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KIRCHHOFF TYPE PROBLEMS WITH POTENTIAL WELL AND INDEFINITE POTENTIAL

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Abstract. In this article, we study the Kirchhoff type problem

$$-\Big(\alpha\int_{\mathbb{R}^3}|\nabla u|^2dx+1\Big)\Delta u+(\lambda a(x)+a_0)u=|u|^{p-2}u \quad \text{in } \mathbb{R}^3,$$
$$u\in H^1(\mathbb{R}^3),$$

where $4<p<6$, α and λ are two positive parameters, $a_0\in\mathbb{R}$ is a (possibly negative) constant and $a(x)\geq 0$ is the potential well. Using the variational method, we show the existence of nontrivial solutions. We also obtain the concentration behavior of the solutions as $\lambda\to+\infty$.

1. Introduction

In this article, we will study the Kirchhoff type problem

$$\begin{gathered}-\Big(\alpha\int_{\mathbb{R}^3}|\nabla u|^2dx+1\Big)\Delta u+(\lambda a(x)+a_0)u=|u|^{p-2}u \quad \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\end{gathered}\tag{1.1}$$

where $4<p<6$, α and λ are two positive parameters, $a_0\in\mathbb{R}$ is a constant and $a(x)$ is a potential satisfying some conditions to be specified later.

The Kirchhoff type problems in bounded domains is one of most popular nonlocal problems in the study areas of elliptic equations (cf. [5, 6, 16, 18, 22, 23, 28] and the references therein). One motivation comes from the very important application to such problems in physics. Indeed, The Kirchhoff type problem in bounded domains is related to the stationary analogue of the model

$$\begin{gathered}u_{tt}-\Big(\alpha\int_{\Omega}|\nabla u|^2dx+\beta\Big)\Delta u=h(x,u) \quad \text{in } \Omega\times(0,T),\\ u=0 \quad \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x), \quad u_t(x,0)=u^*(x),\end{gathered}\tag{1.2}$$

where $T>0$ is a constant, u_0,u^* are continuous functions. Such model was first proposed by Kirchhoff in 1883 as an extension of the classical D'Alembert's wave equations for free vibration of elastic strings, Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.2), u denotes the displacement, $h(x,u)$ the external force and β the initial tension while

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α is related to the intrinsic properties of the string (such as Youngs modulus). For more details on the physical background of Kirchhoff type problems, we refer the readers to [1, 13].

The Kirchhoff type nonlocal term was introduced to the elliptic equations in $\mathbb{R}^3$ by He and Zou in [11], where, by using the variational method, some existence results of the nontrivial solutions were obtained. Since then, many papers have been devoted to such topic, see for example [2, 10, 12, 15, 17, 24, 26] and the references therein. In particular, in a recent article [24], Sun and Wu have studied the Kirchhoff type problem

$$-\Big(\mu\int_{\mathbb{R}^3}|\nabla u|^2dx+\nu\Big)\Delta u+\lambda a(x)u=f(x,u)\quad \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^N),$$

where $\mu,\nu,\lambda>0$ are parameters and $a(x)$ satisfies the following conditions:

- (A1) $a(x)\in C(\mathbb{R}^3)$ and $a(x)\geq 0$ on $\mathbb{R}^3$.
- (A2) There exists $a_\infty>0$ such that $|\mathcal{A}_\infty|<+\infty$, where $\mathcal{A}_\infty=\{x\in\mathbb{R}^3: a(x)<a_\infty\}$ and $|\mathcal{A}_\infty|$ is the Lebesgue measure of the set $\mathcal{A}_\infty$.
- (A3) $\Omega=\text{int}a^{-1}(0)$ is a bounded domain and has smooth boundaries with $\overline{\Omega}=a^{-1}(0)$.

Using the variational method, they obtain some existence and non-existence results of the nontrivial solutions when $f(x,u)$ is 1-asymptotically linear, 3-asymptotically linear or 4-asymptotically linear at infinity.

Under the conditions (A1)–(A3), $\lambda a(x)$ is called as the steep potential well for λ sufficiently large and the depth of the well is controlled by the parameter λ. Such potentials were first introduced by Bartsch and Wang in [3] for the scalar Schrödinger equations. An interesting phenomenon for this kind of Schrödinger equations is that, one can expect to find the solutions which are concentrated at the bottom of the wells as the depth goes to infinity. Because this interesting property, such topic for the scalar Schrödinger equations was studied extensively in the past decade. We refer the readers to [4, 7, 14, 21, 25] and the references therein. Recently, the steep potential well was also considered for some other elliptic equations and systems, see for example [8, 9, 19, 27, 29] and the references therein. To our best knowledge, most of the literatures on this topic are devoted to the definite case while the indefinite case was only considered in [4, 7] for the the scalar Schrödinger equations and in [29] for the Schrödinger-Poisson systems.

Inspired by the above facts, we wonder what will happen for the Kirchhoff type problem with steep potential wells in the indefinite case of $a<0$? To our best knowledge, this kind of problems has not been studied yet in the literatures. Thus, the purpose of this paper is to explore the preceding problems.

Before stating our results, we shall introduce some notation. By condition (A3), it is well known that in the case of $a_0\neq 0$, all the eigenvalues $\{\gamma_i\}$ of the problem

$$-\Delta u=\gamma|a_0|u\quad u\in H_0^1(\Omega) \tag{1.3}$$

satisfy $\gamma_1<\gamma_2<\gamma_3<\dots<\gamma_i<\dots$ with $\gamma_i\to+\infty$ as $i\to\infty$ and the multiplicity of γ_i is finite for every $i\in\mathbb{N}$. In particular, γ_1 is simple. For each $i\in\mathbb{N}$, denote the corresponding eigenfunctions and the eigenspace of γ_i by $\{\varphi_{i,j}\}_{j=1,2,\dots,k_i}$ and $\mathcal{N}_i=\text{span}\{\varphi_{i,j}\}_{j=1,2,\dots,k_i}$ respectively, where k_i are the multiplicity of γ_i, then $\varphi_{i,j}$ can be chosen so that $\|\varphi_{i,j}\|_{L^2(\Omega)}=\frac{1}{|a_0|^2}$ and $\{\varphi_{i,j}\}$ can form a basis of $H_0^1(\Omega)$.

Let

$$k_0^* = \inf\{k : \gamma_k > 1\}, \tag{1.4}$$

then our main result in this paper can be stated as follows.

Theorem 1.1. *Suppose that* (A1)–(A3) *hold. If either* $a_0 \geq 0$ *or* $a_0 < 0$ *with* $\gamma_{k_0^*-1} < 1$ *then there exist positive constants* α_* *and* Λ_* *such that* $(\mathcal{P}_{\alpha,\lambda})$ *has a nontrivial solution* $u_{\alpha,\lambda}$ *for all* $\lambda > \Lambda_*$ *and* $\alpha \in (0, \alpha_*)$. *Moreover,* $u_{\alpha,\lambda} \to u_\alpha$ *strongly in* $H^1(\mathbb{R}^3)$ *as* $\lambda \to +\infty$ *up to a subsequence and* u_α *is a nontrivial solution of the following Kirchhoff type problem:*

$$\begin{gathered} -\Big(\alpha \int_\Omega |\nabla u|^2 dx + 1\Big)\Delta u + a_0 u = |u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

Remark 1.2. (a) If $a_0 < 0$ with $|a_0|$ large enough then it is easy to see that $k_0^* > 1$. It follows that (1.1) is indefinite in a suitable Hilbert space (see Lemma 2.5 for more details). To out best knowledge, Theorem 1.1 is the first result for the Kirchhoff type problem in $\mathbb{R}^3$ for the indefinite case.

