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ANALYTICITY OF THE GLOBAL ATTRACTOR FOR THE 3D REGULARIZED MHD EQUATIONS

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Abstract. We study the three-dimensional (3D) regularized magnetohydrodynamics (MHD) equations. Using the method of splitting of the asymptotic approximate solutions into higher and lower Fourier components, we prove that the global attractor of the 3D regularized MHD equations consists of real analytic functions, whenever the forcing terms are analytic.

1. Introduction

In this article, we investigate the regularized magnetohydrodynamics (MHD) equations

$$\partial_t(u-\alpha^2\Delta u)-\nu\Delta u+(u\cdot\nabla)u-(b\cdot\nabla)b+\nabla p=f,\quad (x,t)\in\Omega\times\mathbb{R}_+, \tag{1.1}$$

$$\partial_t(b-\beta^2\Delta b)-\mu\Delta b+(u\cdot\nabla)b-(b\cdot\nabla)u=g,\quad (x,t)\in\Omega\times\mathbb{R}_+, \tag{1.2}$$

$$\nabla\cdot u=\nabla\cdot b=0,\quad (x,t)\in\Omega\times\mathbb{R}_+, \tag{1.3}$$

$$u(x,0)=u^{\rm in},\quad b(x,0)=b^{\rm in},\quad x\in\Omega, \tag{1.4}$$

which are regularizations in both the velocity and the magnetic field of the following large eddy simulation model for the turbulent flow of a magnetofluid (see [9]):

$$\partial_t u-\nu\Delta u+(u\cdot\nabla)u-(b\cdot\nabla)b+\nabla p=f,\quad (x,t)\in\Omega\times\mathbb{R}_+, \tag{1.5}$$

$$\partial_t b-\mu\Delta b+(u\cdot\nabla)b-(b\cdot\nabla)u=g,\quad (x,t)\in\Omega\times\mathbb{R}_+, \tag{1.6}$$

$$\nabla\cdot u=\nabla\cdot b=0,\quad (x,t)\in\Omega\times\mathbb{R}_+, \tag{1.7}$$

$$u(x,0)=u^{\rm in},\quad b(x,0)=b^{\rm in},\quad x\in\Omega, \tag{1.8}$$

where the velocity field $u=(u_1,u_2,u_3)$, the magnetic field $b=(b_1,b_2,b_3)$ and the total pressure $p(x,t)$ are the unknown terms, ν is the kinematic viscosity and μ is the constant magnetic resistivity, f represents volume force applied to the fluid, g is usually zero when Maxwell's displacement currents are ignored. We will assume the constants ν,μ,α and β are all positive. We consider the case $\Omega=[0,L]^3\subset\mathbb{R}^3$ $(L>0)$ and assume space-periodic conditions on the initial data so that the corresponding solutions are space-periodic.

Compared to equations (1.5) and (1.6), equations (1.1) and (1.2) contain the extra regularizing terms $(-\alpha^2\partial_t\Delta u)$ and $(-\beta^2\partial_t\Delta b)$, respectively. These two terms

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have two main effects. On the one hand, they regularize the equations in a way that the 3D equations (1.1)-(1.4) become now globally wellposed (see Lemma 2.2 in Section 2 or [9]). On the other hand, they change the parabolic character of the MHD equations to the regularized one. For this reason, we call equations (1.1)-(1.2) as regularized MHD equations.

We can view equations (1.1)-(1.4), also called MHD-Voight equations, as the approximate equations for the MHD equations (1.5)-(1.8) as $\alpha, \beta \to 0$. Equations (1.1)-(1.4) are the exact equations for a class of visco-elastic fluid known as Kelvin-Voight fluids (the corresponding equations for nonmagnetic fluids were first introduced by Oskolkov in [30]). Equations (1.5)-(1.8) (or the relevant equations) have been widely studied, see e.g. [10, 36] for the existence and uniqueness of the solutions and [7, 41, 47, 48] for the regularity criteria. Also, equations (1.1)-(1.4) have been deeply studied, see e.g., Catania and Secchi [8, 10], Catania [9], Larios and Titi [21, 22], Levant, Ramos and Titi [23]. Particularly, Catania [9] studied the global attractor and determining modes for the solutions of equations (1.1)-(1.4) in $[0, 2\pi L]^3$ $(L > 0)$ with space-periodic conditions.

Attractor is an important concept in the study of infinite-dimensional dynamical systems. There are some monographs concerning this subject, see, e.g., [12, 34, 35, 39]. At the same time, the theory and method within these monographs have been extensively applied to many concrete partial differential equations arising in mathematical physics. For instance, we can refer to [44, 45, 46]. We note that Foias, Manley, Rosa and Temam in the monograph [16] researched the determining modes and nodes, as well as the Gevrey regularity of the global attractor for the Navier-Stokes equations. The Gevrey class regularity of the attractor reveals that the solutions lying in the attractor are analytic with values in a Gevrey class of analytic functions in space.

Nowadays, there are some papers investigating the Gevrey class regularity of the solutions. For example, we can refer to [2, 3, 6, 15, 25] for the Navier-Stokes equations; to [42, 43] for the Navier-Stokes-α equations; to [31, 32] for the second-grade fluid equations; to [4] for a class of dissipative equations; to [5] for a class of hypoelliptic equations; to [11] for the time-dependent Ginzburg-Landau equations; to [14] for nonlinear analytic parabolic equations; to [18] for a class of water-wave models; to [20, 24] for the Euler equations; to [26, 29] for the Bénard equation; to [27] for the laser equations; to [28] for the weakly damped driven nonlinear Schrödinger equation; to [33] for the Kuramoto-Sivashinsky equation; to [37] for the micropolar fluid equations, so on and so forth.

