Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 176, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

BOUNDEDNESS IN A THREE-DIMENSIONAL ATTRACTION-REPULSION CHEMOTAXIS SYSTEM WITH NONLINEAR DIFFUSION AND LOGISTIC SOURCE

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ABSTRACT. This article concerns the attraction-repulsion chemotaxis system with nonlinear diffusion and logistic source,

$$\begin{gathered} u_t = \nabla\cdot((u+1)^{m-1}\nabla u) - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w) + ru - \mu u^\eta, \\ x\in\Omega,\ t>0, \\ v_t = \Delta v + \alpha u - \beta v, \quad x\in\Omega,\ t>0, \\ w_t = \Delta w + \gamma u - \delta w, \quad x\in\Omega,\ t>0 \end{gathered}$$

under Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary. We show that if the diffusion is strong enough or the logistic dampening is sufficiently powerful, then the corresponding system possesses a global bounded classical solution for any sufficiently regular initial data. Moreover, it is proved that if $r=0$, $\beta>\frac{1}{2(\eta-1)}$ and $\delta>\frac{1}{2(\eta-1)}$ for the latter case, then $u(\cdot,t)\to 0$, $v(\cdot,t)\to 0$ and $w(\cdot,t)\to 0$ in $L^\infty(\Omega)$ as $t\to\infty$.

1. Introduction

We consider the attraction-repulsion chemotaxis system with nonlinear diffusion and logistic source,

$$\begin{gathered} u_t = \nabla\cdot((u+1)^{m-1}\nabla u) - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w) + ru - \mu u^\eta, \\ x\in\Omega,\ t>0, \\ v_t = \Delta v + \alpha u - \beta v, \quad x\in\Omega,\ t>0, \\ w_t = \Delta w + \gamma u - \delta w, \quad x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), \quad x\in\Omega, \end{gathered} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, where $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative on $\partial\Omega$, $m\ge 1$, $r\ge 0$, $\mu>0$, $\eta>1$, $\chi>0$, $\xi>0$, $\alpha>0$, $\beta>0$, $\gamma>0$ and $\delta>0$ are prescribed parameters. Here $u(x,t)$ denotes the cell density, $v(x,t)$ and $w(x,t)$ represent the chemoattractant concentration and the chemorepellent concentration, respectively. In (1.1) we assume that cell kinetics

2010 *Mathematics Subject Classification.* 35K55, 35Q35, 35Q92, 92C17.
Key words and phrases. Chemotaxis; attraction-repulsion; boundedness; nonlinear diffusion; logistic source.

Submitted February 8, 2016. Published July 6, 2016.

follows a logistic-type law determined by parameters r, μ and η. The first cross-diffusive term and the second in the first equation reflect the attractive migration and repulsive movement of cells, respectively. The second and the third equations imply that the attractive and repulsive signals are produced by cells themselves. System (1.1) is a generalized version of the classical Keller-Segel model [9] and can describe the aggregation of microglia in Alzheimer's disease due to the interaction of chemoattractant and chemorepellent [11]. System (1.1) can also address the quorum effect in the chemotactic process [18]. The mathematical studies on system (1.1) are much harder than the classical Keller-Segel system due to lacking a useful Lyapunov functional.

To motivate our study, we recall some related works on system (1.1). In the case $m = 1$, $r = 0$ and $\mu = 0$, there have been a series of development: Wang *et al.* [16, 6] proved that system (1.1) possesses a unique global bounded solution when $n = 1$; when $n = 2$ and the repulsion prevails over the attraction in the sense that $\xi\gamma - \chi\alpha > 0$, Jin [4] and Tao *et al.* [15] independently proved that system (1.1) admits a unique global bounded solution; however, when $n = 3$ and $\xi\gamma - \chi\alpha > 0$, the global existence of classical solutions for (1.1) is still open; Jin [4] only proved that system (1.1) possesses a global weak solution; when $n = 2, 3$ and the repulsion cancels the attraction (i.e. $\xi\gamma = \chi\alpha$), the authors in [13, 5] proved that system (1.1) possesses a unique global bounded classical solution and the large time behavior of solutions is considered for the bounded domain and the whole space. The authors in [14] investigated the pattern formation of (1.1) analytically and numerically for $n = 1$. For the parabolic-elliptic case, the global solvability, critical mass phenomenon, blow-up, and large time behavior were investigated for the bounded domain and the whole space (see [1, 21, 22, 12, 7]). In [26, 27], the global solvability and the uniform boundedness were considered for the attraction-repulsion chemotaxis system with nonlinear diffusion.

In the case $m = 1$, $r > 0$ and $\mu > 0$, there are only few results: Li *et al.* [17, 10] proved that (1.1) with $\eta \geq 1$ for $n = 1$ or $\eta \geq 2$ for $n = 2$ admits a unique global bounded solution. For $n \geq 3$, however, the global existence of classical solutions of (1.1) is still open. In [28], the global solvability, boundedness and large time behavior were only investigated for a quasilinear attraction-repulsion chemotaxis model with logitic source for parabolic-elliptic type in a bounded domain.

To the best of our knowledge, there is no rigorous result on the quasilinear attraction-repulsion chemotaxis model with logistic source for parabolic-parabolic type except [10], where the author only considered two-dimensional case and proved that when $m > 1$ or $\eta \geq 2$, system (1.1) possesses a global bounded classical solution. Thus the goal of this paper is to explore the interactions between the nonlinear diffusion and logistic source on the solutions of system (1.1) for three-dimensional case and partially answer the above open problem. Throughout this paper, we assume that the initial data satisfy

$$
\begin{gathered}
u_0 \in W^{1,\infty}(\Omega), \quad u_0 > 0 \quad \text{on} \quad \bar{\Omega}, \\
v_0 \in W^{1,\infty}(\Omega), \quad v_0 \geq 0 \quad \text{on} \quad \bar{\Omega}, \\
w_0 \in W^{1,\infty}(\Omega), \quad w_0 \geq 0 \quad \text{on} \quad \bar{\Omega}.
\end{gathered} \tag{1.2}
$$

We now state the main results of this article.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and initial data (u_0, v_0, w_0) satisfy (1.2). Suppose that $m \geq 1$, $r \geq 0$, $\mu > 0$, $\eta > 1$, $\chi > 0$, $\xi > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\delta > 0$. If one of the following two cases holds:*

- (i) *$\eta \geq 2$ and $\mu > \max\{1 + 4\chi^2 + 11\alpha^2 + 2\chi^2(9 + \alpha^2 + \gamma^2),\ 1 + 4\xi^2 + 11\gamma^2 + 2\xi^2(9 + \alpha^2 + \gamma^2)\}$;*
- (ii) *$\eta \in (1, 2)$ and $m > 4/3$,*

then there exists a triplet (u, v, w) belonging to $C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ which solves system (1.1) classically. Moreover, u, v and w are bounded in $\Omega \times (0, \infty)$ in the sense that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq M \quad \forall t \in (0, \infty), \tag{1.3}$$

where $M > 0$ is a constant independent of t.

Remark 1.2. Theorem 1.1 answers the open problem in [10, Remark 1.4] and [17, Remark 1.3] for three-dimensional case.

Remark 1.3. For the case $n \geq 4$, if $m > 2 - \frac{2}{n}$, then Theorem 1.1 holds. This can be proved by the same arguments of case (ii) in Theorem 1.1.

Next, we consider the large time behavior of solutions to (1.1) for a special case $r = 0$, $\eta \geq 2$, $\mu > \max\{1 + 4\chi^2 + 11\alpha^2 + 2\chi^2(9 + \alpha^2 + \gamma^2),\ 1 + 4\xi^2 + 11\gamma^2 + 2\xi^2(9 + \alpha^2 + \gamma^2)\}$, $\beta > \frac{1}{2(\eta-1)}$ and $\delta > \frac{1}{2(\eta-1)}$ by using the ideas in [24]. Here $r = 0$ means that either cells are a prior unable to reproduce themselves, or the considered time scales are much smaller than those of cell proliferation (see [2, 8] for details).

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and initial data (u_0, v_0, w_0) satisfy (1.2). Suppose that $m \geq 1$, $\chi > 0$, $\xi > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\delta > 0$. If $r = 0$, $\eta \geq 2$, $\mu > \max\{1 + 4\chi^2 + 11\alpha^2 + 2\chi^2(9 + \alpha^2 + \gamma^2),\ 1 + 4\xi^2 + 11\gamma^2 + 2\xi^2(9 + \alpha^2 + \gamma^2)\}$, $\beta > \frac{1}{2(\eta-1)}$ and $\delta > \frac{1}{2(\eta-1)}$, then the global solution (u, v, w) of system (1.1) satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|v(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|w(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{1.4}$$

This article is organized as follows. In Section 2, we state the existence of local solutions to (1.1) and some preliminary inequalities. In Section 3, we give some fundamental estimates for solutions to (1.1) and prove Theorem 1.1. In Section 4, we consider the large time behavior of solutions to (1.1) for a special case and prove Theorem 1.4.

