

EXISTENCE OF SOLUTIONS TO BIHARMONIC SYSTEMS WITH SINGULAR NONLINEARITY

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ABSTRACT. In this article we prove the existence of positive solutions of non-linear singular biharmonic elliptic systems in smooth bounded domains, with coupling of the equations, under Navier boundary condition. Under some suitable assumptions on the nonlinearity, we prove a uniqueness result. The existence result is based on the Schauder's fixed point theorem.

1. INTRODUCTION

In this article, we study the existence and uniqueness of positive solutions to the biharmonic elliptic system

$$\begin{aligned}\Delta^2 u &= \frac{A(x)}{u^\alpha} + \frac{B(x)}{(u+v)^{r_1}} && \text{in } \Omega, \\ \Delta^2 v &= \frac{C(x)}{v^\beta} + \frac{D(x)}{(u+v)^{r_2}} && \text{in } \Omega, \\ u, v &> 0 && \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 && \text{on } \partial\Omega, \\ v = 0, \quad \Delta v &= 0 && \text{on } \partial\Omega,\end{aligned}\tag{1.1}$$

where Δ^2 is the biharmonic operator, α, β, r_1, r_2 are positive constants, $\Omega \subset \mathbb{R}^N$ ($1 \leq N$) is a smooth bounded domain and $A, B, C, D \in C(\bar{\Omega})$. The condition on the boundary is known as Navier boundary condition.

System (1.1) appears as a natural extension of the single singular problem

$$\begin{aligned}\Delta^2 u &= \frac{A(x)}{u^\alpha} && \text{in } \Omega, \\ u = 0, \quad \mathfrak{B}(u) &= 0 && \text{on } \partial\Omega,\end{aligned}\tag{1.2}$$

where $0 < \alpha < 1$, which has been considered, among other works, in [13] (when $\mathfrak{B}(u) = \Delta u$). The problem (1.2) was also studied under Dirichlet boundary condition (that is, when $\mathfrak{B}(u) = \partial_\nu u$), see [10]. In both references, the existence result was obtained by means of Schauder fixed point theorem. The study of singular elliptic problems is greatly justified in view of some basic aspects of mathematical

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research. They arise in several physical situations such as fluids, biological pattern formation and so on. As a physical illustration we describe briefly a practical problem which leads to a singular problem as it has been done in Fulks-Maybe [8].

The single fourth order elliptic equations arises in the study of traveling waves in suspension bridges [16]. In recent years, fourth order nonlinear differential equations have become increasingly popular due to their possible applications in the fields of image and signal processing, nuclear physics, and engineering, see e.g. [4, 5, 20]. The current knowledge of fourth order elliptic equations has considerably grown in recent times [9], but still it is not comparable to the stage of development of the theory concerning harmonic boundary value problems.

Scalar elliptic problems of the type

$$\Delta^2 u = h(x, u) \quad \text{in } \Omega, \quad (1.3)$$

where $\Omega \subset \mathbb{R}^N$, with appropriate boundary conditions have been studied by many authors. When $\Omega = \mathbb{R}^3$, $h(x, u) = -u^{-q}$, with $q > 0$, problem (1.3) is related to a fourth order analog of Yamabes equation. We refer to [5, 14, 17]. In both references the authors studied the existence and properties of solutions. When $\Omega = B_R$, the ball in \mathbb{R}^N of radius R centered at the origin, and $h(x, u) = \lambda \frac{f(x)}{(1-u)^q}$, problem (1.3) also arises in the study of MEMS. We refer to [3, 12], and references therein. For general domains, and $h(x, u) = \lambda f(u)$, where the nonlinearity f could be superlinear or singular, we refer to [6], where the regularity of the extremal solution of eigenvalue problem (1.3) is considered.

Elliptic systems of the type

$$\begin{aligned} \Delta^2 u + c\Delta u &= f(x, u, v) \quad \text{in } \Omega, \\ \Delta^2 v + c\Delta v &= g(x, u, v) \quad \text{in } \Omega, \end{aligned} \quad (1.4)$$

without singularity conditions or with appropriate singularity built into f and g , $c \geq 0$ and appropriate boundary conditions have been studied by the some authors, see [7, 15] and references therein. In [7] the author studied the existence result for problem (1.4), under Navier boundary condition, where

$$f(x, u, v) = \frac{A(x)}{u^\alpha} + \frac{B(x)}{(u+v)^{r_1}}, \quad g(x, u, v) = \frac{C(x)}{v^\beta} + \frac{D(x)}{(u+v)^{r_2}},$$

$1 \leq N \leq 3$ and $r_1 = r_2$. The author also proved a uniqueness result when $B(x) = D(x)$ and, if $A(x) = C(x) \equiv 0$, the assumption $B = D$ is not necessary to establish the uniqueness, see [7, Remark 3]. The main result is obtained by using a version of approximating process and Brouwer's fixed point theorem known as Galerking's method. In [15] the authors used degree theory to study problem (1.4) with $f(x, u, v) = f(u+v)$ and $g(x, u, v) = g(u+v)$, with Dirichlet boundary condition and without considering any singularities.

In [2], the authors studied the system

$$\begin{aligned} \Delta(|\Delta u_i|^{p-2} \Delta u_i) &= \lambda_i w_i(x) f_i(u_1, \dots, u_m), \quad \text{in } B_1, \\ u_i = \Delta u_i &= 0, \quad x \in \partial B_1, \quad i = 1, \dots, m, \end{aligned} \quad (1.5)$$

where B_1 is the unit ball in \mathbb{R}^2 centered at the origin, $p > 1$ and $m \geq 1$ are integers, w_i is radially symmetric, f_i is a positive continuous function and $f_i(y_1, \dots, y_m)$ may be singular at $y_i = 0$. Under suitable conditions, the authors discuss the existence, uniqueness and dependence of solutions on the parameters λ_i .