Recently, Levant and Titi proved in [19] the Gevrey regularity for the attractor of the 3D Navier-Stokes-Voight equations. The method of the proof in [19] is the splitting of the velocity into higher and lower Fourier components. The method of splitting of the unknown functions into higher and lower Fourier components has been used before in the context of the weakly damped driven nonlinear Schrödinger equation in Oliver and Titi [28] and a model of Béard convection in a porous medium in Oliver and Titi [29] (see also Goubet [17]). Recently, the authors of Chueshov *et al.* [13] followed the same methods to prove the Gevrey regularity of the global attractor of the generalized Benjamin-Bona-Mahony equation.

The purpose of this paper is to establish the Gevrey regularity of the global attractor for the 3D MHD equations (1.1)-(1.4). Our result reveals that all solutions within the global attractor are real analytic functions, whenever the forcing terms are analytic.

We want to point that the idea of this paper originates from article [19]. Different to the Navier-Stokes-Voight equations studied in [19], equations (1.1)-(1.4) contain the coupled Maxwell's equations which rule the magnetic field. Observing the coupled structure of the addressed equations, we expect to extend the method of the proof in [19]–splitting both the velocity field $u(x,t)$ and the magnetic field $b(x,t)$ into higher and lower Fourier components. In terms of these splitting, we then construct the asymptotic approximations of the solutions in the Sobolev space V_m and the Gevrey space G^1_τ (see Section 2 for notations). When doing so, however, we need give proper decomposition of the equations. At the same time, we find that the coupled structure of the regularized MHD equations plays an important role when we estimate the relevant nonlinear terms in the asymptotic approximate solutions.

This article is organized as follows. In the next section, we introduce some notations and operators, as well as some lemmas that will be used frequently in our proof. In Section 3, we construct the asymptotic approximation of the solution of the regularized MHD equations in the Sobolev space V_m for $m \geqslant 2$. In Section 4, we construct the asymptotic approximation of the solution in the Gevrey space G^2_τ for some $\tau > 0$.

2. Preliminaries

Throughout this article, we denoted by $\mathbb{L}^p(\Omega) = L^p(\Omega) \times L^p(\Omega) \times L^p(\Omega)$, for $1 \leqslant p \leqslant \infty$, and $\mathbb{H}^m(\Omega) = H^m(\Omega) \times H^m(\Omega) \times H^m(\Omega)$ the 3D vector Lebesgue and Sobolev spaces (see [1]) of the periodic functions on Ω, respectively. Let $\digamma$ be the set of all 3D vector trigonometric polynomials on the periodic domain Ω, and denote

$$\mathcal{V} = \Big\{\phi \in \digamma : \nabla \cdot \phi = 0 \text{ and} \int_\Omega \phi(x)\mathrm{d}x = 0\Big\},$$
$$H = \text{closure of } \mathcal{V} \text{ in the } \mathbb{L}^2(\Omega) \text{ topology},$$
$$V = \text{closure of } \mathcal{V} \text{ in the } \mathbb{H}^1(\Omega) \text{ topology}.$$

Also, we let $P_\sigma : \mathbb{L}^2(\Omega) \to H$ be the Helmholtz-Leray projection operator and $A = -P_\sigma\Delta$ be the stokes operator subject to the periodic boundary conditions with domain $D(A) = \mathbb{H}^2(\Omega) \cap V$. Notice that in the space-periodic case

$$Au = -P_\sigma \Delta u = -\Delta u, \quad \forall u \in D(A).$$

We know that the operator A^{-1} is a positive definite, self-adjoint, compact operator from H into H. Thus, for any $s \in \mathbb{R}$, we can define the Hilbert space $V_s = D(A^{s/2})$ endowed with the inner product and norm as

$$(u,v)_s = \sum_{j\in\mathbb{Z}^3} (u_j \cdot v_j |j|^{2s}), \quad |u|_s^2 = (u,u)_s,$$

for any $u, v \in V_s$, where u_j, v_j are the corresponding Fourier coefficients of u and v, respectively. Obviously, we have $V = V_1$ and $V_0 = H$. We will denote the inner product and norm in $V_0 = H$ as

$$(u,v) = \sum_{j\in\mathbb{Z}^3} (u_j \cdot v_j), \quad \|u\|^2 = (u,u), \quad \forall u, v \in V_0.$$

To deal with the convective terms in equations (1.1) and (1.2), we introduce the bilinear form
$$B(u,v) = P_\sigma((u\cdot\nabla)v), \quad \forall u,v\in\mathcal{V}.$$
We see (e.g., [19, 40]) that B can be extended to a continuous map $B : V\times V \longmapsto V'$, where $V' = V_{-1}$ is the dual space of V, moreover,
$$|\langle B(u,v),w\rangle_{V',V}| \leqslant c(\Omega)\lambda_1^{-3/4}\|u\|^{1/2}|u|_1^{1/2}|v|_1|w|_1, \quad \forall u,v,w\in V, \tag{2.1}$$
where λ_1 is the first eigenvalue of A, $\langle\cdot,\cdot\rangle_{V',V}$ is the dual product between V' and V, and hereafter $c(\cdot)$ or $c(\cdot,\cdot,\cdots,\cdot) > 0$ denotes the generic constant depending only the quantities appearing in the bracket, which may take different values from one line to the next. Furthermore, we have