(b) Theorem 1.1 also gives the existence of nontrivial solutions to (1.5). Note that (1.5) is also indefinite if $a_0 < 0$ with $|a_0|$ large enough. Thus, to our best knowledge, it is also the first result for the Kirchhoff type problem on bounded domains in the indefinite case.

Through this paper, C and C_i $(i = 1, 2, \dots)$ will be indiscriminately used to denote various positive constants. $o_n(1)$ and $o_\lambda(1)$ will always denote the quantities tending towards zero as $n \to \infty$ and $\lambda \to +\infty$ respectively.

2. Variational setting

By condition (A1), we see that for every $a_0 \in \mathbb{R}$ and $\lambda > \max\{0, \frac{-a_0}{a_\infty}\}$,

$$E = \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} a(x)u^2 dx < +\infty\}$$

equipped with the inner product

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^3} (\nabla u \nabla v + (\lambda a(x) + a_0)^+ uv) dx$$

is a Hilbert space, which we will denote by E_λ, where $(\lambda a(x)+a_0)^+ = \max\{\lambda a(x)+a_0, 0\}$. The corresponding norm on E_λ is

$$\|u\|_\lambda = \Big(\int_{\mathbb{R}^3} (|\nabla u|^2 + (\lambda a(x) + a_0)^+ u^2) dx\Big)^{1/2}.$$

It follows from the Hölder inequality, the Sobolev inequality and the conditions (A1)–(A2) that for every $u \in E_\lambda$ with $\lambda > \max\{0, -a_0/a_\infty\}$,

$$\begin{aligned} \int_{\mathbb{R}^3} u^2 dx &= \int_{\mathcal{A}_\infty} u^2 dx + \int_{\mathbb{R}^3 \setminus \mathcal{A}_\infty} u^2 dx \\ &\leq |\mathcal{A}_\infty|^{2/3} S^{-1} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{a_0 + a_\infty \lambda} \int_{\mathbb{R}^3} (\lambda a(x) + a_0)^+ u^2 dx \\ &\leq \max\big\{|\mathcal{A}_\infty|^{2/3} S^{-1}, \frac{1}{a_0 + a_\infty \lambda}\big\} \int_{\mathbb{R}^3} (|\nabla u|^2 + (\lambda a(x) + a_0)^+ u^2) dx \end{aligned}$$

and

$$\begin{aligned}\Big(\int_{\mathbb{R}^3}|u|^p dx\Big)^{1/p} &\leq S_p^{-1/2}\Big(\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)dx\Big)^{1/2}\\ &\leq S_p^{-1/2}\sqrt{1+\max\{|\mathcal{A}_\infty|^{2/3}S^{-1},\frac{1}{a_0+a_\infty\lambda}\}}\\ &\quad\times\Big(\int_{\mathbb{R}^3}(|\nabla u|^2+(\lambda a(x)+a_0)^+u^2)dx\Big)^{1/2},\end{aligned}$$

where S and S_p are the best Sobolev embedding constant from $D^{1,2}(\mathbb{R}^3)$ to $L^6(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ respectively; that is,

$$S=\inf\{\|\nabla u\|^2_{L^2(\mathbb{R}^3)}:u\in D^{1,2}(\mathbb{R}^3),\|u\|^2_{L^6(\mathbb{R}^3)}=1\}$$

and

$$S_p=\inf\{\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\|u\|^2_{L^2(\mathbb{R}^3)}:u\in H^1(\mathbb{R}^3),\|u\|^2_{L^p(\mathbb{R}^3)}=1\},$$

where $\|\cdot\|_{L^p(\mathbb{R}^3)}$ is the usual norm in $L^p(\mathbb{R}^3)$ for all $p\geq 1$.

Let $d_\lambda=\sqrt{\max\{|\mathcal{A}_\infty|^{2/3}S^{-1},\frac{1}{a_0+a_\infty\lambda}\}}$. Then we have

$$\|u\|_{L^2(\mathbb{R}^N)}\leq d_\lambda\|u\|_\lambda \quad\text{and}\quad \|u\|_{L^p(\mathbb{R}^3)}\leq S_p^{-1/2}\sqrt{1+d_\lambda^2}\|u\|_\lambda, \tag{2.1}$$

which yields that E_λ is embedded continuously into $H^1(\mathbb{R}^3)$ for $\lambda>\max\{0,\frac{-a_0}{a_\infty}\}$. Moreover, by using (2.1), the conditions (A1)–(A2) and by following a standard argument, we can show that corresponding energy functional $J_{\alpha,\lambda}(u)$ to the Problem (1.1), given by

$$J_{\alpha,\lambda}(u)=\frac{\alpha}{4}\|\nabla u\|^4_{L^2(\mathbb{R}^3)}+\frac12\int_{\mathbb{R}^3}(|\nabla u|^2+(\lambda a(x)+a_0)u^2)dx-\frac1p\|u\|^p_{L^p(\mathbb{R}^3)},$$

is C^2 in E_λ for $\lambda>\max\{0,\frac{-a_0}{a_\infty}\}$. For the sake of convenience, we re-write the energy functional $J_\lambda(u)$ by

$$J_{\alpha,\lambda}(u)=\frac{\alpha}{4}\|\nabla u\|^4_{L^2(\mathbb{R}^3)}+\frac12\|u\|^2_\lambda-\frac12\mathcal{D}_\lambda(u,u)-\frac1p\|u\|^p_{L^p(\mathbb{R}^3)},$$

where $\mathcal{D}_\lambda(u,v)=\int_{\mathbb{R}^3}(\lambda a(x)+a_0)^-uvdx$ and $(\lambda a(x)+a_0)^-=\max\{-(\lambda a(x)+a_0),0\}$. In what follows, inspired by [7, 29], we shall make some further observations on the functional $\mathcal{D}_\lambda(u,u)$.

By condition (A1), $\int_{\mathbb{R}^3}(\lambda a(x)+a_0)u^2dx\geq 0$ for all $u\in E_\lambda$ with $\lambda>0$ in the case of $a_0\geq 0$. It follows that $\mathcal{D}_\lambda(u,u)$ is definite on E_λ with $\lambda>0$ in the case of $a_0\geq 0$. Let us consider the case of $a_0<0$ in what follows. Let

$$\mathcal{A}_\lambda:=\{x\in\mathbb{R}^3:\lambda a(x)+a_0<0\},$$

then by the condition (A3), we have $\Omega\subset\mathcal{A}_\lambda$, which means that $\mathcal{A}_\lambda\neq\emptyset$ for every $\lambda>0$, and moreover, by the conditions (A1)–(A2), the real number

$$\Lambda_0:=\inf\{\lambda>0:|\mathcal{A}_\lambda|<+\infty\}.$$

satisfies $0<\Lambda_0\leq\frac{-a_0}{a_\infty}$. For $\lambda>\Lambda_0$, we define

$$\mathcal{F}_\lambda:=\{u\in E_\lambda:\operatorname{supp}u\subset\mathbb{R}^3\backslash\mathcal{A}_\lambda\}.$$

It follows from the conditions (A1)–(A3) that $\mathcal{F}_\lambda$ is nonempty, closed and convex with $\mathcal{F}_\lambda\neq E_\lambda$. Hence, $E_\lambda=\mathcal{F}_\lambda\oplus\mathcal{F}_\lambda^\perp$ and $\mathcal{F}_\lambda^\perp\neq\emptyset$ for $\lambda>\Lambda_0$ in the case of $a_0<0$, where $\mathcal{F}_\lambda^\perp$ is the orthogonal complement of $\mathcal{F}_\lambda$ in E_λ.