2. Preliminaries

In this section, we first state the local existence of solutions to the system (1.1) and then present some preliminary inequalities. By directly adapting the reasoning in [25, Lemma 2.1], we can derive the following lemma on local existence of classical solutions to (1.1).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and initial data (u_0, v_0, w_0) satisfy (1.2). Suppose that $m \geq 1$, $r \geq 0$, $\mu > 0$, $\eta > 1$, $\chi > 0$, $\xi > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\delta > 0$. Then there exist $T_{\max} \in (0, \infty]$ and a triplet (u, v, w) of nonnegative functions from $C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$*

solving (1.1) *classically in* $\Omega \times (0, T_{\max})$. *These functions satisfy* $u > 0$, $v > 0$ *and* $w > 0$ *in* $\bar{\Omega} \times (0, T_{\max})$ *and, moreover, we have*

$$\begin{aligned} &\text{either } T_{\max} = \infty, \text{ or} \\ &\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \to \infty \text{ as } t \to T_{\max}. \end{aligned} \tag{2.1}$$

Let us state the following basic properties associated with u.

Lemma 2.2. *There exists some constants* $M_1 > 0$ *and* $C > 0$ *such that*

$$\int_\Omega u(x,t)dx \le M_1 \quad \forall t \in (0, T_{\max}), \tag{2.2}$$

$$\int_t^{t+\tau} \int_\Omega u^\eta(x,t)\,dx\,ds \le C \quad \forall t \in (0, T_{\max} - \tau), \tag{2.3}$$

where

$$\tau := \min\{1, \frac{1}{2}T_{\max}\}. \tag{2.4}$$

Proof. Integrating the first equation of (1.1), we obtain

$$\frac{d}{dt}\int_\Omega udx + \mu \int_\Omega u^\eta dx = r\int_\Omega udx \quad \forall t \in (0, T_{\max}). \tag{2.5}$$

If $r = 0$, we have $\frac{d}{dt}\int_\Omega udx = -\mu\int_\Omega u^\eta dx$ for all $t \in (0, T_{\max})$. The nonnegativity of u implies $\frac{d}{dt}\int_\Omega udx < 0$ for all $t \in (0, T_{\max})$. Thus, we have $\int_\Omega udx \le \int_\Omega u_0 dx$ for all $t \in (0, T_{\max})$. If $r > 0$, then by using Young's inequality we have

$$2ru \le \mu u^\eta + \mu^{-\frac{1}{\eta-1}}(2r)^{\frac{\eta}{\eta-1}}.$$

This implies

$$\frac{d}{dt}\int_\Omega udx + r\int_\Omega udx \le \mu^{-\frac{1}{\eta-1}}(2r)^{\frac{\eta}{\eta-1}}|\Omega| \quad \forall t \in (0, T_{\max}).$$

Thus,

$$\int_\Omega udx \le M_1 := \max\Big\{\int_\Omega u_0 dx, \frac{\mu^{-\frac{1}{\eta-1}}(2r)^{\frac{\eta}{\eta-1}}|\Omega|}{r}\Big\}$$

for all $t \in (0, T_{\max})$. Therefore, (2.2) holds. By a time integration of (2.5), we derive that

$$\begin{aligned} &\int_\Omega u(x,t+\tau)dx + \mu\int_t^{t+\tau}\int_\Omega u^\eta(x,s)\,dx\,ds \\ &\le \int_\Omega u(x,t)dx + r\int_t^{t+\tau}\int_\Omega u(x,s)\,dx\,ds \\ &\le M_1 + r\tau M_1 \quad \forall t \in (0, T_{\max} - \tau), \end{aligned}$$

which implies that (2.3) holds. □

The following auxiliary lemma on a boundedness property in an ODI will be used in our analysis. It is a straightforward generalization of a particular case $\tau = 1$ which has been proved in [20, Lemma 3.4].

Lemma 2.3. *Let* $T > 0$, $\tau \in (0, T)$, $a > 0$ *and* $b > 0$, *and suppose that* $y : [0, T) \to [0, \infty)$ *is absolutely continuous and such that*

$$y'(t) + ay(t) \le h(t) \quad \text{for a.e. } t \in (0, T),$$

where $h \in L^1_{loc}([0,T))$ is nonnegative and satisfies

$$\int_t^{t+\tau} h(s)ds \le b \quad \forall t \in [0, T-\tau).$$

Then

$$y(t) \le \max\{y(0)+b, \frac{b}{a\tau}+2b\} \quad \textit{for all } t \in (0,T).$$

As an application of Lemma 2.2 and Lemma 2.3, we can establish the following estimates associated with v and w.

Lemma 2.4. *Let $\eta \ge 2$, then there exists a constant $C > 0$ such that*

$$\int_\Omega v(x,t)dx \le C \quad \forall t \in (0, T_{\max}), \tag{2.6}$$

$$\int_\Omega |\nabla v|^2 dx \le C \quad \forall t \in (0, T_{\max}), \tag{2.7}$$

$$\int_\Omega w(x,t)dx \le C \quad \forall t \in (0, T_{\max}), \tag{2.8}$$

$$\int_\Omega |\nabla w|^2 dx \le C \quad \forall t \in (0, T_{\max}). \tag{2.9}$$

Proof. We first prove the estimates associated with v. Integrating the second equation in (1.1) over Ω and then using (2.2), we obtain that

$$\frac{d}{dt}\int_\Omega v dx = -\beta\int_\Omega v dx + \alpha\int_\Omega u dx \le -\beta\int_\Omega v dx + \alpha M_1 \quad \forall t \in (0, T_{\max}),$$

which implies (2.6) by an ODE comparison. We multiply the second equation in (1.1) by $-\Delta v$ and integrate by parts over Ω to derive that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \int_\Omega |\Delta v|^2 dx + \beta\int_\Omega |\nabla v|^2 dx \\ &= -\alpha\int_\Omega u\Delta v \\ &\le \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \frac{\alpha^2}{2}\int_\Omega u^2 dx \quad \forall t \in (0, T_{\max}). \end{aligned} \tag{2.10}$$

Letting $y(t) := \int_\Omega |\nabla v|^2 dx$ and $h(t) := \alpha^2\int_\Omega u^2 dx$ for $t \in (0, T_{\max})$, we have

$$y'(t) + 2\beta y(t) \le h(t) \quad \forall t \in (0, T_{\max}).$$

Since (2.3) holds for all $\eta \ge 2$, we have $\int_t^{t+\tau} h(s)ds \le C_1$ for any $t \in (0, T_{\max}-\tau)$ with some constant $C_1 > 0$ and $\tau = \min\{1, \frac{1}{2}T_{\max}\}$. Thus, according to Lemma 2.3, we establish (2.7). Taking a similar procedure for w, we can also establish (2.8) and (2.9). □

3. Global existence and boundedness of classical solutions

In this section, we investigate the global existence and boundedness of classical solutions to system (1.1). In Section 3.1, we establish the global existence for the strong logistic source case (i.e., $\eta \ge 2$ and $\mu > \max\{1+4\chi^2+11\alpha^2+2\chi^2(9+\alpha^2+\gamma^2), 1+4\xi^2+11\gamma^2+2\xi^2(9+\alpha^2+\gamma^2)\}$). In Section 3.2, we establish it for the sufficiently strong diffusion case (i.e., $m > \frac{4}{3}$). The key idea is to deduce a uniform estimate for $\|u(\cdot,t)\|_{L^p(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} + \|\nabla w(\cdot,t)\|_{L^q(\Omega)}$ for sufficiently large p and q.

A priori estimates for the strong logistic source case. Inspired by Tao and Winkler [24], we first establish an estimate for an appropriate linear combination of the functions $\int_\Omega (u+1)^{m+1}dx$, $\int_\Omega |\nabla v|^4 dx$, $\int_\Omega |\nabla w|^4 dx$, $\int_\Omega u|\nabla v|^2 dx$ and $\int_\Omega u|\nabla w|^2 dx$. As a beginning, let us derive differential inequalities for $\int_\Omega (u+1)^{m+1}dx$, $\int_\Omega |\nabla v|^4 dx$ and $\int_\Omega |\nabla w|^4 dx$.