Lemma 2.1. *The bilinear form $B(\cdot,\cdot)$ satisfies*

(I) *If $u,b\in V$, then all $B(u,u)$, $B(b,b)$, $B(u,b)$, $B(b,u)$ belong to $V_{-1/2}$, and*
$$|B(u,u)|_{-1/2} \leqslant c(\Omega)\lambda_1^{-3/4}|u|_1^2,$$
$$|B(b,b)|_{-1/2} \leqslant c(\Omega)\lambda_1^{-3/4}|b|_1^2,$$
$$|B(u,b)|_{-1/2} \leqslant c(\Omega)\lambda_1^{-3/4}|u|_1|b|_1,$$
$$|B(b,u)|_{-1/2} \leqslant c(\Omega)\lambda_1^{-3/4}|u|_1|b|_1.$$

(II) *If $u,b\in V_{3/2}$, then all $B(u,u)$, $B(b,b)$, $B(u,b)$, $B(b,u)$ belong to H, and*
$$\|B(u,u)\| \leqslant c(\Omega)\lambda_1^{-3/4}|u|_1|u|_{3/2},$$
$$\|B(b,b)\| \leqslant c(\Omega)\lambda_1^{-3/4}|b|_1|b|_{3/2},$$
$$\|B(u,b)\| \leqslant c(\Omega)\lambda_1^{-3/4}|u|_1|b|_{3/2},$$
$$\|B(b,u)\| \leqslant c(\Omega)\lambda_1^{-3/4}|u|_{3/2}|b|_1.$$

(III) *For any $m\geqslant 1$, if $u,b\in V_{m+1}$, then all $B(u,u)$, $B(b,b)$, $B(u,b)$, $B(b,u)$ belong to V_m, and*
$$|B(u,u)|_m \leqslant c(m,\Omega)\lambda_1^{-7/8}|u|_1^{1/4}|u|_2^{3/4}|u|_{m+1},$$
$$|B(b,b)|_m \leqslant c(m,\Omega)\lambda_1^{-7/8}|b|_1^{1/4}|b|_2^{3/4}|b|_{m+1},$$
$$|B(u,b)|_m \leqslant c(m,\Omega)\lambda_1^{-7/8}|u|_1^{1/4}|u|_2^{3/4}|b|_{m+1},$$
$$|B(b,u)|_m \leqslant c(m,\Omega)\lambda_1^{-7/8}|b|_1^{1/4}|b|_2^{3/4}|u|_{m+1}.$$

Proof. The conclusions of this lemma can be obtained by the standard interpolation estimates and the Gagliardo-Nirenberg inequity (see e.g., [40, 38]). Here we omit the details. □

With the above notation and definitions, we can write equations (1.1)-(1.4) in the equivalent functional form
$$u_t + \nu Au + \alpha^2 Au_t + B(u,u) - B(b,b) = f, \tag{2.2}$$
$$b_t + \mu Ab + \beta^2 Ab_t + B(u,b) - B(b,u) = g, \tag{2.3}$$
$$u(x,0) = u^{\text{in}}, \quad b(x,0) = b^{\text{in}}. \tag{2.4}$$
For the global well-posedness of the solutions and existence of the global attractor for equations (2.2)-(2.4), we have

Lemma 2.2 ([9]). *Assume that the initial data and the forcing terms f and g are space-periodic with zero-spatial-mean vector fields that satisfy $f, g \in L^\infty(\mathbb{R}_+; H)$, $(u^{\rm in}, b^{\rm in}) \in V \times V$, and $\nabla \cdot u^{\rm in} = \nabla \cdot b^{\rm in} = 0$. Then equations (2.2)-(2.4) have a unique global solution (u, b) such that, for each $T > 0$, one has*

$$u, b \in L^\infty([0,T]; V) \cap L^2([0,T]; \mathbb{H}^2(\Omega)).$$

If furthermore f and g are independent of time t, then the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by the solution operators possesses a unique global attractor $\mathcal{A} \subset V \times V$.

The motivation of this paper is to investigate the analyticity of the solutions within the global attractor $\mathcal{A}$. We will employ the concept of the Gevrey class regularity. For some given $\tau > 0$ and $r \geqslant 0$, we define the Gevrey space as

$$G_\tau^r = D(A^{r/2} e^{\tau A^{1/2}}) = \big\{ u \in H \big| \, \|A^{r/2} e^{\tau A^{1/2}} u\|^2 = \sum_{j\in\mathbb{Z}^3} |u_j|^2 |j|^{2r} e^{2\tau|j|} < \infty \big\},$$

which is endowed with the inner product and norm as follows

$$(u, v)_{r,\tau} = \Big(A^{r/2} e^{\tau A^{1/2}} u, A^{r/2} e^{\tau A^{1/2}} v \Big) = \sum_{j\in\mathbb{Z}^3} u_j \cdot v_j |j|^{2r} e^{2\tau|j|},$$
$$|u|_{r,\tau} = \|A^{r/2} e^{\tau A^{1/2}} u\|, \quad u, v \in G_\tau^r,$$

where u_j and v_j are the corresponding Fourier coefficients of u and v, respectively. Note that Levermore and Oliver [24] proved that the space of real analyticity functions $\mathcal{C}^\omega(\Omega)$ has the following characterization

$$\mathcal{C}^\omega(\Omega) = \cup_{\tau>0} G_\tau^r.$$

We end this section with some technique lemmas that will be used in the proof of our main results. First, let $\lambda > 0$ and denote by P_λ the H-orthogonal projection onto the span of eigenfunctions of A corresponding to eigenvalues of the magnitude less than or equal to λ. Set $Q_\lambda = I - P_\lambda$. The following Poincaré-type inequality holds.