Lemma 2.1. *Let*
$$\beta(\lambda) := \inf_{u\in\mathcal{F}_\lambda^\perp\cap\mathbb{D}_\lambda}\|u\|_\lambda^2,$$
where $\mathbb{D}_\lambda := \{u \in E_\lambda : \mathcal{D}_\lambda(u,u) = 1\}$*. If the conditions (A1)–(A3) hold, then* $\beta(\lambda)$ *is nondecreasing as the function of* λ *on* $(\Lambda_0,+\infty)$ *and* $\beta(\lambda)$ *can be attained by some* $e(\lambda) \in \mathcal{F}_\lambda^\perp$*. Furthermore,* $(e(\lambda),\beta(\lambda)) \to (\varphi_1,\gamma_1)$ *strongly in* $H^1(\mathbb{R}^3)\times\mathbb{R}$ *as* $\lambda\to+\infty$ *up to a subsequence.*

Proof. First, thanks to the definition of Λ_0, we see that $\mathcal{D}_\lambda(u,u)$ and $\|u\|_\lambda^2$ are weakly continuous and weakly low semi-continuous on $\mathcal{F}_\lambda^\perp$ respectively. Thus, we can use a standard argument to show that $\beta(\lambda)$ can be attained by some $e(\lambda)\in\mathcal{F}_\lambda^\perp\cap\mathbb{D}_\lambda$ for all $\lambda>\Lambda_0$.

Next, we show that $\beta(\lambda)$ is nondecreasing as the function of λ on $(\Lambda_0,+\infty)$. Indeed, let $\lambda_1\ge\lambda_2$, then by the definition of E_λ, we have $E_{\lambda_1}=E_{\lambda_2}$ in the sense of sets. It follows that $\mathcal{F}_{\lambda_2}\subset\mathcal{F}_{\lambda_1}$, which implies $\mathcal{F}_{\lambda_1}^\perp\subset\mathcal{F}_{\lambda_2}^\perp$. Note that $\|u\|_{\lambda_1}^2\ge\|u\|_{\lambda_2}^2$ and $\mathcal{D}_{\lambda_1}(u,u)\le\mathcal{D}_{\lambda_2}(u,u)$ for all $u\in E_{\lambda_1}$ by the condition (A1). Thus, due to the definition of $\beta(\lambda_1)$ and $\beta(\lambda_2)$, we can see that $\beta(\lambda_2)\le\beta(\lambda_1)$; that is, $\beta(\lambda)$ is nondecreasing as a functional of λ on $(\Lambda_0,+\infty)$.

Finally, we shall prove that $(e(\lambda),\beta(\lambda))\to(\varphi_1,\gamma_1)$ strongly in $H^1(\mathbb{R}^3)\times\mathbb{R}$ as $\lambda\to+\infty$ up to a subsequence. In fact, since $\int_{\mathbb{R}^3}(\lambda a(x)+a_0)^-[e(\lambda)]^2dx=1$, it implies that
$$\lim_{\lambda\to+\infty}\int_{\mathbb{R}^3}(a(x)+\frac{a_0}{\lambda})^+[e(\lambda)]^2dx=0. \tag{2.2}$$
Note that $H_0^1(\Omega)\subset\mathcal{F}_\lambda^\perp$ for all $\lambda>\Lambda_0$ due to the condition (A3), we can easily show that $0<\beta(\lambda)\le\gamma_1$ for all $\lambda>\Lambda_0$. It follows from (2.1) that $\{e(\lambda)\}$ is bounded in $H^1(\mathbb{R}^3)$ for λ. Without loss of generality, we assume that $e(\lambda)\rightharpoonup e^*$ weakly in $H^1(\mathbb{R}^3)$ and $\beta(\lambda)\to\beta^*$ as $\lambda\to+\infty$. By the Sobolev embedding theorem, the condition (A2) and (2.2), we must have $(e^*,\beta^*)\in H_0^1(\Omega)\times\mathbb{R}^+$ satisfying $e^*\equiv0$ outside Ω and $|a_0|^2\int_\Omega|e^*|^2dx=1$ and $(e(\lambda),\beta(\lambda))\to(e^*,\beta^*)$ strongly in $L^2(\mathbb{R}^3)\times\mathbb{R}$ as $\lambda\to+\infty$ up to a subsequence, which gives
$$\begin{aligned}\gamma_1&\ge\int_{\mathbb{R}^3}(|\nabla e(\lambda)|^2+(\lambda a(x)+a_0)^+[e(\lambda)]^2)dx\\&\ge\int_\Omega|\nabla e^*|^2dx+o_\lambda(1)\\&\ge\gamma_1+o_\lambda(1).\end{aligned}\tag{2.3}$$
Hence, $(e(\lambda),\beta(\lambda))\to(e^*,\gamma_1)$ strongly in $H^1(\mathbb{R}^3)\times\mathbb{R}$ as $\lambda\to+\infty$ up to a subsequence and (e^*,γ_1) satisfies
$$\gamma_1=\int_\Omega|\nabla e^*|^2dx=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2dx}{|a_0|^2\int_\Omega|u|^2dx}.$$
Thus, $e_\alpha^*=\varphi_1$ and the proof is complete. □

We re-denote the above $(e(\lambda),\beta(\lambda))$ by $(e_1(\lambda),\beta_1(\lambda))$ and define
$$\mathcal{F}_{\lambda,1}^\perp:=\big\{u\in\mathcal{F}_\lambda^\perp:\frac{\|u\|_\lambda^2}{\mathcal{D}_\lambda(u,u)}=\beta_1(\lambda)\big\}.$$
Since $\gamma_2>\gamma_1\ge\beta_1(\lambda)$ for $\lambda>\Lambda_0$, it is easy to see that $\mathcal{F}_{\lambda,1}^\perp\ne\mathcal{F}_\lambda^\perp$. Thus, we have $\mathcal{F}_\lambda^\perp=\mathcal{F}_{\lambda,1}^\perp\oplus\mathcal{F}_{\lambda,1}^{\perp,*}$, where $\mathcal{F}_{\lambda,1}^{\perp,*}$ is the orthogonal complement of $\mathcal{F}_{\lambda,1}^\perp$ in $\mathcal{F}_\lambda^\perp$.

Lemma 2.2. *Suppose that* (A1)–(A3) *hold. Then there exists* $\Lambda_1 \geq \Lambda_0$ *such that* $\mathcal{F}_{\lambda,1}^{\perp} = \operatorname{span}\{e_1(\lambda)\}$ *and* $\beta_2(\lambda)$ *can be attained by some* $e_2(\lambda) \in \mathcal{F}_{\lambda,1}^{\perp,*}$ *for* $\lambda > \Lambda_1$, *where*

$$\beta_2(\lambda) := \inf_{\mathcal{F}_{\lambda,1}^{\perp,*} \cap \mathbb{D}_\lambda} \|u\|_\lambda^2.$$

Furthermore, $(e_2(\lambda), \beta_2(\lambda)) \to (\varphi_{2,j}, \gamma_2)$ *strongly in* $H^1(\mathbb{R}^3) \times \mathbb{R}$ *as* $\lambda \to +\infty$ *up to a subsequence for some* $j \in \mathbb{N}$ *with* $1 \leq j \leq k_2$.