In this article we generalize the result by Hernandez and Choi [13] for the system case and, in cases $3 < N$, the result by Faria in [7]. We also obtain uniqueness results in some situations that were not considered in [7].

The existence of solutions for problem (1.1) is obtained since the functions A , B , C and D satisfy the assumptions

$$\begin{aligned} z_1 &= \max \left\{ \min_{x \in \bar{\Omega}} \{A(x)\}, \min_{x \in \bar{\Omega}} \{B(x)\} \right\} > 0, \\ z_2 &= \max \left\{ \min_{x \in \bar{\Omega}} \{C(x)\}, \min_{x \in \bar{\Omega}} \{D(x)\} \right\} > 0. \end{aligned} \quad (1.6)$$

Our main result concerning (1.1) is the following.

Theorem 1.1. *Assuming that $\alpha, \beta, r_1, r_2 \in (0, 1)$ and $A, B, C, D \in C(\bar{\Omega})$ are non-negative functions satisfying (1.6), there exists a classical solution $U = (u, v)$ of (1.1).*

By a (classical) solution of (1.1) we mean a pair of functions $U = (u, v) \in (C^4(\Omega) \cap C^3(\bar{\Omega}))^2$ satisfying the system (1.1).

Remark 1.2. In this paper we prove that if $U = (u, v)$ is a classical solution to problem (1.1), then there exists $\delta > 0$ (where δ depends on the sup norm $|(u, v)|_\infty = |u|_\infty + |v|_\infty$) so that $u \geq \delta\varphi_1$ and $v \geq \delta\varphi_1$, where $\varphi_1 > 0$ is the first eigenfunction of the negative Laplacian operator subject to zero Dirichlet boundary conditions.

Remark 1.3. If $1 \leq N \leq 3$, there exists a positive constant δ so that $u \geq \delta\varphi_1$ and $v \geq \delta\varphi_1$ for all classical solutions (u, v) of (1.1). Here δ does not depend on (u, v) , see Remark 5.1 in the Appendix.

Theorem 1.4. *If we assume that one of the following conditions is verified, then problem (1.1) has a unique solution.*

- (i) $B \equiv 0$;
- (ii) $D \equiv 0$;
- (iii) $r_1 = r_2 = r \in (0, 1)$ and $B \equiv D$;
- (iv) $r_1 = r_2 = r \in (0, 1)$ and $A = C = 0$;
- (v) $r_1 = r_2 = r \in (0, 1)$, $1 \leq N \leq 3$ and there exists a constant Γ such that

$$\frac{|B(x) - D(x)|}{\varphi_1^{r+1}(x)} < \Gamma \quad \text{and} \quad \frac{r\Gamma C_\Omega^2}{2\delta^{r+1}} < 1,$$

where δ is as in Remark 5.1 (see also Lemma 2.2), C_Ω is the best constant in Sobolev embedding $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ (see [19]).

In the prove of Theorem 1.1, one of the main results is to prove the existence of solutions to a family of approximate problems to problem (1.1). The proof of this result is based on the Schauder's fixed point theorem.

The organization of this article is the following. Section 2 contains the notation used, important lemmas that will be used, and the study of a family of approximate problems to problem (1.1). Section 3 is devoted to the proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.4. Section 5 is devoted to obtaining a priori estimates (in the L^∞ sense) on the classical solutions of (1.1), in the cases $1 \leq N \leq 3$.

2. NOTATION AND AUXILIARY RESULTS

In this section we collect useful results regarding problem (1.1). Let $(x_1, \dots, x_m), (y_1, \dots, y_m) \in \mathbb{R}^m$. We use $(x_1, \dots, x_m) \leq (y_1, \dots, y_m)$ to denote $x_i \leq y_i, i = 1, \dots, m$. Let φ_1 be the first eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$. Therefore, φ_1 satisfies

$$\begin{aligned} -\Delta\varphi_1 &= \lambda_1\varphi_1 \quad \text{in } \Omega, \\ \varphi_1 &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where λ_1 is the first eigenvalue of $(-\Delta)$ with zero Dirichlet boundary conditions.

It is well known that φ_1 has constant sign in Ω , so by suitable normalization we may assume $\varphi_1 > 0$ in Ω and $|\varphi_1|_\infty = 1$. From Hopf's lemma [18], there exists $\sigma > 0$ such that $-\frac{\partial\varphi_1}{\partial\eta} \geq \sigma$ for all $x \in \partial\Omega$, where η is the outer unit normal to $\partial\Omega$. Thus, $|\nabla\varphi_1(x)| \geq \sigma$ for all $x \in \partial\Omega$, and there exists $c > 0$ such that

$$c\delta_0(x) \leq \varphi_1(x) \leq \frac{1}{c}\delta_0(x), \tag{2.2}$$

where $\delta_0(x) = \text{dist}(x, \partial\Omega)$.