Lemma 3.1. *Let $\eta \geq 2$ and the other assumptions in Lemma 2.1 hold. Then we have*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^{m+1}dx + \frac{m(m+1)}{2}\int_\Omega (u+1)^{2(m-1)}|\nabla u|^2 dx \\
&\leq \chi^2 m(m+1)\int_\Omega u^2|\nabla v|^2 dx + \xi^2 m(m+1)\int_\Omega u^2|\nabla w|^2 dx \\
&\quad + r(m+1)\int_\Omega (u+1)^{m+1}dx - \frac{\mu(m+1)}{2^\eta}\int_\Omega (u+1)^{m+\eta}dx \\
&\quad + \mu(m+1)\int_\Omega (u+1)^m dx \quad \forall t\in(0,T_{\max}).
\end{aligned} \tag{3.1}
$$

Proof. Multiplying the first equation in (1.1) by $(u+1)^m$ and integrating by parts over Ω, we obtain

$$
\begin{aligned}
&\frac{1}{m+1}\frac{d}{dt}\int_\Omega (u+1)^{m+1}dx \\
&= -m\int_\Omega (u+1)^{2(m-1)}|\nabla u|^2 dx + \chi m\int_\Omega u(u+1)^{m-1}\nabla v\cdot\nabla u dx \\
&\quad - \xi m\int_\Omega u(u+1)^{m-1}\nabla w\cdot\nabla u dx + r\int_\Omega u(u+1)^m dx - \mu\int_\Omega u^\eta (u+1)^m dx \\
&\leq -\frac{m}{2}\int_\Omega (u+1)^{2(m-1)}|\nabla u|^2 dx + \chi^2 m\int_\Omega u^2|\nabla v|^2 dx \\
&\quad + \xi^2 m\int_\Omega u^2|\nabla w|^2 dx + r\int_\Omega (u+1)^{m+1}dx - \mu\int_\Omega u^\eta (u+1)^m dx
\end{aligned}
$$

for all $t\in(0,T_{\max})$. Due to the nonnegativity of u, we have $(u+1)^\eta \leq 2^\eta(u^\eta+1)$ and then obtain $u^\eta \geq \frac{1}{2^\eta}(u+1)^\eta - 1$. Inserting it into the above inequality, we obtain the desired result (3.1). □

Lemma 3.2. *Let the assumptions in Lemma 3.1 hold. Then we have*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^4 dx + \frac{3}{2}\int_\Omega |\nabla|\nabla v|^2|^2 dx \\
&\leq 2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dx + 11\alpha^2\int_\Omega u^2|\nabla v|^2 dx, \quad \forall t\in(0,T_{\max}),
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla w|^4 dx + \frac{3}{2}\int_\Omega |\nabla|\nabla w|^2|^2 dx \\
&\leq 2\int_{\partial\Omega} |\nabla w|^2\frac{\partial|\nabla w|^2}{\partial\nu}dx + 11\gamma^2\int_\Omega u^2|\nabla w|^2 dx, \quad \forall t\in(0,T_{\max}).
\end{aligned} \tag{3.3}
$$

Proof. Differentiating the second equation in (1.1) and then multiplying $2\nabla v$, we obtain that

$$
(|\nabla v|^2)_t = \Delta|\nabla v|^2 - 2|D^2 v|^2 - 2\beta|\nabla v|^2 + 2\alpha\nabla u\cdot\nabla v \quad \forall t\in(0,T_{\max}), \tag{3.4}
$$

where we have used the identity $\Delta|\nabla v|^2 = 2\nabla v\cdot\nabla\Delta v + 2|D^2v|^2$. Testing (3.4) against $2|\nabla v|^2$ yields

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^4 dx &= 2\int_\Omega \Delta|\nabla v|^2|\nabla v|^2 dx - 4\int_\Omega |D^2v|^2|\nabla v|^2 dx\\
&\quad - 4\beta\int_\Omega |\nabla v|^4 dx + 4\alpha\int_\Omega (\nabla u\cdot\nabla v)|\nabla v|^2 dx\\
&= 2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} dx - 2\int_\Omega |\nabla|\nabla v|^2|^2 dx\\
&\quad - 4\int_\Omega |D^2v|^2|\nabla v|^2 dx - 4\beta\int_\Omega |\nabla v|^4 dx\\
&\quad - 4\alpha\int_\Omega u\Delta v|\nabla v|^2 dx - 4\alpha\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2 dx
\end{aligned} \tag{3.5}$$

for all $t\in(0,T_{\rm max})$. Since $|\Delta v|\le\sqrt{3}|D^2v|$, by Young's inequality we have

$$\begin{aligned}
-4\alpha\int_\Omega u\Delta v|\nabla v|^2 dx &\le 4\sqrt{3}\alpha\int_\Omega u|D^2v||\nabla v|^2 dx\\
&\le 4\int_\Omega |D^2v|^2|\nabla v|^2 dx + 3\alpha^2\int_\Omega u^2|\nabla v|^2 dx
\end{aligned} \tag{3.6}$$

and

$$-4\alpha\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2 dx \le \frac12\int_\Omega |\nabla|\nabla v|^2|^2 dx + 8\alpha^2\int_\Omega u^2|\nabla v|^2 dx \tag{3.7}$$

for all $t\in(0,T_{\rm max})$. Inserting (3.6) and (3.7) into (3.5), we infer that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^4 dx &\le 2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} dx - \frac32\int_\Omega |\nabla|\nabla v|^2|^2 dx\\
&\quad + 11\alpha^2\int_\Omega u^2|\nabla v|^2 dx - 4\beta\int_\Omega |\nabla v|^4 dx\\
&\le 2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} dx - \frac32\int_\Omega |\nabla|\nabla v|^2|^2 dx\\
&\quad + 11\alpha^2\int_\Omega u^2|\nabla v|^2 dx \quad \forall t\in(0,T_{\rm max}).
\end{aligned} \tag{3.8}$$

Thus, we obtain (3.2). Taking a same procedure for the third equation in (1.1), we can derive (3.3). □

To deal with the integrals $\int_\Omega u^2|\nabla v|^2 dx$ and $\int_\Omega u^2|\nabla w|^2 dx$ on the right-hand side of (3.1)-(3.3), we shall establish the following differential inequality related to them.

Lemma 3.3. *Let the assumptions in Lemma 3.1 hold. Then we have*

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega u|\nabla v|^2 dx + \int_\Omega u|\nabla w|^2 dx\Big\}\\
&+ (\mu-1-4\chi^2)\int_\Omega u^2|\nabla v|^2 dx + (\mu-1-4\xi^2)\int_\Omega u^2|\nabla w|^2 dx\\
&\le \frac12\int_\Omega |\nabla|\nabla v|^2|^2 dx + \frac12\int_\Omega |\nabla|\nabla w|^2|^2 dx
\end{aligned}$$

$$+ (8+\alpha^2+\gamma^2)\int_\Omega (u+1)^{2(m-1)}|\nabla u|^2dx + (r-2\beta)\int_\Omega u|\nabla v|^2dx$$
$$+ (r-2\delta)\int_\Omega u|\nabla w|^2dx + \int_{\partial\Omega} u\frac{\partial|\nabla v|^2}{\partial\nu}dx + \int_{\partial\Omega} u\frac{\partial|\nabla w|^2}{\partial\nu}dx + C \tag{3.9}$$

for all $t\in(0,T_{\max})$, where C is a positive constant.

Proof. By employing the identity $\nabla v\cdot\nabla\Delta v = \frac{1}{2}\Delta|\nabla v|^2 - |D^2v|^2$, we take a straightforward computation on $\frac{d}{dt}\int_\Omega u|\nabla v|^2dx$ to obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u|\nabla v|^2dx\\
&= \int_\Omega |\nabla v|^2\left\{\nabla\cdot((u+1)^{m-1}\nabla u - \chi u\nabla v + \xi u\nabla w) + ru - \mu u^\eta\right\}dx\\
&\quad + 2\int_\Omega u\nabla v\cdot\nabla(\Delta v - \beta v + \alpha u)dx\\
&= -\int_\Omega (u+1)^{m-1}\nabla u\cdot\nabla|\nabla v|^2dx + \chi\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2dx\\
&\quad - \xi\int_\Omega u\nabla w\cdot\nabla|\nabla v|^2dx - \mu\int_\Omega u^\eta|\nabla v|^2dx\\
&\quad + 2\int_\Omega u(\frac{1}{2}\Delta|\nabla v|^2 - |D^2v|^2)dx + (r-2\beta)\int_\Omega u|\nabla v|^2dx\\
&\quad + 2\alpha\int_\Omega u\nabla v\cdot\nabla u dx
\end{aligned} \tag{3.10}$$

for all $t\in(0,T_{\max})$. We now estimate the integrals on the right of (3.10). By Young's inequality, we have

$$\begin{aligned}
&-\int_\Omega (u+1)^{m-1}\nabla u\cdot\nabla|\nabla v|^2dx\\
&\le \frac{1}{8}\int_\Omega \big|\nabla|\nabla v|^2\big|^2dx + 2\int_\Omega (u+1)^{2(m-1)}|\nabla u|^2dx,
\end{aligned} \tag{3.11}$$

$$\chi\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2dx \le \frac{1}{8}\int_\Omega \big|\nabla|\nabla v|^2\big|^2dx + 2\chi^2\int_\Omega u^2|\nabla v|^2dx, \tag{3.12}$$