Proof. Since $\mathcal{D}_\lambda(u,u)$ and $\|u\|_\lambda^2$ are weakly continuous and weakly low semi-continuous on $\mathcal{F}_{\lambda,1}^{\perp,*}$ respectively, by the fact that $\mathcal{F}_{\lambda,1}^{\perp,*}$ is weakly closed for $\lambda > \Lambda_0$, we can also use a standard argument to show that $\beta_2(\lambda)$ can be attained by some $e_2(\lambda) \in \mathcal{F}_{\lambda,1}^{\perp,*}$ for $\lambda > \Lambda_0$. For the sake of clarity, the remaining proof will be performed through the following steps.

Step 1. We prove that there exists $\Lambda_1 \geq \Lambda_0$ such that $\mathcal{F}_{\lambda,1}^{\perp} = \operatorname{span}\{e_1(\lambda)\}$ for $\lambda > \Lambda_1$. Indeed, suppose on the contrary that there exist $e_1^*(\lambda_n), e_1^0(\lambda_n) \in \mathcal{F}_{\alpha,\lambda,1}^{\perp}$ with

$$\langle e_1^*(\lambda_n), e_1^0(\lambda_n)\rangle_{E_{\lambda_n},E_{\lambda_n}} = 0,$$
$$\int_{\mathbb{R}^3} (\lambda_n a(x) + a_0)^- [e_1^*(\lambda_n)]^2 dx = \int_{\mathbb{R}^3} (\lambda_n a(x) + a_0)^- [e_1^0(\lambda_n)]^2 dx = 1$$

for $\{\lambda_n\}$ satisfying $\lambda_n \to +\infty$ as $n \to \infty$. By Lemma 2.1, we can see that $e_1^*(\lambda_n) \to \varphi_1$ and $e_1^0(\lambda_n) \to \varphi_1$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$ up to a subsequence. It follows from (2.3) and Lemma 2.1 once more that

$$\begin{aligned} 2\gamma_1 &= 2\beta_1(\lambda_n) + o_n(1) \\ &= \|e_1^*(\lambda_n)\|_{\lambda_n}^2 + \|e_1^0(\lambda_n)\|_{\lambda_n}^2 + o_n(1) \\ &= \|\nabla(e_1^*(\lambda_n) - e_1^0(\lambda_n))\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) = o_n(1), \end{aligned} \tag{2.4}$$

which is a contradiction.

Step 2. We show that $\limsup_{\lambda\to+\infty} \beta_2(\lambda) \leq \gamma_2$. In fact, by Step 1, we have $\varphi_{2,1} = d_\lambda^* e_1(\lambda) + \varphi_{2,1,\lambda}^*$, where d_λ^* is a constant and $\varphi_{2,1,\lambda}^*$ is the projection of $\varphi_{2,1}$ in $\mathcal{F}_{\lambda,1}^{\perp,*}$. Thus, $\langle e_1(\lambda), \varphi_{2,1}\rangle_{E_\lambda,E_\lambda} = d_\lambda^* \|e_1(\lambda)\|_\lambda^2$. It follows from the condition (A3) and Lemma 2.1 that $d_\lambda^* \to 0$ as $\lambda \to +\infty$ up to a subsequence. Now, by the definition of $\beta_2(\lambda)$, we can see from Lemma 2.1, (2.3) and a variant of the Lebesgue dominated convergence theorem (cf. [20, Theorem 2.2]) that

$$\begin{aligned} \limsup_{\lambda\to+\infty} \beta_2(\lambda) &\leq \limsup_{\lambda\to+\infty} \frac{\|\varphi_{2,1,\lambda}^*\|_\lambda^2}{\mathcal{D}_\lambda(\varphi_{2,1,\lambda}^*, \varphi_{2,1,\lambda}^*)} \\ &= \limsup_{\lambda\to+\infty} \frac{\|\varphi_{2,1} - d_\lambda^* e_1(\lambda)\|_\lambda^2}{\mathcal{D}_\lambda(\varphi_{2,1} - d_\lambda^* e_1(\lambda), \varphi_{2,1} - d_\lambda^* e_1(\lambda))} \\ &= \frac{\|\nabla\varphi_{2,1}\|_{L^2(\mathbb{R}^3)}^2}{|a_0|^2 \|\varphi_{2,1}\|_{L^2(\mathbb{R}^3)}^2} = \gamma_2. \end{aligned}$$

Step 3. We prove that $\limsup_{\lambda\to+\infty} \beta_2(\lambda) \geq \gamma_2$ and $(e_2(\lambda), \beta_2(\lambda)) \to (\varphi_{2,j}, \gamma_2)$ strongly in $H^1(\mathbb{R}^3) \times \mathbb{R}$ as $\lambda \to +\infty$ up to a subsequence for some $j \in \mathbb{N}$ with $1 \leq j \leq k_2$. Actually, by Step 2, we know that $\{e_2(\lambda)\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$. Similarly as in the proof of Lemma 2.1, we can see that $(e_2(\lambda), \beta_2(\lambda)) \to (e_2^*, \beta_2^*)$ strongly in

$L^2(\mathbb{R}^3)\times\mathbb{R}$ as $\lambda\to+\infty$ up to a subsequence with $e_2^*\in H_0^1(\Omega)$ and $e_2^*\equiv 0$ outside Ω. Since $\mathcal{D}_\lambda(u,u)$ is weakly continuous on $\mathcal{F}_\lambda^\perp$, we also have $|a_0|^2\int_\Omega |e_2^*|^2dx=1$. Furthermore, by the theory of Lagrange multipliers, we can also see that (e_2^*,β_2^*) satisfies (1.3). It follows from a variant of the Lebesgue dominated convergence theorem (cf. [20, Theorem 2.2]) that

$$\begin{aligned}
\|\nabla e_2^*\|_{L^2(\mathbb{R}^3)}^2 &= \beta_2^*|a_0|\|e_2^*\|_{L^2(\mathbb{R}^3)}^2\\
&= \beta_2(\lambda)\mathcal{D}_\lambda(e_2(\lambda),e_2^*(\lambda))+o_\lambda(1)\\
&= \int_{\mathbb{R}^3}|\nabla e_2(\lambda)|^2dx+o_\lambda(1)\\
&\geq \|\nabla e_2^*\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

Thus, $(e_2(\lambda),\beta_2(\lambda))\to(e_2^*,\beta_2^*)$ strongly in $H^1(\mathbb{R}^3)\times\mathbb{R}$ as $\lambda\to+\infty$ up to a subsequence. By Step 2, we must have $\beta_2^*=\gamma_1$ or $\beta_2^*=\gamma_2$. If $\limsup_{\lambda\to+\infty}\beta_2(\lambda)<\gamma_2$ then there exists $\{\lambda_n\}$ such that $(e_2(\lambda_n),\beta_2(\lambda_n))\to(\varphi_1,\gamma_1)$ strongly in $L^2(\mathbb{R}^3)\times\mathbb{R}$ as $n\to\infty$ up to a subsequence. It follows from Lemma 2.1, (2.3) and Step 2 that

$$0=\langle e_1(\lambda_n),e_2(\lambda_n)\rangle_{\lambda_n,\lambda_n}=\|\nabla\varphi_1\|_{L^2(\mathbb{R}^3)}^2+o_n(1),$$

which is a contradiction. ☐

Let

$$\mathcal{F}_{\lambda,2}^\perp:=\big\{u\in\mathcal{F}_\lambda^\perp:\frac{\|u\|_\lambda^2}{\mathcal{D}_\lambda(u,u)}=\beta_2(\lambda)\big\}.$$

Since $\gamma_3>\gamma_2$, it yields from Lemma 2.2 and condition (A3) that $\mathcal{F}_{\lambda,1}^\perp\oplus\mathcal{F}_{\lambda,2}^\perp\neq\mathcal{F}_\lambda^\perp$.