We denote by $G(\cdot, \cdot)$ the Green's function associated with the negative Laplacian operator subject to zero Dirichlet boundary conditions. It is known that G is non-negative. If $h \in C(\overline{\Omega})$, the problem

$$-\Delta w = h(x) \text{ in } \Omega, \quad w|_{\partial\Omega} = 0, \tag{2.3}$$

has solution

$$w(x) = \int_{\Omega} G(x, y)h(y)dy. \tag{2.4}$$

Now, let ϕ_0 be the function that satisfies

$$-\Delta\phi_0 = 1 \text{ in } \Omega, \quad \phi_0|_{\partial\Omega} = 0. \tag{2.5}$$

By the maximum principle we obtain $\phi_0(x) > 0$ in Ω . Therefore,

$$\begin{aligned} \varphi_1(x) &= \lambda_1 \int_{\Omega} G(x, y)\varphi_1(y)dy, \\ \phi_0(x) &= \int_{\Omega} G(x, y)dy, \end{aligned}$$

which, as a consequence of the normalization of φ_1 , leads to

$$\varphi_1 \leq \lambda_1\phi_0. \tag{2.6}$$

The next lemma, due Hernandez and Choi [13], gives an estimate which will be useful in proving our results.

Lemma 2.1. *Given $0 < \xi < 1$, there exists a constant $C = C(\xi) > 0$, such that for all $x \in \Omega$,*

$$\int_{\Omega} \frac{G(x, y)}{\varphi_1(y)^\xi} dy \leq C(\xi).$$

Now, for each $\epsilon \in (0, 1)$ fixed, consider the auxiliary problem

$$\begin{aligned} \Delta^2 u &= \frac{A(x)}{|u|^\alpha + \epsilon} + \frac{B(x)}{|u+v|^{r_1} + \epsilon} & \text{in } \Omega, \\ \Delta^2 v &= \frac{C(x)}{|v|^\beta + \epsilon} + \frac{D(x)}{|u+v|^{r_2} + \epsilon} & \text{in } \Omega, \\ u &= 0, \quad \Delta u = 0 & \text{on } \partial\Omega, \\ v &= 0, \quad \Delta v = 0 & \text{on } \partial\Omega, \end{aligned} \tag{2.7}$$

where Ω is a smooth domain in \mathbb{R}^N ($1 \leq N$), $A, B, C, D \in C(\overline{\Omega})$ are nonnegative functions satisfying (1.6) and $\alpha, \beta, r_1, r_2 \in (0, 1)$.

The following result will be used to assure us, under some suitable assumptions on the nonlinearity, the uniqueness result.

Lemma 2.2. *Suppose that (1.6) holds. Let (u_ϵ, v_ϵ) , $\epsilon \in (0, 1)$, a classical solution of (2.7). If there exists $K > 0$, independent of ϵ , such that $|(u_\epsilon, v_\epsilon)|_\infty = |u_\epsilon|_\infty + |v_\epsilon|_\infty \leq K$, then there exist positive constants δ_1 and δ_2 (independent of ϵ) such that*

$$(u_\epsilon, v_\epsilon) \geq (\delta_1 \varphi_1, \delta_2 \varphi_1).$$

Proof. Let (u_ϵ, v_ϵ) be a solution of (2.7), $\epsilon \in (0, 1)$, $K > 0$ such that $|(u_\epsilon, v_\epsilon)|_\infty = |u_\epsilon|_\infty + |v_\epsilon|_\infty \leq K$, $A_0 = \min_{x \in \overline{\Omega}} A(x)$, $B_0 = \min_{x \in \overline{\Omega}} B(x)$, $C_0 = \min_{x \in \overline{\Omega}} C(x)$, $D_0 = \min_{x \in \overline{\Omega}} D(x)$. Let $(\omega_1, \omega_2) = (u_\epsilon - \delta_1 \varphi_1, v_\epsilon - \delta_2 \varphi_1)$, where

$$\begin{aligned} 0 < \delta_1 &< \frac{1}{\lambda_1^2} \left(\frac{A_0}{K^\alpha + 1} + \frac{B_0}{(2K)^{r_1} + 1} \right), \\ 0 < \delta_2 &< \frac{1}{\lambda_1^2} \left(\frac{C_0}{K^\beta + 1} + \frac{D_0}{(2K)^{r_2} + 1} \right). \end{aligned}$$

] The choice of δ_1, δ_2 is always possible by (1.6). Then $(\Delta^2 \omega_1, \Delta^2 \omega_2) \geq (0, 0)$ in Ω , and $\omega_1 = \omega_2 = \Delta \omega_1 = \Delta \omega_2 = 0$ on $\partial\Omega$. By using the Maximum Principle, we obtain $(\omega_1, \omega_2) \geq (0, 0)$, and so

$$(u_\epsilon, v_\epsilon) \geq (\delta_1 \varphi_1, \delta_2 \varphi_1) \quad \text{in } \overline{\Omega}.$$

□

System (2.7) can be written as the system of equations

$$\begin{aligned} \Delta u + \lambda_1 w &= 0 & \text{in } \Omega, \\ \Delta w + \frac{1}{\lambda_1} \frac{A(x)}{|u|^\alpha + \epsilon} + \frac{1}{\lambda_1} \frac{B(x)}{|u+v|^{r_1} + \epsilon} &= 0 & \text{in } \Omega, \\ \Delta v + \lambda_1 z &= 0 & \text{in } \Omega, \\ \Delta z + \frac{1}{\lambda_1} \frac{C(x)}{|v|^\beta + \epsilon} + \frac{1}{\lambda_1} \frac{D(x)}{|u+v|^{r_2} + \epsilon} &= 0 & \text{in } \Omega, \\ u = v = w = z &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{2.8}$$

Let

$$\begin{aligned} \mathcal{A} &= \{(u, w, v, z) \in (C(\overline{\Omega}))^4 : (\tau_1 \varphi_1, \tau_1 \varphi_1, \tau_2 \varphi_1, \tau_2 \varphi_1) \\ &\leq (u, w, v, z) \leq (K_1, K_2, K_1, K_2)\}. \end{aligned}$$