$$-\xi\int_\Omega u\nabla w\cdot\nabla|\nabla v|^2dx \le \frac{1}{8}\int_\Omega \big|\nabla|\nabla v|^2\big|^2dx + 2\xi^2\int_\Omega u^2|\nabla w|^2dx, \tag{3.13}$$

$$-\mu\int_\Omega u^\eta|\nabla v|^2dx \le \mu\int_\Omega |\nabla v|^2dx - \mu\int_\Omega u^2|\nabla v|^2dx, \tag{3.14}$$

$$\begin{aligned}
&2\int_\Omega u(\frac{1}{2}\Delta|\nabla v|^2 - |D^2v|^2)dx\\
&= \int_\Omega u\Delta|\nabla v|^2dx - 2\int_\Omega u|D^2v|^2dx\\
&= \int_{\partial\Omega} u\frac{\partial|\nabla v|^2}{\partial\nu}dx - \int_\Omega \nabla|\nabla v|^2\cdot\nabla u dx - 2\int_\Omega u|D^2v|^2dx\\
&\le \int_{\partial\Omega} u\frac{\partial|\nabla v|^2}{\partial\nu}dx + \frac{1}{8}\int_\Omega \big|\nabla|\nabla v|^2\big|^2dx + 2\int_\Omega |\nabla u|^2dx,
\end{aligned} \tag{3.15}$$

$$2\alpha\int_\Omega u\nabla v\cdot\nabla u dx \le \int_\Omega u^2|\nabla v|^2dx + \alpha^2\int_\Omega |\nabla u|^2dx \tag{3.16}$$

for all $t \in (0, T_{\max})$. Substituting (3.11)-(3.16) into (3.10) and using (2.7), we obtain that there exists a positive constant C_1 satisfying

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u|\nabla v|^2 dx + (\mu - 1 - 2\chi^2)\int_\Omega u^2|\nabla v|^2 dx \\ &\le \frac{1}{2}\int_\Omega \big|\nabla|\nabla v|^2\big|^2 dx + (4+\alpha^2)\int_\Omega (u+1)^{2(m-1)}|\nabla u|^2 dx \\ &\quad + 2\xi^2\int_\Omega u^2|\nabla w|^2 dx + (r-2\beta)\int_\Omega u|\nabla v|^2 dx + \int_{\partial\Omega} u\frac{\partial|\nabla v|^2}{\partial\nu}dx + C_1 \end{aligned} \tag{3.17}$$

for all $t \in (0, T_{\max})$. By taking a similar procedure, we can derive that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u|\nabla w|^2 dx + (\mu - 1 - 2\xi^2)\int_\Omega u^2|\nabla w|^2 dx \\ &\le \frac{1}{2}\int_\Omega \big|\nabla|\nabla w|^2\big|^2 dx + (4+\gamma^2)\int_\Omega (u+1)^{2(m-1)}|\nabla u|^2 dx \\ &\quad + 2\chi^2\int_\Omega u^2|\nabla v|^2 dx + (r-2\delta)\int_\Omega u|\nabla w|^2 dx + \int_{\partial\Omega} u\frac{\partial|\nabla w|^2}{\partial\nu}dx + C_2 \end{aligned} \tag{3.18}$$

with a constant $C_2 > 0$ for all $t \in (0, T_{\max})$. Adding (3.18) to (3.17) yields (3.9). □

Multiplying (3.1) by $\frac{2(9+\alpha^2+\gamma^2)}{m(m+1)}$ and then combining (3.2), (3.3) and (3.9), we obtain the following result.

Corollary 3.4. *Let the assumptions in Lemma 3.1 hold. Then we have*

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{2(9+\alpha^2+\gamma^2)}{m(m+1)}\int_\Omega (u+1)^{m+1}dx + \int_\Omega |\nabla v|^4 dx + \int_\Omega |\nabla w|^4 dx \\ &+ \int_\Omega u|\nabla v|^2 dx + \int_\Omega u|\nabla w|^2 dx\Big\} + \int_\Omega \big|\nabla|\nabla v|^2\big|^2 dx + \int_\Omega \big|\nabla|\nabla w|^2\big|^2 dx \\ &+ [\mu - 1 - 4\chi^2 - 11\alpha^2 - 2\chi^2(9+\alpha^2+\gamma^2)]\int_\Omega u^2|\nabla v|^2 dx \\ &+ [\mu - 1 - 4\xi^2 - 11\gamma^2 - 2\xi^2(9+\alpha^2+\gamma^2)]\int_\Omega u^2|\nabla w|^2 dx \\ &+ \int_\Omega (u+1)^{2(m-1)}|\nabla u|^2 dx + \frac{2\mu(9+\alpha^2+\gamma^2)}{2^\eta m}\int_\Omega (u+1)^{m+\eta}dx \\ &\le (r-2\beta)\int_\Omega u|\nabla v|^2 dx + (r-2\delta)\int_\Omega u|\nabla w|^2 dx \\ &\quad + \frac{2(9+\alpha^2+\gamma^2)}{m}\Big(r\int_\Omega (u+1)^{m+1}dx + \mu\int_\Omega (u+1)^m dx\Big) \\ &\quad + \int_{\partial\Omega} u\frac{\partial|\nabla v|^2}{\partial\nu}dx + \int_{\partial\Omega} u\frac{\partial|\nabla w|^2}{\partial\nu}dx + 2\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dx \\ &\quad + 2\int_{\partial\Omega}|\nabla w|^2\frac{\partial|\nabla w|^2}{\partial\nu}dx + C \end{aligned} \tag{3.19}$$

for all $t \in (0, T_{\max})$, where C is a positive constant.

We now are in a position to establish the estimates on $\int_\Omega u^{m+1}dx$, $\int_\Omega |\nabla v|^4 dx$ and $\int_\Omega |\nabla w|^4 dx$. We will show that if μ is taken large enough, then these integrals are uniformly bounded for all $t \in (0, T_{max})$.

Lemma 3.5. *Suppose that* $\mu > \max\{1 + 4\chi^2 + 11\alpha^2 + 2\chi^2(9 + \alpha^2 + \gamma^2), 1 + 4\xi^2 + 11\gamma^2 + 2\xi^2(9 + \alpha^2 + \gamma^2)\}$. *Let* $\eta \geq 2$ *and let* (u, v, w) *be a solution to system* (1.1). *Then we can find a constant* $C > 0$ *such that*

$$\int_\Omega (u + 1)^{m+1} dx \leq C \quad \forall t \in (0, T_{\max}), \tag{3.20}$$

$$\int_\Omega |\nabla v|^4 dx \leq C \quad \forall t \in (0, T_{\max}), \tag{3.21}$$

$$\int_\Omega |\nabla w|^4 dx \leq C \quad \forall t \in (0, T_{\max}). \tag{3.22}$$

Proof. Let $y(t) := \frac{2(9+\alpha^2+\gamma^2)}{m(m+1)} \int_\Omega (u + 1)^{m+1} dx + \int_\Omega |\nabla v|^4 dx + \int_\Omega |\nabla w|^4 dx + \int_\Omega u|\nabla v|^2 dx + \int_\Omega u|\nabla w|^2 dx$. Since $\mu > \max\{1 + 4\chi^2 + 11\alpha^2 + 2\chi^2(9 + \alpha^2 + \gamma^2), 1 + 4\xi^2 + 11\gamma^2 + 2\xi^2(9 + \alpha^2 + \gamma^2)\}$, from Corollary 3.4 we have

$$\begin{aligned}
&y'(t) + y(t) + \int_\Omega \big|\nabla |\nabla v|^2\big|^2 dx + \int_\Omega \big|\nabla |\nabla w|^2\big|^2 dx \\
&+ \int_\Omega (u + 1)^{2(m-1)} |\nabla u|^2 dx + \frac{2\mu(9 + \alpha^2 + \gamma^2)}{2^\eta m} \int_\Omega (u + 1)^{m+\eta} dx \\
&\leq (r + 1 - 2\beta) \int_\Omega u|\nabla v|^2 dx + (r + 1 - 2\delta) \int_\Omega u|\nabla w|^2 dx \\
&\quad + \frac{2(9 + \alpha^2 + \gamma^2)[r(m + 1) + 1]}{m(m + 1)} \int_\Omega (u + 1)^{m+1} dx \\
&\quad + \int_{\partial\Omega} u \frac{\partial |\nabla v|^2}{\partial \nu} dx + \int_{\partial\Omega} u \frac{\partial |\nabla w|^2}{\partial \nu} dx + 2\int_{\partial\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dx \\
&\quad + 2\int_{\partial\Omega} |\nabla w|^2 \frac{\partial |\nabla w|^2}{\partial \nu} dx + \frac{2\mu(9 + \alpha^2 + \gamma^2)}{m} \int_\Omega (u + 1)^m dx \\
&\quad + \int_\Omega |\nabla v|^4 dx + \int_\Omega |\nabla w|^4 dx + C_0 \quad \forall t \in (0, T_{\max}),
\end{aligned} \tag{3.23}$$