Lemma 2.3. *Suppose that* (A1)–(A3) *hold. Then there exists* $\Lambda_2\geq\Lambda_1$ *such that* $dim(\mathcal{F}_{\lambda,2}^\perp)\leq k_2$ *for* $\lambda>\Lambda_2$.

Proof. Let $e_2(\lambda),e_2'(\lambda)\in\mathcal{F}_{\lambda,2}^\perp$. By Lemma 2.2, $e_2(\lambda)\to\varphi_{2,j}$ and $e_2'(\lambda)\to\varphi_{2,j'}$ strongly in $H^1(\mathbb{R}^3)\times\mathbb{R}$ as $\lambda\to+\infty$ up to a subsequence for some $j,j'\in\mathbb{N}$ with $1\leq j,j'\leq k_2$. Clearly, two cases may occur:

(1) $\varphi_{2,j}=\varphi_{2,j'}$;
(2) $\varphi_{2,j}\neq\varphi_{2,j'}$ and $\int_\Omega\varphi_{2,j}\varphi_{2,j'}dx=0$.

If case (1) happens then by a similar argument used in the proof of (2.4), we can get that $\gamma_2=0$, which is a contradiction. Thus, we must have case (2). It follows that there exists $\Lambda_2\geq\Lambda_1$ such that $\dim(\mathcal{F}_{\lambda,2}^\perp)\leq k_2$ for $\lambda>\Lambda_2$. ☐

Now, by iterating, for $m=3,4,\dots$, we can define $\beta_m(\lambda)$ as follows:

$$\beta_m(\lambda):=\inf_{\mathcal{F}_{\lambda,m}^{\perp,*}\cap\mathbb{D}_\lambda}\|u\|_\lambda^2,$$

where

$$\mathcal{F}_{\lambda,m}^{\perp,*}:=\{u\in\mathcal{F}_\lambda^\perp:\langle u,v\rangle_\lambda=0,\ \text{for all } v\in\oplus_{i=1}^{m-1}\mathcal{F}_{\lambda,i}^\perp\},$$

$$\mathcal{F}_{\lambda,i}^\perp:=\Big\{u\in\mathcal{F}_\lambda^\perp:\frac{\|u\|_\lambda^2}{\mathcal{D}_\lambda(u,u)}=\beta_i(\lambda)\Big\}.$$

Similarly as Lemmas 2.2 and 2.3, we can obtain the following result.

Lemma 2.4. *Suppose that* (A1)–(A3) *hold. Then there exists* $\Lambda_m \geq \Lambda_{m-1}$ *such that* $\beta_m(\lambda)$ *can be attained by some* $e_m(\lambda) \in \mathcal{F}^{\perp,*}_{\lambda,m}$ *for* $\lambda > \Lambda_m$. *Furthermore,* $(e_m(\lambda), \beta_m(\lambda)) \to (\varphi_{m,j}, \gamma_m)$ *strongly in* $H^1(\mathbb{R}^3) \times \mathbb{R}$ *as* $\lambda \to +\infty$ *up to a subsequence for some* $j \in \mathbb{N}$ *with* $1 \leq j \leq k_m$ *and* $dim(\mathcal{F}^{\perp}_{\lambda,m}) \leq k_m$ *for* $\lambda > \Lambda_m$, *where*

$$\mathcal{F}^{\perp}_{\lambda,m} := \Big\{u \in \mathcal{F}^{\perp}_{\lambda} : \frac{\|u\|^2_\lambda}{\mathcal{D}_\lambda(u,u)} = \beta_m(\lambda)\Big\}.$$

Let k_0^* be given by (1.4), then by Lemmas 2.1, 2.2 and 2.4, $\oplus_{i=1}^{k_0^*-1}\mathcal{F}^{\perp}_{\lambda,i}$ and $\mathcal{F}^{\perp,*}_{\lambda,k_0^*}$ are well defined for $\lambda > \Lambda_{k_0^*}$.

Lemma 2.5. *Suppose that* (A1)–(A3) *hold. If* $\gamma_{k_0^*-1} < 1$ *then there exists* $\widetilde{\Lambda}_{k_0^*} \geq \Lambda_{k_0^*}$ *such that for* $\lambda > \widetilde{\Lambda}_{k_0^*}$, *it holds that*

1. $\|u\|^2_\lambda - \mathcal{D}_\lambda(u,u) \leq \frac{1}{2}(1 - \frac{1}{\gamma_{k_0^*-1}})\|u\|^2_\lambda$ *in* $\oplus_{i=1}^{k_0^*-1}\mathcal{F}^{\perp}_{\lambda,i}$;
2. $\|u\|^2_\lambda - \mathcal{D}_\lambda(u,u) \geq \frac{1}{2}(1 - \frac{1}{\gamma_{k_0^*}})\|u\|^2_\lambda$ *in* $\mathcal{F}^{\perp,*}_{\lambda,k_0^*}$.

The proof of the above lemma follows immediately from Lemmas 2.1, 2.2 and 2.4.

Remark 2.6. By Lemmas 2.2–2.4, we also have $\oplus_{i=1}^{k_0^*-1}\mathcal{F}^{\perp}_{\lambda,i} = \emptyset$ in the case of $\gamma_1 > 1$ while $\oplus_{i=1}^{k_0^*-1}\mathcal{F}^{\perp}_{\lambda,i} \neq \emptyset$ and $\dim(\oplus_{i=1}^{k_0^*-1}\mathcal{F}^{\perp}_{\lambda,i}) \leq \sum_{i=1}^{k_0^*-1} k_i$ in the case of $\gamma_1 < 1$.

3. Nontrivial solution

We first consider the case of $a_0 < 0$. By the decomposition of E_λ, we will find the nontrivial solution by the linking theorem. Let us first verify that $J_{\alpha,\lambda}(u)$ has a linking structure in E_λ in the case of $a_0 < 0$.

Lemma 3.1. *Suppose that* (A1)–(A3) *hold and* $a_0 < 0$. *For every* $\alpha > 0$, *if* $\beta_{k_0^*-1} < 1$ *then there exists* $\rho > 0$ *independent of* λ *such that*

$$\inf_{\mathcal{F}^{\perp,*}_{\lambda,k_0^*}\cap\mathbb{S}_{\lambda,\rho}} J_{\alpha,\lambda}(u) \geq d_0 \tag{3.1}$$

for all $\lambda > \widetilde{\Lambda}_{k_0^*}$, *where* $\mathbb{S}_{\lambda,\rho} := \{u \in E_\lambda : \|u\|_\lambda = \rho\}$ *and* d_0 *is a constant independent of* λ *and* α.

