liouville theorems for anisotropic quasilinear inequalities*, Nonlinear Anal., 70 (2009), no. 8, 2855–2869.

[8] L. D'Ambrosio, E. Mitidieri; *Nonnegative solutions of some quasilinear elliptic inequalities and applications* , Mat. Sb., 201 (2010) no. 6, 75–92.

[9] L. D'Ambrosio, E. Mitidieri; *A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities*, Adv. Math., 224 (2010), no. 3, 967–1020.

[10] E. Mitidieri, S. I. Pohozaev; *The absence of global positive solutions to quasilinear elliptic inequalities*, Dokl. Math., 57 (1998), 250–253.

[11] E. Mitidieri, S. I. Pohozaev; *Absence of positive solutions for quasilinear elliptic problems in* $\mathbb{R}^N$, Proc. Steklov Inst. Math., 227 (1999), 186–216.

[12] E. Mitidieri, S. I. Pohozaev; *Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on* $\mathbb{R}^n$, J. Evol. Equ., 1 (2001), 189–220.

[13] R. Filippucci; *Nonexistence of positive weak solutions of elliptic inequalities*, Nonlinear Anal., 70 (2009), no. 8, 2903–2916.

[14] R. Filippucci, P. Pucci, M. Rigoli; *Non-existence of entire solutions of degenerate elliptic inequalities with weights*, Arch. Ration. Mech. Anal., 188 (2008), no. 1, 155–179.

[15] R. Filippucci, P. Pucci, M. Rigoli; *On weak solutions of nonlinear weighted p-Laplacian elliptic inequalities*, Nonlinear Anal., 70 (2009), no. 8, 3008–3019.

[16] R. Filippucci, P. Pucci, M. Rigoli; *Nonlinear weighted p-Laplacian elliptic inequalities with gradient terms*, Commun. Contemp. Math., 12(2010), 501–535.

[17] E. Galakhov; *Some nonexistence results for quasilinear PDE's*, Commun. Pure Appl. Anal., 6 (2007), 141–161.

[18] M. Ghergu, V. Rădulescu; *Existence and nonexistence of entire solutions to the logistic differential equation*, Abstr. Appl. Anal., 2003, no. 17, 995–1003.

[19] G. Caristi; *On the absence of solutions of systems of quasilinear elliptic inequalities*, Differ. Equ., 38 (2002), 375–383.

[20] G. G. Laptev; *On the absence of solutions to a class of singular semilinear differential inequalities*, Proc. Steklov Inst. Math., 232 (2001), 216–228.

[21] Y. Naito, H. Usami; *Entire solutions of the inequality* $\operatorname{div}(A(|Du|)Du) \geq f(u)$, Math. Z., 225 (1997), no. 1, 167–175.

[22] Y. Naito, H. Usami; *Nonexistence results of positive entire solutions for quasilinear elliptic inequalities*, Canad. Math. Bull., 40 (1997), no. 2, 244–253.

[23] R. Osserman; *On the inequality* $\Delta u \geq f(u)$, Pacific J. Math., 7 (1957) 1641–1647.

[24] S. I. Pohozaev; *A general approach to the theory of the nonexistence of global solutions of nonlinear partial differential equations and inequalities*, Proc. Steklov Inst. Math., 236 (2002), 273–284.

[25] S. I. Pohozaev, A. Tesei; *Nonexistence of local solutions to semilinear partial differential inequalities*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 21 (2004), 487–502.

[26] P. Pucci, J. Serrin; *The strong maximum principle revisited*, J. Differential Equations, 196 (2004), no. 1, 1–66.

[27] P. Pucci, J. Serrin; *The maximum principle*, Progress in Nonlinear Differential Equations and their Applications, 73. Birkhuser Verlag, Basel, 2007.

[28] J. Serrin; *Entire solutions of quasilinear elliptic equations*, J. Math. Anal. Appl., 352 (2009), no. 1, 3–14.
[29] J. Serrin, H. Zou; *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math., 189 (2002), 79–142.

Xiaohong Li
College of Mathematics and Information Science, Shandong Institute of Business and Technology, Yantai, Shandong 264005, China
E-mail address: `xh0535@sina.com`