where C_0 is a positive constant. We next use the dissipated quantities on the left of (3.23) to estimate all integrals on the right. By the Gagliardo-Nirenberg inequality, we obtain that

$$\begin{aligned}
\int_\Omega |\nabla v|^4 dx &= \||\nabla v|^2\|^2_{L^2(\Omega)} \\
&\leq C_1 \|\nabla |\nabla v|^2\|^{\frac{6}{5}}_{L^2(\Omega)} \||\nabla v|^2\|^{\frac{4}{5}}_{L^1(\Omega)} + C_1 \||\nabla v|^2\|^2_{L^1(\Omega)}
\end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max})$, with some constant $C_1 > 0$. From Lemma 2.4, we know that $\int_\Omega |\nabla v|^2 \leq C_2$ for all $t \in (0, T_{\max})$ with some $C_2 > 0$. Thus by Young's inequality we can find a constant $C_3 > 0$ such that

$$\int_\Omega |\nabla v|^4 dx \leq \frac{1}{8} \|\nabla |\nabla v|^2\|^2_{L^2(\Omega)} + C_3 \quad \forall t \in (0, T_{\max}). \tag{3.25}$$

Similarly, we can also have

$$\int_\Omega |\nabla w|^4 dx \leq \frac{1}{8} \|\nabla |\nabla w|^2\|^2_{L^2(\Omega)} + C_4 \quad \forall t \in (0, T_{\max}) \tag{3.26}$$

with some constant $C_4 > 0$. By Young's inequality, we obtain

$$(r+1-2\beta)\int_\Omega u|\nabla v|^2 dx \le \frac{(r+1-2\beta)^2}{4}\int_\Omega u^2 dx + \int_\Omega |\nabla v|^4 dx \tag{3.27}$$

and hence, upon (3.25) we derive that

$$\begin{aligned}
&(r+1-2\beta)\int_\Omega u|\nabla v|^2 dx + \int_\Omega |\nabla v|^4 dx \\
&\le \frac{(r+1-2\beta)^2}{4}\int_\Omega u^2 dx + 2\int_\Omega |\nabla v|^4 dx \\
&\le \frac{(r+1-2\beta)^2}{4}\int_\Omega u^2 dx + \frac{1}{4}\|\nabla|\nabla v|^2\|^2_{L^2(\Omega)} + 2C_3 \\
&\le \frac{\mu(9+\alpha^2+\gamma^2)}{2^{\eta+1}m}\int_\Omega (u+1)^{m+\eta} dx + \frac{1}{4}\|\nabla|\nabla v|^2\|^2_{L^2(\Omega)} + C_5
\end{aligned} \tag{3.28}$$

with some constant $C_5 > 0$ for all $t \in (0, T_{\max})$. Similarly, we can obtain

$$\begin{aligned}
&(r+1-2\delta)\int_\Omega u|\nabla w|^2 dx + \int_\Omega |\nabla w|^4 dx \\
&\le \frac{\mu(9+\alpha^2+\gamma^2)}{2^{\eta+1}m}\int_\Omega (u+1)^{m+\eta} dx + \frac{1}{4}\|\nabla|\nabla w|^2\|^2_{L^2(\Omega)} + C_6
\end{aligned} \tag{3.29}$$

with some $C_6 > 0$ for all $t \in (0, T_{\max})$. Using Young's inequality again, we obtain that there exist positive constants C_7 and C_8 fulfilling

$$\begin{aligned}
&\frac{2(9+\alpha^2+\gamma^2)[r(m+1)+1]}{m(m+1)}\int_\Omega (u+1)^{m+1} dx \\
&\le \frac{\mu(9+\alpha^2+\gamma^2)}{2^{\eta+1}m}\int_\Omega (u+1)^{m+\eta} dx + C_7,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&\frac{2\mu(9+\alpha^2+\gamma^2)}{m}\int_\Omega (u+1)^{m} dx \\
&\le \frac{\mu(9+\alpha^2+\gamma^2)}{2^{\eta+1}m}\int_\Omega (u+1)^{m+\eta} dx + C_8
\end{aligned} \tag{3.31}$$

for all $t \in (0, T_{\max})$. For the boundary integrals in (3.23), by taking similar arguments in [24, (3.20)] we obtain that there exist some positive constants C_9 and C_{10} fulfilling

$$\begin{aligned}
&\int_{\partial\Omega} u\frac{\partial|\nabla v|^2}{\partial\nu} dx + 2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} dx \\
&\le \frac{1}{4}\int_\Omega \big|\nabla|\nabla v|^2\big|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + C_9,
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
&\int_{\partial\Omega} u\frac{\partial|\nabla w|^2}{\partial\nu} dx + 2\int_{\partial\Omega} |\nabla w|^2\frac{\partial|\nabla w|^2}{\partial\nu} dx \\
&\le \frac{1}{4}\int_\Omega \big|\nabla|\nabla w|^2\big|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + C_{10}
\end{aligned} \tag{3.33}$$

for all $t \in (0, T_{\max})$. Substituting (3.28)-(3.33) into (3.23), we obtain that

$$\begin{aligned} &y'(t) + y(t) + \int_\Omega \big|\nabla|\nabla v|^2\big|^2 dx + \int_\Omega \big|\nabla|\nabla w|^2\big|^2 dx \\ &+ \int_\Omega (u+1)^{2(m-1)}|\nabla u|^2 dx + \frac{2\mu(9+\alpha^2+\gamma^2)}{2^\eta m}\int_\Omega (u+1)^{m+\eta}dx \\ &\le \frac{1}{2}\int_\Omega \big|\nabla|\nabla v|^2\big|^2 dx + \frac{1}{2}\int_\Omega \big|\nabla|\nabla w|^2\big|^2 dx \\ &\quad + \int_\Omega |\nabla u|^2 dx + \frac{2\mu(9+\alpha^2+\gamma^2)}{2^\eta m}\int_\Omega (u+1)^{m+\eta}dx + C_{11} \end{aligned} \tag{3.34}$$

for all $t \in (0, T_{\max})$, where $C_{11} := C_0 + \sum_{i=5}^{10} C_i$. Hence we have

$$y'(t) + y(t) \le C_{11} \quad \forall t \in (0, T_{\max}). \tag{3.35}$$

Thus, by Gronwall's inequality, we have

$$\begin{aligned} y(t) &= \frac{2(9+\alpha^2+\gamma^2)}{m(m+1)}\int_\Omega u^{m+1}dx + \int_\Omega |\nabla v|^4 dx + \int_\Omega |\nabla w|^4 dx \\ &\quad + \int_\Omega u|\nabla v|^2 dx + \int_\Omega u|\nabla w|^2 dx \\ &\le \max\{y(0), C_{11}\} \end{aligned}$$

for all $t \in (0, T_{\max})$, which implies (3.20), (3.21) and (3.22). □

We now use the L^{m+1} estimate for $u+1$ and the L^4 estimate for ∇v and ∇w from Lemma 3.5 to establish higher regularity estimates for (u, v, w).

Lemma 3.6. *Suppose that $\mu > \max\{1 + 4\chi^2 + 11\alpha^2 + 2\chi^2(9+\alpha^2+\gamma^2), 1 + 4\xi^2 + 11\gamma^2 + 2\xi^2(9+\alpha^2+\gamma^2)\}$. Let $\eta \ge 2$ and let (u, v, w) be a solution to system* (1.1)*. Then for all $p > \frac{3m-1}{2}$ and each $q > 1$ there exists a constant $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C \quad \forall t \in (0, T_{\max}), \tag{3.36}$$

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le C \quad \forall t \in (0, T_{\max}), \tag{3.37}$$

$$\|\nabla w(\cdot,t)\|_{L^q(\Omega)} \le C \quad \forall t \in (0, T_{\max}). \tag{3.38}$$