Proof. By (2.1) and Lemma 2.5, for every $u \in \mathcal{F}^{\perp,*}_{\lambda,k_0^*}$, we have

$$\begin{aligned} J_{\alpha,\lambda}(u) &= \frac{\alpha}{4}\|\nabla u\|^4_{L^2(\mathbb{R}^3)} + \frac{1}{2}\|u\|^2_\lambda - \frac{1}{2}\mathcal{D}_\lambda(u,u) - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R}^3)} \\ &\geq \frac{1}{4}(1 - \frac{1}{\gamma_{k_0^*}})\|u\|^2_\lambda - S_p^{-\frac{p}{2}}(1+d_\lambda^2)^{\frac{p}{2}}\|u\|^p_\lambda \\ &\geq \|u\|^2_\lambda\Big(\frac{1}{4}(1 - \frac{1}{\gamma_{k_0^*}}) - S_p^{-\frac{p}{2}}(1+d_\lambda^2)^{\frac{p}{2}}\|u\|^{p-2}_\lambda\Big). \end{aligned} \tag{3.2}$$

Note that $d_\lambda = \sqrt{\max\{|\mathcal{A}_\infty|^{2/3}S^{-1}, \frac{1}{a_0+a_\infty\lambda}\}}$, so that

$$d_\lambda \leq \sqrt{\max\{|\mathcal{A}_\infty|^{2/3}S^{-1}, \frac{1}{a_0+a_\infty\widetilde{\Lambda}_{k_0^*}}\}}$$

for $\lambda > \widetilde{\Lambda}_{k_0^*}$. It follows from (3.2) that there exists $\rho > 0$ independent on λ such that (3.1) holds for all $\lambda > \widetilde{\Lambda}_{k_0^*}$. $\square$

Let

$$\mathcal{Q}_{\lambda,k_0^*} := \{u = v + te_{k_0^*}(\lambda) : t \geq 0 \text{ and } v \in \oplus_{i=1}^{k_0^*-1}\mathcal{F}_{\lambda,i}^{\perp}\}.$$

Lemma 3.2. *Suppose that* (A1)–(A3) *hold and* $a_0 < 0$*. If* $\gamma_{k_0^*-1} < 1$ *then there exist* $\alpha_0 > 0$ *and* $R_0 > \rho$ *independent of* λ *such that*

$$\sup_{\partial\mathcal{Q}_{\lambda,k_0^*}^{R_0}} J_{\alpha,\lambda}(u) \leq \frac{1}{2}d_0$$

for all $\lambda > \widetilde{\Lambda}_{k_0^*}$ *in the case of* $\alpha \in (0, \alpha_0)$*, where* d_0 *is given in lemma 3.1,* $\mathcal{Q}_{\lambda,k_0^*}^{R_0} := \mathcal{Q}_{\lambda,k_0^*} \cap \mathbb{B}_{\lambda,R_0}$ *and* $\mathbb{B}_{\lambda,R_0} := \{u \in E_\lambda : \|u\|_\lambda \leq R_0\}$.

Proof. Let $u_\lambda \in \partial\mathcal{Q}_{\lambda,k_0^*}^{R}$. Then one of the following two cases must happen:

(a) $u_\lambda = R\widetilde{u}_\lambda$ with $\widetilde{u}_\lambda \in \oplus_{i=1}^{k_0^*-1}\mathcal{F}_{\lambda,i}^{\perp}$ and $\|\widetilde{u}_\lambda\|_\lambda \leq 1$.

(b) $u_\lambda = R\widetilde{u}_\lambda$ with $\widetilde{u}_\lambda \in \mathcal{Q}_{\lambda,k_0^*}^{1} \setminus \oplus_{i=1}^{k_0^*-1}\mathcal{F}_{\lambda,i}^{\perp}$ and $\|\widetilde{u}_\lambda\|_\lambda = 1$.

If the case (b) happens then by Lemma 2.5, we deduce that

$$J_{\alpha,\lambda}(u_\lambda) = J_{\alpha,\lambda}(R\widetilde{u}_\lambda) \leq \frac{\alpha}{4}R^4 + \frac{1}{2}(1 - \frac{1}{\gamma_{k_0^*}})R^2 - \frac{1}{p}\|R\widetilde{u}_\lambda\|_{L^p(\mathbb{R}^3)}^p. \tag{3.3}$$

Since $\widetilde{u}_\lambda \in \oplus_{i=1}^{k_0^*}\mathcal{F}_{\lambda,i}^{\perp}$, by Lemmas 2.1, 2.2 and 2.4, $\widetilde{u}_\lambda = \widetilde{u} + o_\lambda(1)$ strongly in $H^1(\mathbb{R}^3)$ for some $\widetilde{u} \in \text{span}\{\varphi_{i,j}\}_{j=1,2,\dots,k_i}^{i=1,2,\dots,k_0^*}$ and $\int_\Omega |\nabla\widetilde{u}|^2dx = 1$. Thus, $\|\widetilde{u}_\lambda\|_{L^p(\mathbb{R}^3)}^p = \|\widetilde{u}\|_{L^p(\mathbb{R}^3)}^p + o_\lambda(1)$ by the Sobolev embedding theorem.

Note that $\dim\text{span}\{\varphi_{i,j}\}_{j=1,2,\dots,k_i}^{i=1,2,\dots,k_0^*}) \leq \sum_{i=1}^{k_0^*-1} k_i + 1$ for all $\lambda > \widetilde{\Lambda}_{k_0^*}$ by Remark 2.6. Therefore, there exists a constant $M > 0$ such that $\|u\|_{L^p(\mathbb{R}^3)} \geq M$ for all $u \in\text{span}\{\varphi_{i,j}\}_{j=1,2,\dots,k_i}^{i=1,2,\dots,k_0^*}$ with $\int_\Omega |\nabla u|^2dx = 1$. In particular, $\|\widetilde{u}\|_{L^p(\mathbb{R}^3)} \geq M$. It follows from $4 < p < 6$ and (3.3) that there exists a constant $R_0(> \rho)$ such that $J_{\alpha,\lambda}(R_0\widetilde{u}_\lambda) \leq 0$ for all $\lambda > \widetilde{\Lambda}_{k_0^*}$. Now, we consider the case of (a). By Lemma 2.5 once more, we know that

$$J_{\alpha,\lambda}(u_\lambda) = J_{\alpha,\lambda}(R\widetilde{u}_\lambda) \leq \frac{\alpha}{4}R_0^4.$$

Thus, there exists $\alpha_0 > 0$ such that $J_{\alpha,\lambda}(u_\lambda) \leq \frac{1}{2}d_0$ for $\lambda > \widetilde{\Lambda}_{k_0^*}$ and $\alpha \in (0, \alpha_0)$. $\square$

From Lemmas 3.1 and 3.2, we can see that $J_{\alpha,\lambda}(u)$ has a linking structure in E_λ with $\lambda > \widetilde{\Lambda}_{k_0^*}$ and $\alpha \in (0, \alpha_0)$ in the case of $a_0 < 0$. By the linking theorem, there exists $\{u_n\} \subset E_\lambda$ such that $(1 + \|u_n\|_\lambda)J'_{\alpha,\lambda}(u_n) = o_n(1)$ strongly in E_λ^* and $J_{\alpha,\lambda}(u_n) = c_{\alpha,\lambda} + o_n(1)$, where E_λ^* is the dual space of E_λ. Furthermore, $c_{\alpha,\lambda} \in [d_0, \frac{\alpha}{4}R_0^4 + \frac{1}{2}(1 - \frac{1}{\gamma_{k_0^*}})R_0^2]$. Note that in the special case $\gamma_1 > 1$, the linking structure is actually the mountain pass geometry. Thus, the linking theorem can be replaced by the mountain pass theorem and we can also obtain a sequence $\{u_n\} \subset E_\lambda$ such that $(1 + \|u_n\|_\lambda)J'_{\alpha,\lambda}(u_n) = o_n(1)$ strongly in E_λ^* and $J_{\alpha,\lambda}(u_n) = c_{\alpha,\lambda} + o_n(1)$. In the case $a_0 \geq 0$, since $4 < p < 6$ and the fact that $\mathcal{D}_\lambda(u, u) = 0$ in E_λ, by using a standard argument, we can verify that $J_{\alpha,\lambda}(u)$ has a mountain pass geometry in E_λ for $\lambda > 0$; that is,

(a) $\inf_{\mathbb{S}_{\lambda,\overline{\rho}}} J_{\alpha,\lambda}(u) \geq C$ for some $\overline{\rho} > 0$;

(b) $J_{\alpha,\lambda}(\overline{R}_0\phi)\le 0$ for some $\overline{R}_0>\overline{\rho}$ and $\phi\in H_0^1(\Omega)$.