Lemma 2.3 ([19]). *Let $\bar{u} \in P_\lambda G_\tau^{r+1}$ and $\check{u} \in Q_\lambda G_\tau^{r+1}$. Then*

$$|\overline{u}|_{r+1,\tau} \leqslant e^{\tau\lambda^{1/2}} |\overline{u}|_{r+1} \quad \text{and} \quad |\check{u}|_{r,\tau} \leqslant \lambda^{-1/2} |\check{u}|_{r+1,\tau}.$$

Lemma 2.4 ([19]). *For any $\tau > 0$, $u, w \in G_\tau^2$, and $v \in G_\tau^1$, the following inequality holds*

$$\big| \big(B(u,v), w\big)_{1,\tau} \big| \leqslant c(\Omega) \lambda_1^{-3/4} |u|_{1,\tau}^{1/2} |u|_{2,\tau}^{1/2} |v|_{1,\tau} |w|_{2,\tau}.$$

Lemma 2.5 ([19]). *Let $s \in \mathbb{R}$. Assume that $h(t) \in L^\infty([0,T]; V_{s-2})$ for some $T \in (0, \infty)$. Then the linear problem*

$$z_t + \nu A z + \alpha^2 A z_t = h(t), \quad z(0) = 0,$$

has a unique solution $z(t) \in \mathcal{C}([0,T]; V_s)$. In addition, the following estimate holds

$$|z(t)|_s \leqslant \frac{\|h(t)\|_{L^\infty([0,T];V_{s-2})}}{\alpha\nu\sqrt{(1/\lambda_1 + \alpha^2)^{-1}}}, \tag{2.5}$$

for all $t \in [0, T]$.

Lemma 2.6 ([19]). *Consider a nonnegative function $\psi(t)$ satisfying for all $t \geqslant t_0$, for some t_0, the following inequality*

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} \leqslant -a\psi + d\psi^{3/2} + C, \quad \psi(t_0) = 0,$$

where the positive real coefficient a, d, C obey

$$dC^{1/2} < (a/2)^{3/2}. \tag{2.6}$$

Then for all $t \geqslant t_0$, there holds $\psi(t) \leqslant 2C/a$.

Finally, we will use the following lemma from Foias *et al.* [16].

Lemma 2.7 ([16]). *Let $\psi(t)$ and $\phi(t)$ be locally integrable functions on $(0, +\infty)$ which satisfy some $T > 0$ the conditions*

$$\begin{gathered}
\liminf_{t\to+\infty} \frac{1}{T}\int_t^{t+T} \psi(\tau)\mathrm{d}\tau > 0, \\
\limsup_{t\to+\infty} \frac{1}{T}\int_t^{t+T} \psi^-(\tau)\mathrm{d}\tau < \infty, \\
\limsup_{t\to+\infty} \frac{1}{T}\int_t^{t+T} \phi^+(\tau)\mathrm{d}\tau = 0,
\end{gathered}$$

where $\psi^- = \max\{-\psi, 0\}$ and $\phi^+ = \max\{\phi, 0\}$. Suppose that $\varphi(t)$ is a nonnegative, absolutely continuous function on $[0, +\infty)$ that satisfies the following inequality a.e. on $[0, +\infty)$,

$$\varphi'(t) + \psi(t)\varphi(t) \leqslant \phi(t).$$

Then $\varphi(t) \longrightarrow 0$ as $t \to +\infty$.

3. Asymptotic approximation in V_m

The aim of this section is to construct an asymptotic approximation of the solution of equations (2.2)-(2.4) in the space V_m, for every $m \geqslant 2$. Note that

$$\cap_{m=1}^{\infty} V_m \subset \mathcal{C}^{\infty}(\Omega).$$

Theorem 3.1. *Let $m \geqslant 2$ be an integer, and let $(u(x,t), b(x,t))$ be a solution of equations (2.2)-(2.4) corresponding to the initial condition $(u^{\mathrm{in}}, b^{in}) \in V \times V$ with the forcing terms $(f, g) \in V_{m-2} \times V_{m-2}$. Then there exist two functions*

$$v^{(m)}(x,t) \in L^{\infty}([0,+\infty); V_m) \quad \text{and} \quad \xi^{(m)}(x,t) \in L^{\infty}([0,+\infty); V_m)$$

satisfying

$$\lim_{t\to+\infty} |u(t) - v^{(m)}(t)|_1 = \lim_{t\to+\infty} |b(t) - \xi^{(m)}(t)|_1 = 0.$$