Let $(u, w, v, z) \in \mathcal{A}$, define

$$T_\epsilon \begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} = \begin{pmatrix} \lambda_1 \int_\Omega G(x, y) w(y) dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{A(y)}{u(y)^{\alpha+\epsilon}} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{B(y)}{(u(y)+v(y))^{r_1+\epsilon}} dy \\ \lambda_1 \int_\Omega G(x, y) z(y) dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{C(y)}{v(y)^{\beta+\epsilon}} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{D(y)}{(u(y)+v(y))^{r_2+\epsilon}} dy \end{pmatrix}. \quad (2.9)$$

Lemma 2.3. *Suppose that (1.6) holds. There exist K_1, K_2, τ_1 and τ_2 such that T_ϵ maps \mathcal{A} into \mathcal{A} .*

Proof. Let A_0, B_0, C_0, D_0 be as in Lemma 2.2, let $C(\alpha), C(\beta), C(r_1), C(r_2)$ be as defined in Lemma 2.1, and define $A_\infty = \max_{x \in \bar{\Omega}} A(x)$, $B_\infty = \max_{x \in \bar{\Omega}} B(x)$, $C_\infty = \max_{x \in \bar{\Omega}} C(x)$, $D_\infty = \max_{x \in \bar{\Omega}} D(x)$, $m_0 = \max_{x \in \bar{\Omega}} \phi_0(x)$. Choose K_1 such that

$$\frac{A_0}{\lambda_1^2[(K_1^\alpha) + 1]} + \frac{B_0}{\lambda_1^2[(2K_1)^{r_1} + 1]} > \max \left\{ \frac{(2A_\infty m_0 C(\alpha))^{1/\alpha}}{K_1^{1/\alpha}}, \frac{(2B_\infty m_0 C(r_1))^{1/r_1}}{K_1^{1/r_1}} \right\},$$

$$\frac{C_0}{\lambda_1^2[(K_1^\beta) + 1]} + \frac{D_0}{\lambda_1^2[(2K_1)^{r_2} + 1]} > \max \left\{ \frac{(2C_\infty m_0 C(\beta))^{1/\beta}}{K_1^{1/\beta}}, \frac{(2D_\infty m_0 C(r_2))^{1/r_2}}{K_1^{1/r_2}} \right\}$$

which are always possible since $\alpha, \beta, r_1, r_2 \in (0, 1)$ and by (1.6). Now choose τ_1 and τ_2 such that

$$\frac{A_0}{\lambda_1^2[(K_1^\alpha) + 1]} + \frac{B_0}{\lambda_1^2[(2K_1)^{r_1} + 1]} > \tau_1 > \max \left\{ \frac{(2A_\infty m_0 C(\alpha))^{1/\alpha}}{K_1^{1/\alpha}}, \frac{(2B_\infty m_0 C(r_1))^{1/r_1}}{K_1^{1/r_1}} \right\},$$

$$\frac{C_0}{\lambda_1^2[(K_1^\beta) + 1]} + \frac{D_0}{\lambda_1^2[(2K_1)^{r_2} + 1]} > \tau_2 > \max \left\{ \frac{(2C_\infty m_0 C(\beta))^{1/\beta}}{K_1^{1/\beta}}, \frac{(2D_\infty m_0 C(r_2))^{1/r_2}}{K_1^{1/r_2}} \right\}.$$

Then

$$\frac{A_\infty m_0 C(\alpha)}{\tau_1^\alpha} + \frac{B_\infty m_0 C(r_1)}{\tau_1^{r_1}} < K_1,$$

$$\frac{C_\infty m_0 C(\beta)}{\tau_2^\beta} + \frac{D_\infty m_0 C(r_2)}{\tau_2^{r_2}} < K_1,$$

$$\tau_1 < \frac{A_0}{\lambda_1^2[(K_1^\alpha) + 1]} + \frac{B_0}{\lambda_1^2[(2K_1)^{r_1} + 1]}$$

$$\tau_2 < \frac{C_0}{\lambda_1^2[(K_1^\beta) + 1]} + \frac{D_0}{\lambda_1^2[(2K_1)^{r_2} + 1]}.$$

Finally choose

$$K_2 = \frac{K_1}{\lambda_1 m_0}.$$

With such choices of K_1, K_2, τ_1 and τ_2 in \mathcal{A} , we prove that T_ϵ maps \mathcal{A} into \mathcal{A} by the following calculations. Without loss of generality, we take $0 < \epsilon < 1$.

Step one. Let us obtain an estimate from below for $T_\epsilon(u, w, v, z)$.

$$\begin{aligned}
 T_\epsilon \begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} &= \begin{pmatrix} \lambda_1 \int_\Omega G(x, y)w(y)dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{A(y)}{u(y)^{\alpha+\epsilon}} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{B(y)}{(u(y)+v(y))^{r_1+\epsilon}} dy \\ \lambda_1 \int_\Omega G(x, y)z(y)dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{C(y)}{v(y)^{\beta+\epsilon}} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{D(y)}{(u(y)+v(y))^{r_2+\epsilon}} dy \end{pmatrix} \\
 &\geq \begin{pmatrix} \lambda_1 \tau_1 \int_\Omega G(x, y)\varphi_1(y)dy \\ \left(\frac{A_0}{\lambda_1[(K_1^\alpha)+1]} + \frac{B_0}{\lambda_1[(2K_1)^{r_1}+1]} \right) \int_\Omega G(x, y)dy \\ \lambda_1 \tau_2 \int_\Omega G(x, y)\varphi_1(y)dy \\ \left(\frac{C_0}{\lambda_1[(K_1^\beta)+1]} + \frac{D_0}{\lambda_1[(2K_1)^{r_2}+1]} \right) \int_\Omega G(x, y)dy \end{pmatrix} \\
 &\geq \begin{pmatrix} \lambda_1 \tau_1 \int_\Omega G(x, y)\varphi_1(y)dy \\ \left(\frac{A_0}{\lambda_1[(K_1^\alpha)+1]} + \frac{B_0}{\lambda_1[(2K_1)^{r_1}+1]} \right) \phi_0(x) \\ \lambda_1 \tau_2 \int_\Omega G(x, y)\varphi_1(y)dy \\ \left(\frac{C_0}{\lambda_1[(K_1^\beta)+1]} + \frac{D_0}{\lambda_1[(2K_1)^{r_2}+1]} \right) \phi_0(x) \end{pmatrix}.
 \end{aligned}$$