This also gives the existence of a sequence $\{u_n\}\subset E_\lambda$ such that

$$(1+\|u_n\|_\lambda)J'_{\alpha,\lambda}(u_n)=o_n(1)$$

strongly in E_λ^* and $J_{\alpha,\lambda}(u_n)=c_{\alpha,\lambda}+o_n(1)$ with $c_{\alpha,\lambda}\in[C_\alpha,C'_\alpha]$, where C_α,C'_α are two positive constants independent of λ. In a word, in both cases of $a_0<0$ and $a_0\ge 0$, for $\lambda>\widetilde{\Lambda}_{k_0^*}$, there exists $\{u_n\}\subset E_\lambda$ such that $(1+\|u_n\|_\lambda)J'_{\alpha,\lambda}(u_n)=o_n(1)$ strongly in E_λ^* and $J_{\alpha,\lambda}(u_n)=c_{\alpha,\lambda}+o_n(1)$ with $c_{\alpha,\lambda}\in[C_\alpha,C'_\alpha]$.

Lemma 3.3. *Suppose that* (A1)–(A3) *hold. For every $\alpha>0$, if either $a_0\ge 0$ or $a_0<0$ with $\beta_{k_0^*-1}<1$ then $\{\|u_n\|_\lambda\}$ is bounded.*

Proof. Since $\lambda>\widetilde{\Lambda}_{k_0^*}$, by the condition (A2) and the Hölder and the Sobolev inequalities, we obtain that

$$\mathcal{D}_\lambda(u_n,u_n)\le |a_0|\int_{\mathcal{A}_\infty}|u_n|^2dx\le |a_0||\mathcal{A}_\infty|^{\frac{2}{3}}S^{-1}\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)}.$$

Note that $(1+\|u_n\|_\lambda)J'_{\alpha,\lambda}(u_n)=o_n(1)$ strongly in E_λ^* and $J_{\alpha,\lambda}(u_n)=c_{\alpha,\lambda}+o_n(1)$, by the Young inequality and the fact that $4<p<6$, we deduce that

$$\begin{aligned}
&c_{\alpha,\lambda}+o_n(1)\\
&=J_{\alpha,\lambda}(u_n)-\frac{1}{p}\langle J'_{\alpha,\lambda}(u_n),u_n\rangle_{E_\lambda^*,E_\lambda}\\
&=\alpha(\frac{1}{4}-\frac{1}{p})\|\nabla u_n\|^4_{L^2(\mathbb{R}^3)}+(\frac{1}{2}-\frac{1}{p})\|u_n\|^2_\lambda-(\frac{1}{2}-\frac{1}{p})\mathcal{D}_\lambda(u_n,u_n)\\
&\ge\frac{p-4}{4p}(\alpha\|\nabla u_n\|^4_{L^2(\mathbb{R}^3)}+\|u_n\|^2_\lambda)-\frac{p-2}{2p}|a_0||\mathcal{A}_\infty|^{\frac{2}{3}}S^{-1}\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)}\\
&\ge\frac{p-4}{8p}(\alpha\|\nabla u_n\|^4_{L^2(\mathbb{R}^3)}+\|u_n\|^2_\lambda)-\frac{2(p-2)^2}{\alpha(p-4)p}|a_0|^2|\mathcal{A}_\infty|^{\frac{4}{3}}S^{-2},
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{E_\lambda^*,E_\lambda}$ is the duality pairing of E_λ^* and E_λ. The preceding inequality, together with $c_{\alpha,\lambda}\in[C_\alpha,C'_\alpha]$ and $4<p<6$, implies $\{\|u_n\|_\lambda\}$ is bounded. □

By Lemma 3.3, we can see that $u_n=u_{\alpha,\lambda}+o_n(1)$ weakly in E_λ for some $u_{\alpha,\lambda}\in E_\lambda$ up to a subsequence. Without loss of generality, we may assume that $u_n=u_{\alpha,\lambda}+o_n(1)$ weakly in E_λ.

Lemma 3.4. *Suppose that* (A1)–(A3) *hold. For every $\alpha>0$, if either $a_0\ge 0$ or $a_0<0$ with $\beta_{k_0^*-1}<1$ then there exists $\overline{\Lambda}_{k_0^*}>\widetilde{\Lambda}_{k_0^*}$ such that $u_{\alpha,\lambda}$ is a nontrivial solution of* (1.1) *for $\lambda>\overline{\Lambda}_{k_0^*}$.*

Proof. We first prove that $u_{\alpha,\lambda}\ne 0$ in E_λ. Indeed, suppose on the contrary, then by the Sobolev embedding theorem, we can see that $u_n=o_n(1)$ strongly in $L^2_{loc}(\mathbb{R}^3)$, which, together with (A2), implies $u_n=o_n(1)$ strongly in $L^2(\mathcal{A}_\infty)$. It follows from Lemma 3.3, conditions (A1)–(A2) and the Hölder and the Sobolev inequality that

$$\begin{aligned}
\int_{\mathbb{R}^3}|u_n|^pdx&\le\Big(\int_{\mathbb{R}^3}|u_n|^2dx\Big)^{\frac{6-p}{4}}\Big(\int_{\mathbb{R}^3}|u_n|^6dx\Big)^{\frac{p-2}{4}}\\
&\le S^{-\frac{3(p-2)}{4}}\|\nabla u_n\|^{\frac{3(p-2)}{2}}_{L^2(\mathbb{R}^3)}\Big(\int_{\mathbb{R}^3\setminus\mathcal{A}_\infty}|u_n|^2dx+o_n(1)\Big)^{\frac{6-p}{4}}\\
&\le S^{-\frac{3(p-2)}{4}}(C_1+o_n(1))^{\frac{5p-10}{4}}\Big(\frac{1}{a_0+a_\infty\lambda}\Big)^{\frac{6-p}{4}}\|u_n\|^2_\lambda+o_n(1).
\end{aligned}\tag{3.4}$$

On the other hand, by conditions (A1)–(A2) once more, we have

$$\mathcal{D}_\lambda(u_n,u_n)\le |a_0|\int_{\mathcal{A}_\infty}|u_n|^2dx=o_n(1). \tag{3.5}$$

Therefore, we deduce from the fact that $(1+\|u_n\|_\lambda)J'_{\alpha,\lambda}(u_n)=o_n(1)$ strongly in E_λ^* that

$$\begin{aligned}&\alpha\|\nabla u_n\|^4_{L^2(\mathbb{R}^2)}+\|u_n\|^2_\lambda\\ &\le S^{-3(p-2)}(C_1+o_n(1))^{\frac{5p-10}{4}}\Big(\frac{1}{a_0+a_\infty\lambda}\Big)^{\frac{6-p}{4}}\|u_n\|^2_\lambda+o_n(1),\end{aligned}$$

which yields that there exists $\overline{\Lambda}_{k_0^*}>\widetilde{\Lambda}_{k_0^*}$ dependent of α such that $u_n=o_n(1)$ strongly in E_λ with $\lambda>\overline{\Lambda}_{k_0^*}$. It is impossible since $c_{\alpha,\lambda}\ge C_\alpha>0$ for all $\lambda>\widetilde{\Lambda}_{k_0^*}$. Therefore $u_{\alpha,\lambda}\ne 0$ in E_λ. It remains to show that $J'_{\alpha,\beta}(u_{\alpha,\beta})=0$ in E_λ^*. In fact, without loss of generality, we may assume that $\|u_n\|^2_{L^2(\mathbb{R}^3)}=A+o_n(1)$ and consider the following energy functional

$$I_{\alpha,\lambda}(u)=\frac{\alpha A}{2}\|u\|^2_{L^2(\mathbb{R}^3)}+\frac12\|u\|^2_\lambda-\frac12\mathcal{D}_\lambda(u,u)-\frac1p\|u\|^p_{L^p(\mathbb{R}^3)}.$$