Proof. Let us fixed $m \geqslant 2$, and let $(u^{\mathrm{in}}, b^{\mathrm{in}}) \in V \times V$. Firstly, we can write the solution as

$$\begin{aligned} u(t) &= v(t) + w(t) \\ b(t) &= \xi(t) + \eta(t), \end{aligned} \tag{3.1}$$

where $(v(t), w(t))$ and $(\xi(t), \eta(t))$ satisfy the coupled equations

$$v_t + \nu Av + \alpha^2 Av_t = f - B(u,u) + B(b,b), \quad v(0) = 0, \tag{3.2}$$

$$w_t + \nu Aw + \alpha^2 Aw_t = 0, \quad w(0) = u^{\mathrm{in}}, \tag{3.3}$$

and

$$\xi_t + \mu A\xi + \beta^2 A\xi_t = g - B(u,b) + B(b,u), \quad \xi(0) = 0, \tag{3.4}$$

$$\eta_t + \mu A\eta + \beta^2 A\eta_t = 0, \quad \eta(0) = b^{\rm in}, \tag{3.5}$$

respectively. By using that both $u(x,t)$ and $b(x,t)$ belong to $L^\infty([0,+\infty);V)$, and applying Lemma 2.1(I) and Lemma 2.5, we see that

$$\begin{gathered} v(t) \in L^\infty([0,+\infty);V_{3/2}) \\ \xi(t) \in L^\infty([0,+\infty);V_{3/2}). \end{gathered} \tag{3.6}$$

Now, by (3.3) and (3.5), we conclude that

$$\|w(t)\|^2 + \alpha^2|w|_1^2 \leqslant e^{-\delta_1 t}(\|u^{\rm in}\|^2 + \alpha^2|u^{\rm in}|_1^2),$$
$$\|\eta(t)\|^2 + \beta^2|\eta|_1^2 \leqslant e^{-\delta_2 t}(\|b^{\rm in}\|^2 + \beta^2|b^{\rm in}|_1^2),$$

where[$\delta_1 := (1/\lambda_1 + \alpha^2)^{-1}$, $\delta_2 := (1/\lambda_1 + \beta^2)^{-1}$. Therefore,

$$\lim_{t\to+\infty} |u(t) - v(t)|_1 = \lim_{t\to+\infty} |w(t)|_1 = 0, \tag{3.7}$$

$$\lim_{t\to+\infty} |b(t) - \xi(t)|_1 = \lim_{t\to+\infty} |\eta(t)|_1 = 0. \tag{3.8}$$

At the next step, we consider $v^{(2)}(x,t)$, the solution of the equations

$$v_t^{(2)} + \nu A v^{(2)} + \alpha^2 A v_t^{(2)} = f - B(v,v) + B(\xi,\xi), \tag{3.9}$$

$$v^{(2)}(0) = 0. \tag{3.10}$$

By Lemma 2.1 (II) and (3.6), we see that the right-hand side of equality (3.9) lies in $L^\infty([0,+\infty);H)$. Hence, applying Lemma 2.5, we find that the unique solution of (3.9)-(3.10) satisfies

$$v^{(2)}(t) \in L^\infty([0,+\infty);V_2). \tag{3.11}$$

Similarly, we consider $\xi^{(2)}(x,t)$, the solution of the equations

$$\xi_t^{(2)} + \mu A\xi^{(2)} + \beta^2 A\xi_t^{(2)} = g - B(v,\xi) + B(\xi,v), \tag{3.12}$$

$$\xi^{(2)}(0) = 0. \tag{3.13}$$

We also can conclude that the unique solution of (3.12)-(3.13) satisfies

$$\xi^{(2)}(t) \in L^\infty([0,+\infty);V_2). \tag{3.14}$$

Now, we set $z^{(2)}(x,t) = v^{(2)}(x,t) - v(x,t)$, which satisfies

$$\begin{aligned} &z_t^{(2)} + \nu A z^{(2)} + \alpha^2 A z_t^{(2)} \\ &= B(u,u-v) + B(u-v,v) - B(b,b-\xi) - B(b-\xi,\xi), \end{aligned} \tag{3.15}$$

$$z^{(2)}(0) = 0. \tag{3.16}$$

At the same time, we set $\eta^{(2)}(x,t) = \xi^{(2)}(x,t) - \xi(x,t)$, which satisfies

$$\begin{aligned} &\eta_t^{(2)} + \mu A\eta^{(2)} + \beta^2 A\eta_t^{(2)} \\ &= B(u-v,\xi) + B(u,b-\xi) - B(b-\xi,v) - B(b,u-v), \end{aligned} \tag{3.17}$$

$$\eta^{(2)}(0) = 0. \tag{3.18}$$

Since $u,b \in L^\infty([0,+\infty);V)$, and v and ξ satisfy (3.6), we conclude from Lemma 2.5 that equations (3.15)-(3.16) have a unique solution $z^{(2)} \in L^\infty([0,+\infty);V_{3/2})$.