Using inequality (2.6), we obtain

$$T_\epsilon \begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} \geq \begin{pmatrix} \tau_1 \varphi_1 \\ \left(\frac{A_0}{\lambda_1^2[(K_1^\alpha)+1]} + \frac{B_0}{\lambda_1^2[(2K_1)^{r_1}+1]} \right) \varphi_1(x) \\ \tau_2 \varphi_1 \\ \left(\frac{C_0}{\lambda_1^2[(K_1^\beta)+1]} + \frac{D_0}{\lambda_1^2[(2K_1)^{r_2}+1]} \right) \varphi_1(x) \end{pmatrix} \geq \begin{pmatrix} \tau_1 \varphi_1 \\ \tau_1 \varphi_1 \\ \tau_2 \varphi_1 \\ \tau_2 \varphi_1 \end{pmatrix}.$$

Step two. Let us obtain an estimate from above for $T_\epsilon(u, w, v, z)$.

$$\begin{aligned}
 T_\epsilon \begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} &= \begin{pmatrix} \lambda_1 \int_\Omega G(x, y)w(y)dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{A(y)}{u(y)^{\alpha+\epsilon}} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{B(y)}{(u(y)+v(y))^{r_1+\epsilon}} dy \\ \lambda_1 \int_\Omega G(x, y)z(y)dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{C(y)}{v(y)^{\beta+\epsilon}} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{D(y)}{(u(y)+v(y))^{r_2+\epsilon}} dy \end{pmatrix} \\
 &\leq \begin{pmatrix} \lambda_1 K_2 \int_\Omega G(x, y)dy \\ \frac{A_\infty}{\lambda_1} \int_\Omega G(x, y) \frac{1}{(\tau_1 \varphi_1)^\alpha} dy + \frac{B_\infty}{\lambda_1} \int_\Omega G(x, y) \frac{1}{(\tau_1 + \tau_2)^{r_1} \varphi^{r_1}} dy \\ \lambda_1 K_2 \int_\Omega G(x, y)dy \\ \frac{C_\infty}{\lambda_1} \int_\Omega G(x, y) \frac{1}{(\tau_2 \varphi_1)^\beta} dy + \frac{D_\infty}{\lambda_1} \int_\Omega G(x, y) \frac{1}{(\tau_1 + \tau_2)^{r_2} \varphi^{r_2}} dy \end{pmatrix} \\
 &\leq \begin{pmatrix} \lambda_1 K_2 m_0 \\ \frac{A_\infty}{\lambda_1 \tau_1^\alpha} \int_\Omega G(x, y) \frac{1}{\varphi_1^\alpha} dy + \frac{B_\infty}{\tau_1 \lambda_1} \int_\Omega G(x, y) \frac{1}{\varphi^{r_1}} dy \\ \lambda_1 K_2 m_0 \\ \frac{C_\infty}{\lambda_1 \tau_2^\beta} \int_\Omega G(x, y) \frac{1}{\varphi_1^\beta} dy + \frac{D_\infty}{\lambda_1 \tau_2} \int_\Omega G(x, y) \frac{1}{\varphi^{r_2}} dy \end{pmatrix}.
 \end{aligned}$$

Using Lemma 2.1, we obtain

$$T_\epsilon \begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} \leq \begin{pmatrix} \lambda_1 K_2 m_0 \\ \frac{A_\infty C(\alpha)}{\lambda_1 \tau_1^\alpha} + \frac{B_\infty C(r_1)}{\lambda_1 \tau_1^{r_1}} \\ \lambda_1 K_2 m_0 \\ \frac{C_\infty C(\beta)}{\lambda_1 \tau_2^\beta} + \frac{D_\infty C(r_2)}{\lambda_1 \tau_2^{r_2}} \end{pmatrix} \leq \begin{pmatrix} \lambda_1 K_2 m_0 \\ \frac{K_1}{m_0 \lambda_1} \\ \lambda_1 K_2 m_0 \\ \frac{K_1}{m_0 \lambda_1} \end{pmatrix} \leq \begin{pmatrix} K_1 \\ K_2 \\ K_1 \\ K_2 \end{pmatrix}.$$

Thus T_ϵ maps \mathcal{A} into \mathcal{A} which complete the proof of Lemma 2.3. □