Proof. Testing the first equation in (1.1) against $(u+1)^{p-1}$ and then integrating by parts over Ω, we derive that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx \\ &= -(p-1)\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 dx + (p-1)\chi\int_\Omega u(u+1)^{p-2}\nabla v\cdot\nabla u dx \\ &\quad - (p-1)\xi\int_\Omega u(u+1)^{p-2}\nabla w\cdot\nabla u dx + r\int_\Omega u(u+1)^{p-1}dx \\ &\quad - \mu\int_\Omega u^\eta (u+1)^{p-1}dx \\ &\le -(p-1)\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 dx + (p-1)\chi\int_\Omega (u+1)^{p-1}|\nabla v||\nabla u|dx \\ &\quad + (p-1)\xi\int_\Omega (u+1)^{p-1}|\nabla w||\nabla u|dx + r\int_\Omega (u+1)^p dx \end{aligned}$$

$$
\begin{aligned}
&- \mu \int_\Omega u^\eta (u+1)^{p-1} dx \\
&\leq -\frac{p-1}{2} \int_\Omega (u+1)^{m+p-3} |\nabla u|^2 dx + (p-1)\chi^2 \int_\Omega (u+1)^{p-m+1} |\nabla v|^2 dx \\
&\quad + (p-1)\xi^2 \int_\Omega (u+1)^{p-m+1} |\nabla w|^2 dx + r \int_\Omega (u+1)^p dx \\
&\quad - \mu \int_\Omega u^\eta (u+1)^{p-1} dx, \quad \forall t \in (0, T_{\max}),
\end{aligned} \tag{3.39}
$$

where we have used Young's inequality in the last step. According to (3.21) and (3.22), we note that there exists a constant $C_1 > 0$ such that $\|\nabla v\|^2_{L^4(\Omega)} \leq C_1$, $\|\nabla w\|^2_{L^4(\Omega)} \leq C_1$ for all $t \in (0, T_{\max})$. Thus, by using Hölder's inequality and the inequality $-u^\eta \leq -\frac{(u+1)^\eta}{2^\eta} + 1$, we obtain that

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega (u+1)^p dx \\
&\leq -\frac{2p(p-1)}{(m+p-1)^2} \int_\Omega |\nabla (u+1)^{\frac{m+p-1}{2}}|^2 dx \\
&\quad + p(p-1)(\chi^2 + \xi^2) C_1 \|(u+1)^{p-m+1}\|_{L^2(\Omega)} \\
&\quad + pr \int_\Omega (u+1)^p dx - \frac{\mu p}{2^\eta} \int_\Omega (u+1)^{p+\eta-1} dx + \mu p \int_\Omega (u+1)^{p-1} dx
\end{aligned} \tag{3.40}
$$

for all $t \in (0, T_{\max})$. Using that $p > \frac{3m-1}{2}$ implies

$$
\frac{2(m+1)}{m+p-1} < \frac{4(p-m+1)}{m+p-1}.
$$

We note that

$$
\frac{4(p-m+1)}{m+p-1} < 6,
$$

because $6(m+p-1) - 4(p-m+1) = 10(m-1) + 2p > 0$. We may invoke the Gagliardo-Nirenberg inequality to estimate

$$
\begin{aligned}
\|(u+1)^{p-m+1}\|_{L^2(\Omega)} &= \|(u+1)^{\frac{m+p-1}{2}}\|^{\frac{2(p-m+1)}{m+p-1}}_{L^{\frac{4(p-m+1)}{m+p-1}}(\Omega)} \\
&\leq C_2 \|\nabla (u+1)^{\frac{m+p-1}{2}}\|^{\frac{2(p-m+1)}{m+p-1}\theta}_{L^2(\Omega)} \|(u+1)^{\frac{m+p-1}{2}}\|^{\frac{2(p-m+1)}{m+p-1}(1-\theta)}_{L^{\frac{2(m+1)}{m+p-1}}(\Omega)} \\
&\quad + C_2 \|(u+1)^{\frac{m+p-1}{2}}\|^{\frac{2(p-m+1)}{m+p-1}}_{L^{\frac{2(m+1)}{m+p-1}}(\Omega)}
\end{aligned}
$$

for all $t \in (0, T_{\max})$ with some $C_2 > 0$ and $\theta \in (0,1)$ determined by

$$
-\frac{3(m+p-1)}{4(p-m+1)} = (1 - \frac{3}{2})\theta - \frac{3(m+p-1)}{2(m+1)}(1-\theta);
$$

that is,

$$
\theta = \frac{\frac{3(m+p-1)}{2}\left(\frac{1}{m+1} - \frac{1}{2(p-m+1)}\right)}{\frac{3(m+p-1)}{2(m+1)} - \frac{1}{2}}.
$$

In view of (3.20), we obtain that there exists a constant $C_3 > 0$ such that

$$
\|(u+1)^{p-m+1}\|_{L^2(\Omega)} \leq C_3 \|\nabla (u+1)^{\frac{m+p-1}{2}}\|^{\frac{2(p-m+1)}{m+p-1}\theta}_{L^2(\Omega)} + C_3 \quad \forall t \in (0, T_{\max}). \tag{3.41}
$$

It is easy to check that

$$\frac{2(p-m+1)}{m+p-1}\theta = 2\frac{3p-\frac{9}{2}m+\frac{3}{2}}{3p+2m-4} < 2,$$

because $(3p+2m-4)-(3p-\frac{9}{2}m+\frac{3}{2}) = m+\frac{11}{2}(m-1) > 0$. Thus, by applying Young's inequality, we have

$$\begin{aligned} & p(p-1)(\chi^2+\xi^2)C_1\|(u+1)^{p-m+1}\|_{L^2(\Omega)} \\ & \leq \frac{2p(p-1)}{(m+p-1)^2}\int_\Omega |\nabla(u+1)^{\frac{m+p-1}{2}}|^2dx + C_4 \end{aligned} \tag{3.42}$$

for all $t \in (0, T_{\max})$ with some $C_4 > 0$. Substituting (3.42) into (3.40) yields that

$$\begin{aligned} & \frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (u+1)^p dx \\ & \leq (pr+1)\int_\Omega (u+1)^p dx - \frac{\mu p}{2^\eta}\int_\Omega (u+1)^{p+\eta-1}dx \\ & \quad + \mu p\int_\Omega (u+1)^{p-1}dx + C_4 \end{aligned} \tag{3.43}$$

for all $t \in (0, T_{\max})$. Using Young's inequality twice again, there exist some positive constants C_5 and C_6 such that

$$(pr+1)\int_\Omega (u+1)^p dx \leq \frac{\mu p}{2^{\eta+1}}\int_\Omega (u+1)^{p+\eta-1}dx + C_5 \quad \forall t \in (0, T_{\max}), \tag{3.44}$$

$$\mu p\int_\Omega (u+1)^{p-1}dx \leq \frac{\mu p}{2^{\eta+1}}\int_\Omega (u+1)^{p+\eta-1}dx + C_6 \quad \forall t \in (0, T_{\max}). \tag{3.45}$$

Substituting (3.44) and (3.45) into (3.43), we have

$$\frac{d}{dt}\int_\Omega (u+1)^p dx + \int_\Omega (u+1)^p dx \leq C_7 \quad \forall t \in (0, T_{\max})$$

with $C_7 := C_4 + C_5 + C_6$. We thus conclude that there exists $C_8 > 0$ such that

$$\int_\Omega (u+1)^p dx \leq C_8 \quad \forall t \in (0, T_{\max}),$$

which implies (3.36).

From the variation-of-constants representation of v, we have

$$v(\cdot,t) = e^{t(\Delta-\beta)}v_0 + \alpha\int_0^t e^{(t-s)(\Delta-\beta)}u(\cdot,s)ds \quad \forall t \in (0, T_{\max}),$$

where $(e^{t\Delta})_{t\geq 0}$ is the Neumann heat semigroup in Ω. Using the $L^p - L^q$ estimate for the Neumann heat semigroup, we can find $C_9 > 0$ such that

$$\begin{aligned} & \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \\ & \leq C_9\|\nabla v_0\|_{L^q(\Omega)} + C_2\int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\beta(t-s)}\|u(\cdot,s)\|_{L^p(\Omega)}ds \end{aligned} \tag{3.46}$$

for all $t \in (0, T_{\max})$. We note that $\|u(\cdot,t)\|_{L^p(\Omega)} \leq C_{10}$ with $C_{10} > 0$ for all $t \in (0, T_{\max})$ and $p > \frac{3m-1}{2}$. In particularly, if we take $p > 3$, then we have

$\frac{1}{2}+\frac{3}{2}(\frac{1}{p}-\frac{1}{q})<1$ for all $q>1$. Thus we can find a constant $C_{11}>0$ such that

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^q(\Omega)} &\le C_9\|\nabla v_0\|_{L^q(\Omega)}+C_2C_{10}\int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\beta(t-s)}ds \\ &\le C_{11}\quad \forall t\in(0,T_{\max}).\end{aligned} \tag{3.47}$$

Similarly, we have $\|\nabla w(\cdot,t)\|_{L^q(\Omega)}\le C_{12}$ for all $t\in(0,T_{\max})$ with some constant $C_{12}>0$. □

In the following subsection, we will deal with the case that the diffusion is strong enough (i.e., $m>\frac{4}{3}$). We can also establish the L^p estimate for u and L^{2q} estimate for ∇v and ∇w for all $p>1$ and $q>1$.

A priori estimates for the sufficiently strong diffusion case.