Similarly, equations (3.17)-(3.18) have a unique solution $\eta^{(2)} \in L^\infty([0,+\infty); V_{3/2})$. Then taking an inner product of (3.15) with $z^{(2)}$ and using (2.1), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|z^{(2)}\|^2 + \alpha^2|z^{(2)}|_1^2) + \nu|z^{(2)}|_1^2 \\
&= B(u, u-v, z^{(2)}) + B(u-v, v, z^{(2)}) - B(b, b-\xi, z^{(2)}) - B(b-\xi, \xi, z^{(2)}) \\
&\leqslant c(\Omega)|u|_1\,|u-v|_1\,|z^{(2)}|_1 + c(\Omega)|u-v|_1\,|v|_1\,|z^{(2)}|_1 \\
&\quad + c(\Omega)|b|_1\,|b-\xi|_1\,|z^{(2)}|_1 + c(\Omega)|b-\xi|_1\,|\xi|_1\,|z^{(2)}|_1 \\
&\leqslant c(\Omega,\nu)\big(|u-v|_1^2(|u|_1^2 + |v|_1^2) + |b-\xi|_1^2(|\xi|_1^2 + |b|_1^2)\big) + \frac{\nu}{2}|z^{(2)}|_1^2,
\end{aligned}$$

which gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|z^{(2)}\|^2 + \alpha^2|z^{(2)}|_1^2) + \frac{\nu\delta_1}{2}(\|z^{(2)}\|^2 + \alpha^2|z^{(2)}|_1^2) \\
&\leqslant c(\Omega,\nu)\big(|u-v|_1^2(|u|_1^2 + |v|_1^2) + |b-\xi|_1^2(|\xi|_1^2 + |b|_1^2)\big).
\end{aligned} \tag{3.19}$$

Analogous to the derivation of (3.19), we can obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\eta^{(2)}\|^2 + \beta^2|\eta^{(2)}|_1^2) + \frac{\mu\delta_2}{2}(\|\eta^{(2)}\|^2 + \beta^2|\eta^{(2)}|_1^2) \\
&\leqslant c(\Omega,\nu)\big(|u-v|_1^2(|b|_1^2 + |\xi|_1^2) + |b-\xi|_1^2(|u|_1^2 + |v|_1^2)\big).
\end{aligned} \tag{3.20}$$

By (3.7)-(3.8) and the fact that all $u(t), v(t), b(t), \xi(t)$ are bounded uniformly in time in the V norm, we conclude from (3.19)-(3.20) and Lemma 2.7 that

$$\begin{gathered}
\lim_{t\to+\infty}|z^{(2)}(t)|_1 = \lim_{t\to+\infty}|v(t) - v^{(2)}(t)|_1 = 0, \\
\lim_{t\to+\infty}|\eta^{(2)}(t)|_1 = \lim_{t\to+\infty}|\xi(t) - \xi^{(2)}|_1 = 0.
\end{gathered}$$

We next continue by induction. Fix $2 \leqslant n \leqslant m$, and suppose that we have constructed $v^{(j)}(t) \in L^\infty([0,+\infty); V_j)$ and $\xi^{(j)}(t) \in L^\infty([0,+\infty); V_j)$, for $j = 1, 2, \cdots, n-1$, such that for any j

$$\lim_{t\to+\infty}|v^{(j-1)}(t) - v^{(j)}(t)|_1 = \lim_{t\to+\infty}|u(t) - v^{(j)}(t)|_1 = 0, \tag{3.21}$$

$$\lim_{t\to+\infty}|\xi^{(j-1)}(t) - \xi^{(j)}(t)|_1 = \lim_{t\to+\infty}|b(t) - \xi^{(j)}(t)|_1 = 0. \tag{3.22}$$

Let us consider the equations

$$v_t^{(n)} + \nu A v^{(n)} + \alpha^2 A v_t^{(n)} = f - B(v^{(n-1)}, v^{(n-1)}) + B(\xi^{(n-1)}, \xi^{(n-1)}), \tag{3.23}$$

$$v^{(n)}(0) = 0, \tag{3.24}$$

and

$$\xi_t^{(n)} + \mu A \xi^{(n)} + \beta^2 A \xi_t^{(n)} = g - B(v^{(n-1)}, \xi^{(n-1)}) + B(\xi^{(n-1)}, v^{(n-1)}), \tag{3.25}$$

$$\xi^{(n)}(0) = 0. \tag{3.26}$$

By Lemma 2.5 and the estimates on the nonlinear terms of Lemma 2.1, the unique solution $v^{(n)}(t)$ of (3.23)-(3.24), and $\xi^{(n)}(t)$ of (3.25)-(3.26), satisfy, respectively

$$\begin{gathered}
v^{(n)}(t) \in L^\infty([0,+\infty); V_n), \\
\xi^{(n)}(t) \in L^\infty([0,+\infty); V_n).
\end{gathered}$$

Furthermore, we denote $z^{(n)}(t) = v^{(n)}(t) - v^{(n-1)}(t)$, which satisfies

$$\begin{aligned} &z_t^{(n)} + \nu A z^{(n)} + \alpha^2 A z_t^{(n)} \\ &= B(v^{(n-2)}, v^{(n-2)} - v^{(n-1)}) + B(v^{(n-2)} - v^{(n-1)}, v^{(n-1)}) \\ &\quad - B(\xi^{(n-2)}, \xi^{(n-2)} - \xi^{(n-1)}) - B(\xi^{(n-2)} - \xi^{(n-1)}, \xi^{(n-1)}), \end{aligned} \tag{3.27}$$

$$z^{(n)}(0) = 0. \tag{3.28}$$

We also set $\eta^{(n)}(x,t) = \xi^{(n)}(x,t) - \xi^{(n-1)}(x,t)$, which satisfies

$$\begin{aligned} &\eta_t^{(n)} + \mu A \eta^{(n)} + \beta^2 A \eta_t^{(n)} \\ &= B(v^{(n-2)} - v^{(n-1)}, \xi^{(n-1)}) + B(v^{(n-2)}, \xi^{(n-2)} - \xi^{(n-1)}) \\ &\quad - B(\xi^{(n-2)} - \xi^{(n-1)}, v^{(n-1)}) - B(\xi^{(n-2)}, v^{(n-2)} - v^{(n-1)}), \end{aligned} \tag{3.29}$$

$$z^{(n)}(0) = 0. \tag{3.30}$$

Taking the inner product of equation (3.27) with $z^{(n)}(t)$ and using Lemma 2.1 and (3.21)-(3.22), we get by Lemma 2.7 that

$$\lim_{t\to+\infty} |z^{(n)}(t)|_1 = \lim_{t\to+\infty} |v^{(n-2)}(t) - v^{(n-1)}(t)|_1 = \lim_{t\to+\infty} |u(t) - v^{(n)}(t)|_1 = 0. \tag{3.31}$$