Clearly, by (2.1), $I_{\alpha,\lambda}(u)$ is of C^2 in E_λ for $\lambda>\overline{\Lambda}_{k_0^*}$. Since $(1+\|u_n\|_\lambda)J'_{\alpha,\lambda}(u_n)=o_n(1)$ strongly in E_λ^*, it is easy to see from $\|u_n\|^2_\lambda=A+o_n(1)$ and $u_n=u_{\alpha,\lambda}+o_n(1)$ weakly in E_λ that $\langle I'_{\alpha,\lambda}(u_n),u_n-u_{\alpha,\beta}\rangle_{E_\lambda^*,E_\lambda}=o_n(1)$ and $I'_{\alpha,\lambda}(u_n)=o_n(1)$ strongly in E_λ^*, so that $I'_{\alpha,\lambda}(u_{\alpha,\lambda})=0$ in E_λ^*. In particular, $\langle I'_{\alpha,\lambda}(u_{\alpha,\lambda}),u_n-u_{\alpha,\beta}\rangle_{E_\lambda^*,E_\lambda}=0$. Now, we obtain

$$\begin{aligned}o_n(1)&=\langle I'_{\alpha,\lambda}(u_n)-I'_{\alpha,\lambda}(u_{\alpha,\lambda}),u_n-u_{\alpha,\beta}\rangle_{E_\lambda^*,E_\lambda}\\ &=\alpha A\|u_n-u_{\alpha,\beta}\|^2_{L^2(\mathbb{R}^3)}+\|u_n-u_{\alpha,\beta}\|^2_\lambda\\ &\quad-\mathcal{D}_\lambda(u_n-u_{\alpha,\beta},u_n-u_{\alpha,\beta})-\|u_n-u_{\alpha,\beta}\|^p_{L^p(\mathbb{R}^3)}.\end{aligned}$$

Since $u_n-u_{\alpha,\beta}=o_n(1)$ weakly in E_λ, by using similar arguments in the proofs of (3.4) and (3.5), we can see that $u_n-u_{\alpha,\beta}=o_n(1)$ strongly in E_λ for λ sufficiently large, say $\lambda>\overline{\Lambda}_{k_0^*}$. Thus, we must have that $J'_{\alpha,\beta}(u_{\alpha,\beta})=0$ in E_λ^* for $\lambda>\overline{\Lambda}_{k_0^*}$. □

The following lemma will give a description on the concentration behavior of the nontrivial solutions $u_{\alpha,\lambda}$ as $\lambda\to+\infty$.

Lemma 3.5. *Suppose that (A1)–(A3) hold. For every $\alpha>0$, if either $a_0\ge 0$ or $a_0<0$ with $\beta_{k_0^*-1}<1$ then we have $u_{\alpha,\lambda}\to u_\alpha$ strongly in $H^1(\mathbb{R}^3)$ as $\lambda\to+\infty$ up to a subsequence. Furthermore, u_α is a nontrivial solution of* (1.5).

Proof. Let u_{α,λ_n} be the nontrivial solution obtained in Lemma 3.4 with $\lambda_n\to+\infty$ as $n\to\infty$. By Lemma 3.3, we can see that

$$\int_{\mathbb{R}^3}(|\nabla u_{\alpha,\lambda_n}|^2+(\lambda_n a(x)+a_0)^+|u_{\alpha,\lambda_n}|^2)dx\le C_1\quad\text{for all } n\in\mathbb{N}.$$

It follows that $\{u_{\alpha,\lambda_n}\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$ for n and

$$\int_{\mathbb{R}^3}(a(x)+\frac{a_0}{\lambda_n})^+|u_{\alpha,\lambda_n}|^2dx=o_n(1).$$

Without loss of generality, we may assume that $u_{\alpha,\lambda_n}=u_\alpha+o_n(1)$ weakly in $D^{1,2}(\mathbb{R}^3)$. Thanks to the Sobolev embedding theorem and conditions (A1)–(A3), we can see that $u_{\alpha,\lambda_n} = u_\alpha + o_n(1)$ strongly in $L^2(\mathbb{R}^3)$ and $u_\alpha \in H_0^1(\Omega)$ with $u_\alpha \equiv 0$ on $\mathbb{R}^3 \backslash \Omega$. Therefore, by the Hölder and the Sobolev inequality, we obtain

$$\begin{aligned}
&\|u_{\alpha,\lambda_n} - u_\alpha\|_{L^p(\mathbb{R}^3)} \\
&\leq \|u_{\alpha,\lambda_n} - u_\alpha\|_{L^2(\mathbb{R}^3)}^{\frac{6-p}{2p}} (\|u_{\alpha,\lambda_n}\|_{L^6(\mathbb{R}^3)} + \|u_\alpha\|_{L^6(\mathbb{R}^3)})^{\frac{3p-6}{2p}} = o_n(1).
\end{aligned}$$

On the other hand, by a variant of the Lebesgue dominated convergence theorem (cf. [20, Theorem 2.2]) and the condition (A1), we also have $\mathcal{D}_{\lambda_n}(u_{\alpha,\lambda_n} - u_\alpha, u_{\alpha,\lambda_n} - u_\alpha) = o_n(1)$. Therefore,

$$\begin{aligned}
\int_\Omega |u_\alpha|^p dx &= \|u_{\alpha,\lambda_n}\|_{L^p(\mathbb{R}^3)}^p + o_n(1) \\
&= \mathcal{D}_{\lambda_n}(u_{\alpha,\lambda_n}, u_{\alpha,\lambda_n}) + \|u_{\alpha,\lambda_n}\|_{\lambda_n}^2 + \alpha \|\nabla u_{\alpha,\lambda_n}\|_{L^2(\mathbb{R}^3)}^4 \\
&\geq \int_\Omega \alpha |\nabla u_\alpha|^4 + |\nabla u_\alpha|^2 + a_0 |u_\alpha|^2 dx + o_n(1).
\end{aligned}$$

Note that $u_\alpha \in H_0^1(\Omega) \subset H^1(\mathbb{R}^3)$, it is easy to see from $J'_{\alpha,\lambda_n}(u_{\alpha,\lambda_n}) = 0$ in $E_{\lambda_n}^*$ that u_α is a solution of (1.5). In particular,

$$\int_\Omega \alpha |\nabla u_\alpha|^4 + |\nabla u_\alpha|^2 + a_0 |u_\alpha|^2 dx = \int_\Omega |u_\alpha|^p dx.$$

Thus, $u_{\alpha,\lambda_n} = u_\alpha + o_n(1)$ strongly in $D^{1,2}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \lambda_n a(x) u_{\alpha,\lambda_n}^2 dx = o_n(1).$$

It follows that $u_{\alpha,\lambda_n} = u_\alpha + o_n(1)$ strongly in $H^1(\mathbb{R}^3)$. Thanks to $c_{\alpha,\lambda} \geq C_\alpha > 0$, u_α must be nonzero. Hence, u_α is a nontrivial solution of (1.5). □

Proof of Theorem 1.1. The statement of the theorem follows immediately from Lemmas 3.4 and 3.5. □

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