3. PROOF OF THEOREM 1.1

This proof will be done by means of Schauder fixed point theorem. By Lemma 2.3, we can define $T_\epsilon : \mathcal{A} \rightarrow \mathcal{A}$. Notice that \mathcal{A} is closed and convex. Now, we want to prove that the map T_ϵ is compact. In fact, let $(u \ w \ v \ z) \in \mathcal{A}$. Considering system (2.8), since

$$\Lambda = \begin{pmatrix} w \\ \frac{A}{|u|^{\alpha+\epsilon}} + \frac{B}{|u+v|^{r_1+\epsilon}} \\ z \\ \frac{C}{|v|^{\beta+\epsilon}} + \frac{D}{|u+v|^{r_2+\epsilon}} \end{pmatrix}$$

belongs to $(C(\bar{\Omega}))^4$, then $\Lambda \in (L^p(\Omega))^4$ for any $1 < p < \infty$. By using elliptic estimates [1], we obtain $T_\epsilon(u, w, v, z) \in (W^{2,p}(\Omega))^4$, for any $1 < p < \infty$. The Sobolev-Morrey's imbedding theorem entails $T_\epsilon(u, w, v, z) \in (C^{1+\rho}(\Omega))^4$ for any $0 < \rho < 1$. This implies that T_ϵ is compact.

Now, rely on Schauder's fixed point theorem we obtain the existence of a fixed point $(u_\epsilon, w_\epsilon, v_\epsilon, z_\epsilon) \in (C^{1+\rho}(\Omega))^4$ of T_ϵ . That is,

$$\begin{pmatrix} u_\epsilon \\ w_\epsilon \\ v_\epsilon \\ z_\epsilon \end{pmatrix} = \begin{pmatrix} \lambda_1 \int_\Omega G(x, y) w_\epsilon(y) dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{A(y)}{|u_\epsilon(y)|^{\alpha+\epsilon}} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{B(y)}{|u_\epsilon(y)+v_\epsilon(y)|^{r_1+\epsilon}} dy \\ \lambda_1 \int_\Omega G(x, y) z_\epsilon(y) dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{C(y)}{|v_\epsilon(y)|^{\beta+\epsilon}} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{D(y)}{|u_\epsilon(y)+v_\epsilon(y)|^{r_2+\epsilon}} dy \end{pmatrix}, \quad (3.1)$$

where $A, B, C, D \in C(\bar{\Omega})$. By bootstrap arguments we obtain $(u_\epsilon, v_\epsilon) \in (C^{4+\rho}(\bar{\Omega}))^2$ and $(w_\epsilon, z_\epsilon) \in (C^{2+\rho}(\bar{\Omega}))^2$. By compactness results, we can extract convergent subsequences in $C^{2+\tilde{\rho}}(\bar{\Omega})$, namely $(u_n), (w_n), (v_n), (z_n)$, of $(u_\epsilon), (w_\epsilon), (v_\epsilon), (z_\epsilon)$, respectively. Since $(u_n, w_n, v_n, z_n) \in \mathcal{A}$, there exist τ_1 and τ_2 , independent of n , such that

$$(\tau_1 \varphi_1, \tau_1 \varphi_1, \tau_2 \varphi_1, \tau_2 \varphi_1) \leq (u_n, w_n, v_n, z_n).$$

By Lemma 2.1, we have

$$\frac{|G(x, y)|}{\varphi_1^s} \in L^1(\Omega), \quad (3.2)$$

for all $s \in (0, 1)$. Using the Theorem of the Dominated Convergence in (3.1), we obtain

$$\begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} = \begin{pmatrix} \lambda_1 \int_\Omega G(x, y) w(y) dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{A(y)}{|u(y)|^\alpha} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{B(y)}{|u(y)+v(y)|^{r_1}} dy \\ \lambda_1 \int_\Omega G(x, y) z(y) dy \\ \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{C(y)}{|v(y)|^\beta} dy + \frac{1}{\lambda_1} \int_\Omega G(x, y) \frac{D(y)}{|u(y)+v(y)|^{r_2}} dy \end{pmatrix}. \quad (3.3)$$

Therefore, according to our construction we have a classical solution $(u, v) \in (C^2(\bar{\Omega}) \cap C^4(\Omega))^2$. To show that $(u, v) \in (C^3(\bar{\Omega}))^2$, we can follow similar ideas of [13], and this completes the proof of the existence.

4. PROOF OF THEOREM 1.4

Before starting the proof of the uniqueness, let us discuss the hypothesis (v). Note that if $B = D \pm \epsilon \Phi \varphi_1^r$ is so that $\Phi \in C_0^2(\bar{\Omega})$ is a solution of (4.1), then there

exists ε_0 such that $|B - D|$ satisfies the hypothesis (v) for all $\varepsilon \in (0, \varepsilon_0)$. In fact, given $f \in C^\infty(\overline{\Omega})$, let $\zeta \in C_0^2(\overline{\Omega})$ be the solution of

$$\begin{aligned} -\Delta\zeta &= f \quad \text{in } \Omega, \\ \zeta &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

By Calderón-Zygmund estimates (see [11]),

$$\|\zeta\|_{W^{2,p'}} \leq C\|f\|_{L^{p'}}.$$

Since $p' > N$, it follows from Morrey's imbedding that

$$\|\zeta/\delta_0\|_{L^\infty} \leq C(\|\zeta\|_{L^\infty} + \|\nabla\zeta\|_{L^\infty}) \leq C\|\zeta\|_{W^{2,p'}}$$

where $\delta_0(x) = \text{dist}(x, \partial\Omega)$. By using (2.2), there exists $\varepsilon_0 > 0$ such that $|B - D|$ satisfies the hypothesis (v) for all $\varepsilon \in (0, \varepsilon_0)$.

Let $\delta = \min\{\delta_1, \delta_2\}$, where δ_1, δ_2 are given in Lemma 2.2 (see also Remark 5.1).