Similarly, taking the inner product of equation (3.29) with $\xi^{(n)}(t)$ and using Lemma 2.1 and (3.21)-(3.22), we get by Lemma 2.7 that

$$\lim_{t\to+\infty} |\eta^{(n)}(t)|_1 = \lim_{t\to+\infty} |\xi^{(n-2)}(t) - \xi^{(n-1)}(t)|_1 = \lim_{t\to+\infty} |b(t) - \xi^{(n)}(t)|_1 = 0. \tag{3.32}$$

The proof of Theorem 3.1 is complete. □

4. Asymptotic approximation in the Gevrey space

The result of Section 3 shows that the global attractor of equation (1.1)-(1.4) lies in $\mathcal{C}^\infty$ whenever the forcing terms f and g are $\mathcal{C}^\infty$. The aim of this section is to show that the global attractor is real analytic, whenever the forcing terms f and g are analytic. To this end, we will borrow the idea of Oliver and Titi [28, 29] and Kalantarov, Levant and Titi [19], to construct the asymptotic approximation of the solution of equations (2.2)-(2.4) in the Gevrey space G^2_τ, for some $\tau > 0$.

Firstly, it can be proved (see Catania[9]) that the solution of the regularized MHD equations satisfies for all $t \geqslant 0$,

$$\begin{aligned} &\|u(t)\|^2 + \alpha^2 |u(t)|_1^2 + \|b(t)\|^2 + \beta^2 |b(t)|_1^2 \\ &\leqslant e^{-\delta_5 t} (\|u^{\rm in}\|^2 + \alpha^2 |u^{\rm in}|_1^2 + \|b^{\rm in}\|^2 + \beta^2 |b^{\rm in}|_1^2 - \frac{\delta_3}{\delta_5}) + \frac{\delta_3}{\delta_5}, \end{aligned}$$

where

$$\delta_3 := \frac{|f|_{-1}^2}{\nu} + \frac{|g|_{-1}^2}{\mu}, \quad \delta_4 := \min\{\mu, \nu\}, \quad \delta_5 := \frac{\delta_4}{2} \min\{\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \lambda_1\}.$$

Therefore, there exists some t_* depending on $\|u^{\rm in}\|$, $|u^{\rm in}|_1$, $\|b^{\rm in}\|$, $|b^{\rm in}|_1$, $|f|_{-1}$, $|g|_{-1}$, μ, ν, α, β and λ_1, such that for all $t \geqslant t_*$

$$|u(t)|_1 \leqslant M_1/\alpha, \tag{4.1}$$

$$|b(t)|_1 \leqslant M_1/\beta, \tag{4.2}$$

where $M_1 := \sqrt{2\delta_3/\delta_5}$. Analogously, there exist two positive constants $M^{(1)}$ and $M^{(2)}$, which depend only on $\|u^{\rm in}\|$, $|u^{\rm in}|_1$, $\|b^{\rm in}\|$, $|b^{\rm in}|_1$, $|f|_{-1}$, $|g|_{-1}$, μ, ν, α, β and λ_1, such that the solutions of (3.9)-(3.10) and (3.12)-(3.13) satisfy

$$|v^{(2)}|_1 \leqslant M_1^{(1)}, \quad \forall\, t \geqslant t_*,$$
$$|\xi^{(2)}|_1 \leqslant M_2^{(2)}, \quad \forall\, t \geqslant t_*.$$

Lemma 4.1. *Let f and g belong to V_{m-2}. Consider $t_* \geqslant 0$, such that the solution of the regularized MHD equations (2.2)-(2.4) satisfies the relations (4.1) and (4.2) for all $t \geqslant t_*$. Then the following statements are true*

(1) *The functions $v(x,t)$, $\xi(x,t) \in L^\infty([0,+\infty); V_{3/2})$, constructed in Theorem 3.1 satisfy for all $t \geqslant t_*$,*

$$|v(t)|_{3/2} \leqslant M_{3/2}^{(1)} := \frac{|f|_{-1/2} + c(\Omega)\lambda_1^{-\frac{3}{4}} M_1^2(\frac{1}{\alpha^2} + \frac{1}{\beta^2})}{\alpha\nu\sqrt{(\lambda_1^{-1} + \alpha^2)^{-1}}},$$

$$|\xi(t)|_{3/2} \leqslant M_{3/2}^{(2)} := \frac{|g|_{-1/2} + c(\Omega)\lambda_1^{-\frac{3}{4}} M_1^2(\frac{1}{\alpha^2} + \frac{1}{\beta^2})}{\beta\mu\sqrt{(\lambda_1^{-1} + \beta^2)^{-1}}}.$$