Lemma 3.7. *Suppose that $m>\frac{4}{3}$. Let (u,v,w) be a solution to system (1.1). Then for all $p>1$ and each $q>1$ there exists a constant $C>0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)}\le C\quad \forall t\in(0,T_{\max}), \tag{3.48}$$
$$\|\nabla v(\cdot,t)\|_{L^{2q}(\Omega)}\le C\quad \forall t\in(0,T_{\max}), \tag{3.49}$$
$$\|\nabla w(\cdot,t)\|_{L^{2q}(\Omega)}\le C\quad \forall t\in(0,T_{\max}). \tag{3.50}$$

Proof. Multiplying the first equation in (1.1) by $(u+1)^{p-1}(p>1)$ and integrating it over Ω and using Young's inequality, we obtain that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx\\ &\le -\frac{p-1}{2}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2dx+(p-1)\chi^2\int_\Omega (u+1)^{p-m+1}|\nabla v|^2dx\\ &\quad+(p-1)\xi^2\int_\Omega (u+1)^{p-m+1}|\nabla w|^2dx+r\int_\Omega (u+1)^p dx\\ &\quad-\mu\int_\Omega u^\eta (u+1)^{p-1}dx\end{aligned} \tag{3.51}$$

for all $t\in(0,T_{\max})$. Since $u^\eta\ge\frac{1}{2^\eta}(u+1)^\eta-1$ and $\eta>1$, by using Young's inequality we obtain

$$\begin{aligned}&r\int_\Omega (u+1)^p dx-\mu\int_\Omega u^\eta(u+1)^{p-1}dx\\ &\le r\int_\Omega (u+1)^pdx+\mu\int_\Omega (u+1)^{p-1}dx-\frac{\mu}{2^\eta}\int_\Omega (u+1)^{p+\eta-1}dx\\ &\le \frac{\mu}{2^{\eta+1}}\int_\Omega (u+1)^{p+\eta-1}dx+C_1+\frac{\mu}{2^{\eta+1}}\int_\Omega (u+1)^{p+\eta-1}dx+C_2\\ &\quad-\frac{\mu}{2^\eta}\int_\Omega (u+1)^{p+\eta-1}dx\\ &=C_1+C_2\quad \forall t\in(0,T_{\max})\end{aligned} \tag{3.52}$$

with some positive constants C_1 and C_2. Combining (3.51) with (3.52) yields that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p dx \le & -\frac{2(p-1)}{(m+p-1)^2}\int_\Omega |\nabla (u+1)^{\frac{m+p-1}{2}}|^2 dx \\ & + (p-1)\chi^2 \int_\Omega (u+1)^{p-m+1}|\nabla v|^2 dx \\ & + (p-1)\xi^2 \int_\Omega (u+1)^{p-m+1}|\nabla w|^2 dx + C_3\end{aligned} \tag{3.53}$$

for all $t \in (0, T_{\max})$, with $C_3 := C_1 + C_2$. In Lemma 3.2, we obtained the identity

$$(|\nabla v|^2)_t = \Delta|\nabla v|^2 - 2|D^2 v|^2 - 2\beta|\nabla v|^2 + 2\alpha\nabla u \cdot \nabla v \quad \forall t \in (0, T_{\max}).$$

Testing this against $|\nabla v|^{2q-2}$ yields

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx + \frac{4(q-1)}{q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2 dx + 2\int_\Omega |D^2 v|^2|\nabla v|^{2q-2} \\ &+ 2\beta\int_\Omega |\nabla v|^{2q} dx \\ &\le \int_{\partial\Omega} |\nabla v|^{2q-2}\frac{\partial|\nabla v|^2}{\partial\nu} dx + 2\alpha\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v dx \quad \forall t \in (0, T_{\max}),\end{aligned} \tag{3.54}$$

where we have used the identity

$$(q-1)\int_\Omega |\nabla v|^{2q-4}\big|\nabla|\nabla v|^2\big|^2 dx = \frac{4(q-1)}{q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2 dx.$$

For the first integral on the right of (3.54), from [3, (3.10)] we have

$$\int_{\partial\Omega} |\nabla v|^{2q-2}\frac{\partial|\nabla v|^2}{\partial\nu} dx \le \frac{q-1}{2q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2 dx + C_4 \quad \forall t \in (0, T_{\max}) \tag{3.55}$$

with some constant $C_4 > 0$. For the second integral on the right of (3.54), we integrate by parts over Ω and use Young's inequality to derive

$$\begin{aligned}&2\alpha\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v\, dx \\ &= -2\alpha(q-1)\int_\Omega u|\nabla v|^{2q-4}\nabla v\cdot\nabla|\nabla v|^2 dx - 2\alpha\int_\Omega u|\nabla v|^{2q-2}\Delta v dx \\ &\le \frac{q-1}{2}\int_\Omega |\nabla v|^{2q-4}\big|\nabla|\nabla v|^2\big|^2 dx + 2\alpha^2(q-1)\int_\Omega u^2|\nabla v|^{2q-2} dx \\ &\quad + \frac{2}{3}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2 dx + \frac{3\alpha^2}{2}\int_\Omega u^2|\nabla v|^{2q-2} dx\end{aligned} \tag{3.56}$$

for all $t \in (0, T_{\max})$. Upon the pointwise inequality $|\Delta v|^2 \le 3|D^2 v|^2$, we have

$$\frac{2}{3}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2 dx \le 2\int_\Omega |\nabla v|^{2q-2}|D^2 v|^2 dx \quad \forall t \in (0, T_{\max}).$$

We thus infer from (3.54)-(3.56) that

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx + \frac{3(q-1)}{2q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2 dx \\ &\le (2q - \frac{1}{2})\alpha^2\int_\Omega u^2|\nabla v|^{2q-2} dx + C_4\end{aligned} \tag{3.57}$$

for all $t \in (0, T_{\max})$. By taking a similar procedure for w, we derive that

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla w|^{2q}dx + \frac{3(q-1)}{2q^2}\int_\Omega \big|\nabla|\nabla w|^q\big|^2 dx \\ &\le (2q-\frac{1}{2})\gamma^2\int_\Omega u^2|\nabla w|^{2q-2}dx + C_5 \end{aligned} \tag{3.58}$$

for all $t \in (0, T_{\max})$, with some constant $C_5 > 0$. Thus, collecting (3.53), (3.55) and (3.56) yields that

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega (u+1)^p dx + \frac{1}{q}\int_\Omega |\nabla v|^{2q}dx + \frac{1}{q}\int_\Omega |\nabla w|^{2q}dx\Big\} \\ &+ \frac{2(p-1)}{(m+p-1)^2}\int_\Omega |\nabla (u+1)^{\frac{m+p-1}{2}}|^2 dx \\ &+ \frac{3(q-1)}{2q^2}\int_\Omega \big|\nabla|\nabla v|^q\big|^2 dx + \frac{3(q-1)}{2q^2}\int_\Omega \big|\nabla|\nabla w|^q\big|^2 dx \\ &\le (p-1)\chi^2\int_\Omega (u+1)^{p-m+1}|\nabla v|^2 dx + (p-1)\xi^2\int_\Omega (u+1)^{p-m+1}|\nabla w|^2 dx \\ &\quad + (2q-\frac{1}{2})\alpha^2\int_\Omega (u+1)^2|\nabla v|^{2q-2}dx \\ &\quad + (2q-\frac{1}{2})\gamma^2\int_\Omega (u+1)^2|\nabla w|^{2q-2}dx + C_6 \end{aligned} \tag{3.59}$$

for all $t \in (0, T_{\max})$ with $C_6 := C_4 + C_5$. Note that (3.59) is similar to [23, (3.14)] and our condition $m > \frac{4}{3}$ satisfies the condition $1-(m-1) < \frac{2}{N}$ ($N = 3$) in [23]. Thus, the remaining computation shall follow closely with minor modification. We omit it for brevity and easily establish (3.48)-(3.50). □

Proof of Theorem 1.1. Using [23, Lemma A.1] and Lemma 3.6 and Lemma 3.7, we obtain that there exists a positive constant $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \forall t \in (0, T_{\max}),$$

which together with the extensibility criterion (2.1) yields that $T_{\max} = +\infty$. By well-known arguments from parabolic regularity theory for the second and third equations in (1.1), we can find some constants $C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned} \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\le C_2 \quad \forall t > 0, \\ \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\le C_3 \quad \forall t > 0. \end{aligned}$$

Thus, we prove that (u, v, w) is a global bounded classical solution to (1.1). □

4. Asymptotic behavior

In this section, according to the ideas from [24, Section 5], we consider the large time behavior of (u, v, w) under the assumptions $r = 0$, $\eta \ge 2$ and $\mu > \max\big\{1 + 4\chi^2 + 11\alpha^2 + 2\chi^2(9 + \alpha^2 + \gamma^2),\ 1 + 4\xi^2 + 11\gamma^2 + 2\xi^2(9 + \alpha^2 + \gamma^2)\big\}$.