Assume condition (i) holds. In this proof we adapt arguments used in [13] as follows. Let $U = (u, v)$ and $\widehat{U} = (\widehat{u}, \widehat{v})$ be two classical solutions to problem (1.1). By [13] we obtain that $u = \widehat{u}$. Let $\delta > 0$ be such that $u, v, \widehat{v} \geq \delta\varphi_1$ in Ω . Define $z_2 = v - \widehat{v}$. By the Mean Value Theorem, we arrive at

$$\Delta^2 z_2 = -\beta \frac{C(x)}{\widetilde{v}^{\beta+1}} z_2 - r_2 \frac{D(x)}{\widetilde{u+v}^{r_2+1}} z_2,$$

where $\widetilde{v}, \widetilde{u+v} \geq \delta\varphi_1$ in Ω . We then multiply the previous equations by z_2 , respectively, and integrate over a smooth domain Ω_1 compactly contained in Ω . After applying the Divergence Theorem, we obtain

$$\begin{aligned} &\int_{\Omega_1} [(\Delta z_2)^2 + \beta \frac{C(x)}{\widetilde{v}^{\beta+1}} z_2^2 + r_2 \frac{D(x)}{\widetilde{u+v}^{r_2+1}} z_2^2] dx \\ &= \int_{\partial\Omega_1} [\Delta z_2 \frac{\partial z_2}{\partial \nu}] ds - \int_{\partial\Omega_1} [z_2 \frac{\partial \Delta z_2}{\partial \nu}] ds, \end{aligned}$$

where ν is the unit outward normal on $\partial\Omega$.

When $\Omega_1 \rightarrow \Omega$, the right-hand side of the equation vanishes. Since $\int_{\Omega} (\Delta z_2)^2 dx < \infty$, we have

$$\int_{\Omega} [\beta \frac{C(x)}{\widetilde{v}^{\beta+1}} z_2^2 + r_2 \frac{D(x)}{\widetilde{u+v}^{r_2+1}} z_2^2] dx$$

is well defined. Hence

$$\int_{\Omega} [(\Delta z_2)^2 + \beta \frac{C(x)}{\widetilde{v}^{\beta+1}} z_2^2 + r_2 \frac{D(x)}{\widetilde{u+v}^{r_2+1}} z_2^2] dx = 0$$

which implies that $z_2 = 0$ in Ω .

Assume condition (ii) holds. The proof is similar to (i).

Assume conditions (iii) or (iv) holds. We can follow the same idea used in [7].

Assume condition (v) holds. Define $u = u_1 - u_2$ and $v = v_1 - v_2$, where $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ are two classical solutions for problem (1.1). Hence $u_1, u_2 \geq \delta\varphi_1$ and $v_1, v_2 \geq \delta\varphi_1$, where δ does not depend of u_i and v_i , $i = 1, 2$ (see Remark 5.1). By the Mean Value Theorem,

$$\begin{aligned} \Delta^2 u &= -\alpha \frac{A(x)}{u^{\alpha+1}} u - r \frac{B(x)}{(u+v)^{r+1}} (u+v) \quad \text{in } \Omega, \\ \Delta^2 v &= -\beta \frac{C(x)}{v^{\beta+1}} v - r \frac{D(x)}{(u+v)^{r+1}} (u+v) \quad \text{in } \Omega, \end{aligned}$$

where $\bar{u} \geq \delta\varphi_1$, $\bar{v} \geq \delta\varphi_1$, $\overline{u+v} \geq \delta\varphi_1$ in Ω . We then multiply the previous equation by u and v , respectively, and integrate over a smooth domain Ω_1 compactly contained in Ω . After applying the Divergence Theorem twice, we obtain

$$\begin{aligned} & \int_{\Omega_1} [(\Delta u)^2 + \alpha \frac{A(x)}{\bar{u}^{\alpha+1}} u^2 + r \frac{B(x)}{(\bar{u} + \bar{v})^{r+1}} (u^2 + uv)] \\ &= \int_{\partial\Omega_1} [\Delta u \frac{\partial u}{\partial \eta}] ds - \int_{\partial\Omega_1} [u \frac{\partial \Delta u}{\partial \eta}] ds, \\ & \int_{\Omega_1} [(\Delta v)^2 + \beta \frac{C(x)}{\bar{v}^{\beta+1}} v^2 + r \frac{D(x)}{(\bar{u} + \bar{v})^{r+1}} (v^2 + uv)] \\ &= \int_{\partial\Omega_1} [\Delta v \frac{\partial v}{\partial \eta}] ds - \int_{\partial\Omega_1} [v \frac{\partial \Delta v}{\partial \eta}] ds, \end{aligned}$$

where η is the unit outward normal on $\partial\Omega$. By adding the last two equations and using the Holder's inequality, we obtain

$$\begin{aligned} & \int_{\Omega_1} [(\Delta u)^2 + (\Delta v)^2 + \alpha \frac{A(x)}{\bar{u}^{\alpha+1}} u^2 + \beta \frac{C(x)}{\bar{v}^{\beta+1}} v^2] dx \\ & \leq \frac{r}{2} \int_{\Omega_1} \frac{|B(x) - D(x)|}{(\bar{u} + \bar{v})^{r+1}} (u^2 + v^2) dx \\ & \quad + \int_{\partial\Omega_1} [\Delta u \frac{\partial u}{\partial \eta} - u \frac{\partial \Delta u}{\partial \eta} + \Delta v \frac{\partial v}{\partial \eta} - v \frac{\partial \Delta v}{\partial \eta}] ds \\ & \leq \frac{r\Gamma}{2\delta^{r+1}} \int_{\Omega_1} (u^2 + v^2) dx \\ & \quad + \int_{\partial\Omega_1} [\Delta u \frac{\partial u}{\partial \eta} - u \frac{\partial \Delta u}{\partial \eta} + \Delta v \frac{\partial v}{\partial \eta} - v \frac{\partial \Delta v}{\partial \eta}] ds. \end{aligned}$$