Lemma 4.1. *Let $r = 0$. Suppose that $\eta \ge 2$ and $\mu > \max\big\{1 + 4\chi^2 + 11\alpha^2 + 2\chi^2(9 + \alpha^2 + \gamma^2),\ 1 + 4\xi^2 + 11\gamma^2 + 2\xi^2(9 + \alpha^2 + \gamma^2)\big\}$. Then the solution of* (1.1) *satisfies*

$$\int_0^\infty \int_\Omega u^\eta(x,t)\,dx\,dt < \frac{1}{\mu}\int_\Omega u_0(x)dx \quad \forall t > 0 \tag{4.1}$$

and there exists a constant $C > 0$ *such that*

$$\int_\Omega u(x,t)dx \le \frac{C}{(t+1)^{\frac{1}{\eta-1}}} \quad \forall t > 0. \tag{4.2}$$

Proof. We integrate the first equation in (1.1) in space to obtain

$$\frac{d}{dt}\int_\Omega u(x,t)dx = -\mu \int_\Omega u^\eta(x,t)dx \quad \forall t > 0. \tag{4.3}$$

Integrating it with respect to t yields

$$\int_\Omega u(x,t)dx + \mu \int_0^t \int_\Omega u^\eta(x,t)\,dx\,dt \le \int_\Omega u_0(x)dx \quad \forall t > 0,$$

which implies (4.1). Using Hölder's inequality, we have

$$\int_\Omega udx \le \Big(\int_\Omega u^\eta dx\Big)^{\frac{1}{\eta}} |\Omega|^{1-\frac{1}{\eta}}$$

and then obtain $-\int_\Omega u^\eta dx \le -\frac{1}{|\Omega|^{\eta-1}}\big(\int_\Omega udx\big)^\eta$. Substituting it into (4.3) yields

$$\frac{d}{dt}\int_\Omega u(x,t)dx \le -\frac{\mu}{|\Omega|^{\eta-1}}\Big(\int_\Omega udx\Big)^\eta \quad \forall t > 0.$$

By integrating in time we have

$$\int_\Omega u(x,t)dx \le \Big(\frac{1}{\frac{1}{\big(\int_\Omega u_0(x)dx\big)^{\eta-1}} + \frac{\mu(\eta-1)}{|\Omega|^{\eta-1}}t}\Big)^{\frac{1}{\eta-1}} \quad \forall t > 0,$$

which implies (4.2). □

Lemma 4.2. *Let the assumptions in Lemma 4.1 hold. Suppose that* $\beta > \frac{1}{2(\eta-1)}$ *and* $\delta > \frac{1}{2(\eta-1)}$. *Then there exists* $C > 0$ *such that*

$$\int_\Omega v(x,t)dx \le \frac{C}{(t+1)^{\frac{1}{\eta-1}}} \quad \forall t > 0, \tag{4.4}$$

$$\int_\Omega w(x,t)dx \le \frac{C}{(t+1)^{\frac{1}{\eta-1}}} \quad \textit{for all } t > 0. \tag{4.5}$$

Proof. Integrating the second equation in (1.1) and letting $y(t) := \int_\Omega v(x,t)dx$, we see that there exists a constant $C_1 > 0$ such that

$$y'(t) = -\beta y(t) + \alpha \int_\Omega u(x,t)dx \le -\beta y(t) + \frac{\alpha C_1}{(t+1)^{\frac{1}{\eta-1}}} \quad \forall t > 0.$$

We define

$$C_2 := \max\Big\{2^{\frac{1}{\eta-1}}\int_\Omega v_0(x)dx,\ \frac{2^{1+\frac{1}{\eta-1}}\alpha C_1}{2\beta - \frac{1}{\eta-1}}\Big\},$$

$$\overline{y}(t) := \frac{C_2}{(t+2)^{\frac{1}{\eta-1}}} \quad \forall t > 0.$$

Then $\overline{y}(0) = \frac{C_2}{2^{\frac{1}{\eta-1}}} \ge \int_\Omega v_0(x)dx = y(0)$ and

$$\overline{y}'(t) + \beta\overline{y}(t) - \frac{\alpha C_1}{(t+1)^{\frac{1}{\eta-1}}}$$

$$
\begin{aligned}
&= -\frac{C_2}{(\eta-1)(t+2)^{1+\frac{1}{\eta-1}}} + \frac{\beta C_2}{(t+2)^{\frac{1}{\eta-1}}} - \frac{\alpha C_1}{(t+1)^{\frac{1}{\eta-1}}} \\
&= \frac{C_2}{(\eta-1)(t+2)^{\frac{1}{\eta-1}}}\big(\frac{1}{2} - \frac{1}{t+2}\big) \\
&\quad + \frac{1}{2(t+2)^{\frac{1}{\eta-1}}}\Big[(2\beta - \frac{1}{\eta-1})C_2 - 2\big(\frac{t+2}{t+1}\big)^{\frac{1}{\eta-1}}\alpha C_1\Big] \\
&\geq \frac{C_2}{(\eta-1)(t+2)^{\frac{1}{\eta-1}}}\big(\frac{1}{2} - \frac{1}{2}\big) + \frac{1}{2(t+2)^{\frac{1}{\eta-1}}}\big[(2\beta - \frac{1}{\eta-1})C_2 - 2^{1+\frac{1}{\eta-1}}\alpha C_1\big] \\
&\geq 0 \quad \forall t > 0,
\end{aligned}
$$

where we have used the assumption $\beta > \frac{1}{2(\eta-1)}$ and the definition of C_2. By comparison, we thus establish (4.4). By taking a similar procedure for w, we can also obtain (4.5). □

Lemma 4.3. *Let $\eta \geq 2$ and $\mu > \max\big\{1 + \chi^2 + 11\alpha^2 + 2\chi^2(9 + \alpha^2 + \gamma^2), 1 + \xi^2 + 11\gamma^2 + 2\xi^2(9 + \alpha^2 + \gamma^2)\big\}$. Then there exists $\theta \in (0,1)$ and a constant $C > 0$ such that*

$$
\|u\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leq C \quad \forall t > 1. \tag{4.6}
$$

Proof. We write the first equation in (1.1) in the form

$$
u_t - \nabla\cdot a(x,t,\nabla u) = b(x,t) \quad x \in \Omega,\ t > 0,
$$

where $a(x,t,\nabla u) := (u+1)^{m-1}\nabla u - h(x,t)$, $h(x,t) := \chi u\nabla v - \xi u\nabla w$ and $b(x,t) := ru(x,t) - \mu u^\eta(x,t)$ for $x \in \Omega$ and $t > 0$. Here we note that

$$
\begin{gathered}
a(x,t,\nabla u)\cdot\nabla u = (u+1)^{m-1}|\nabla u|^2 - h\cdot\nabla u \geq \frac{1}{2}|\nabla u|^2 - \frac{1}{2}|h|^2, \\
|a(x,t,\nabla u)| \leq C_1|\nabla u| + |h|
\end{gathered}
$$

in $\Omega \times (0,\infty)$ with $C_1 := (M+1)^{m-1}$, where we have used (1.3). According to Lemma 3.6, we obtain that h and b belong to $L^\infty((0,\infty); L^q(\Omega))$ for any $q \in (1,\infty)$. By parabolic Hölder regularity [19, Theorem 1.3], this implies (4.6). □

Proof of Theorem 1.4. Let us assume that the first claim in (1.4) does not hold. Then we can find a sequence $(t_j)_{j\in\mathbb{N}} \subset (1,\infty)$ and a constant $C_1 > 0$ such that $t_j \to \infty$ as $j \to \infty$ and

$$
\|u(\cdot,t_j)\|_{L^\infty(\Omega)} \geq C_1 \quad \forall j \in \mathbb{N}. \tag{4.7}
$$

In view of Lemma 4.3 and the Arzelà-Ascoli theorem, we see that $(u(\cdot,t_j))_{j\in\mathbb{N}}$ is relatively compact in $C^0(\overline{\Omega})$. We can extract a subsequence (still denoted by $(u(\cdot,t_j))_{j\in\mathbb{N}}$) such that

$$
u(\cdot,t_j) \to u_\infty \quad \text{in } L^\infty(\Omega) \quad \text{as } j \to \infty
$$

with a certain nonnegative $u_\infty \in C^0(\overline{\Omega})$. However, from Lemma 4.1 we can obtain that $u(\cdot,t) \to 0$ in $L^1(\Omega)$ as $t \to \infty$. Therefore, we have $u_\infty \equiv 0$, which contradicts (4.7). Thus, we prove $\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$. Upon Lemma 3.6 and Lemma 4.2, we can take a similar arguments to prove $\|v(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ and $\|w(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$. □

Acknowledgements. The author is very grateful to the referees for their detailed comments and valuable suggestions, which greatly improved the manuscript. This work was partially supported by the Natural Science Project of Sichuan Province Department of Education (No.16ZB0075).

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