Taking to the limit as $\Omega_1 \rightarrow \Omega$ the right-hand sides of the equations approach

$$\frac{r\Gamma}{2\delta^{r+1}} \int_{\Omega} (u^2 + v^2) dx.$$

Since $\int_{\Omega} (\Delta u)^2 dx$, $\int_{\Omega} (\Delta v)^2 dx$ and $\frac{r\Gamma}{2} \int_{\Omega} (u^2 + v^2) dx < \infty$, it follows that

$$\int_{\Omega} [\alpha \frac{A(x)}{\bar{u}^{\alpha+1}} u^2 + \beta \frac{C(x)}{\bar{v}^{\beta+1}} v^2] dx$$

is well defined. Hence, by Sobolev embedding $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ (see [19]) we have

$$\int_{\Omega} [(\Delta u)^2 + (\Delta v)^2] dx \leq \frac{r\Gamma C_{\Omega}^2}{2\delta^{r+1}} \int_{\Omega} ((\Delta u)^2 + (\Delta v)^2) dx.$$

Therefore,

$$(1 - \frac{r\Gamma C_{\Omega}^2}{2\delta^{r+1}}) \int_{\Omega} [(\Delta u)^2 + (\Delta v)^2] dx \leq 0$$

Since $(1 - \frac{r\Gamma C_{\Omega}^2}{2\delta^{r+1}}) > 0$, we conclude that $u = v = 0$ in Ω .

5. APPENDIX

A priori estimates. This section is devoted to prove a priori estimate for a classical solution of equation (1.1), in the case $1 \leq N \leq 3$.

Let $U = (u, v) \in (C^4(\Omega) \cap C^3(\bar{\Omega}))^2$ ($u, v > 0$ in Ω) be a classical solution for system (1.1). Multiplying the first equation of (1.1) by u and integrating by parts in Ω , we have

$$\int_{\Omega} (\Delta u)^2 dx \leq A_{\infty} \int_{\Omega} u^{1-\alpha} dx + B_{\infty} \int_{\Omega} u^{1-r_1} dx. \quad (5.1)$$

Since $\frac{2}{1-\alpha}, \frac{2}{1-r_1} > 1$ and $u \in L^p(\Omega)$, for each $p \geq 1$, by the Young's inequality we obtain, for each $\epsilon > 0$, that

$$\begin{aligned} \int_{\Omega} u^{1-\alpha} dx &\leq \epsilon \int_{\Omega} u^2 dx + \frac{|\Omega|}{\frac{2}{1+\alpha} \left(\frac{2}{1-\alpha}\right)^{\frac{1-\alpha}{1+\alpha}} \epsilon^{\frac{1-\alpha}{1+\alpha}}}, \\ \int_{\Omega} u^{1-r_1} dx &\leq \epsilon \int_{\Omega} u^2 dx + \frac{|\Omega|}{\frac{2}{1+r_1} \left(\frac{2}{1-r_1}\right)^{\frac{1-r_1}{1+r_1}} \epsilon^{\frac{1-r_1}{1+r_1}}}. \end{aligned}$$

Therefore,

$$\int_{\Omega} (\Delta u)^2 dx \leq 2\epsilon \max\{A_{\infty}, B_{\infty}\} \int_{\Omega} u^2 dx + C_1(\alpha, r_1, \epsilon),$$

where

$$C_1(\alpha, r_1, \epsilon) := \frac{|\Omega|}{\frac{2}{1+\alpha} \left(\frac{2}{1-\alpha}\right)^{\frac{1-\alpha}{1+\alpha}} \epsilon^{\frac{1-\alpha}{1+\alpha}}} + \frac{|\Omega|}{\frac{2}{1+r_1} \left(\frac{2}{1-r_1}\right)^{\frac{1-r_1}{1+r_1}} \epsilon^{\frac{1-r_1}{1+r_1}}}.$$

Using the Sobolev embedding $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ (see [19]), we obtain

$$(1 - 2\epsilon C_{\Omega}^2 \max\{A_{\infty}, B_{\infty}\}) \int_{\Omega} (\Delta u)^2 dx \leq C_1(\alpha, r_1, \epsilon).$$

Taking $\epsilon > 0$ small enough so that $(1 - 2\epsilon C_{\Omega}^2 \max\{A_{\infty}, B_{\infty}\}) > 0$, we obtain $\|u\|_{W^{2,2}(\Omega)} \leq C$. In a similar way we obtain $\|v\|_{W^{2,2}(\Omega)} \leq C$. Hence, since $1 \leq N \leq 3$, by Sobolev embedding $W^{2,2}(\Omega) \hookrightarrow L^{\infty}(\Omega)$, there exists $K > 0$ depending on $(\alpha, \beta, r_1, r_2, \Omega, A, B, C, D)$ such that

$$\|u\|_{L^{\infty}(\Omega)} + \|v\|_{L^{\infty}(\Omega)} \leq K.$$

Remark 5.1. Note that if $1 \leq N \leq 3$, it follows from the previous discussion that all classical solution $U = (u, v)$ of (1.1) is bounded in L^{∞} sense. By Lemma 2.2, there exists a positive constant δ (independent of u and v) such that $(u, v) \geq (\delta\phi_1, \delta\phi_2)$.

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