(2) *The functions $v^{(2)}(x,t)$, $\xi^{(2)}(x,t) \in L^\infty([0,+\infty); V_2)$, constructed in Theorem 3.1 satisfy for all $t \geqslant t_*$,*

$$|v^{(2)}(t)|_2 \leqslant M_2^{(1)} := \frac{|f| + c(\Omega)\lambda_1^{-\frac{3}{4}} M_1(\frac{M_{3/2}^{(1)}}{\alpha} + \frac{M_{3/2}^{(2)}}{\beta})}{\alpha\nu\sqrt{(\lambda_1^{-1} + \alpha^2)^{-1}}},$$

$$|\xi^{(2)}(t)|_2 \leqslant M_2^{(2)} := \frac{|g| + c(\Omega)\lambda_1^{-\frac{3}{4}} M_1(\frac{M_{3/2}^{(2)}}{\alpha} + \frac{M_{3/2}^{(1)}}{\beta})}{\beta\mu\sqrt{(\lambda_1^{-1} + \beta^2)^{-1}}}.$$

(3) *Let m be an integer. The functions $v^{(m)}(x,t)$, $\xi^{(m)}(x,t) \in L^\infty([0,+\infty); V_m)$, constructed in Theorem 3.1 satisfy for all $t \geqslant t_*$,*

$$|v^{(m)}(t)|_m \leqslant M_m^{(1)}, \quad |\xi^{(m)}(t)|_m \leqslant M_m^{(2)},$$

where

$$M_m^{(1)} := \frac{|f|_{m-2} + c(m,\Omega)\lambda_1^{-\frac{7}{8}}\big[(\frac{M_1}{\alpha})^{\frac{1}{4}}(M_2^{(1)})^{\frac{3}{4}} M_{m-1}^{(1)} + (\frac{M_1}{\beta})^{\frac{1}{4}}(M_2^{(2)})^{\frac{3}{4}} M_{m-1}^{(2)}\big]}{\alpha\nu\sqrt{(\lambda_1^{-1} + \alpha^2)^{-1}}},$$

$$M_m^{(2)} := \frac{|f|_{m-2} + c(m,\Omega)\lambda_1^{-\frac{7}{8}}\big[(\frac{M_1}{\alpha})^{\frac{1}{4}}(M_2^{(2)})^{\frac{3}{4}} M_{m-1}^{(1)} + (\frac{M_1}{\beta})^{\frac{1}{4}}(M_2^{(2)})^{\frac{3}{4}} M_{m-1}^{(1)}\big]}{\beta\mu\sqrt{(\lambda_1^{-1} + \beta^2)^{-1}}}.$$

Proof. Recall that $v(x,t)$ and $\xi(x,t)$ satisfy (3.2) and (3.4), respectively. In general, $v^{(m)}(x,t)$ and $\xi^{(m)}(x,t)$, for $m > 2$, satisfy (3.23) and (3.25), respectively. Therefore, the proof of this lemma is an immediate application of Lemma 2.5, in particular relation (2.5), and the inequalities of Lemma 2.1. $\square$

Theorem 4.2. *Let $(u(x,t), b(x,t))$ be a solution of the regularized MHD equations (2.2)-(2.4), corresponding to the initial condition $(u^{\rm in}, b^{\rm in}) \in V \times V$ with forcing* *terms $(f,g)\in G^1_{\tau_0}\times G^1_{\tau_0}$, for some $\tau_0>0$. Let $t_*\geqslant 0$ be as in Lemma 4.1, then there exist two functions*

$$v^\omega(t)\in L^\infty([t_*,+\infty);G^2_\tau), \tag{4.3}$$
$$\xi^\omega(t)\in L^\infty([t_*,+\infty);G^2_\tau), \tag{4.4}$$

for some $\tau>0$, depending only on $|f|_{1,\tau_0}$, $|g|_{1,\tau_0}$, μ, ν, α, β and λ_1, satisfying

$$\lim_{t\to+\infty}|u(t)-v^\omega(t)|_1=0, \tag{4.5}$$
$$\lim_{t\to+\infty}|b(t)-\xi^\omega(t)|_1=0. \tag{4.6}$$

Proof. Let $\lambda>0$ to be chosen later. First, consider $(v^{(2)}(x,t),\xi^{(2)}(x,t))$ the asymptotic approximation of $(u(x,t),b(x,t))$, which is constructed in the proof of Theorem 3.1. Denote

$$\overline{v}(t)=P_\lambda v^{(2)}(t),\quad \overline{\xi}(t)=P_\lambda\xi^{(2)}(t).$$

Consider $\check{v}(t)$ and $\check{\xi}(t)$ for all $t\geqslant t_*$ -the solution of the following equations

$$\check{v}_t+\nu A\check{v}+\alpha^2A\check{v}_t+Q_\lambda B(\overline{v}+\check{v},\overline{v}+\check{v})-Q_\lambda B(\overline{\xi}+\check{\xi},\overline{\xi}+\check{\xi})=\check{f}, \tag{4.7}$$
$$\check{v}(t_*)=0, \tag{4.8}$$

and

$$\check{\xi}_t+\mu A\check{\xi}+\beta^2A\check{\xi}_t+Q_\lambda B(\overline{v}+\check{v},\overline{\xi}+\check{\xi})-Q_\lambda B(\overline{\xi}+\check{\xi},\overline{v}+\check{v})=\check{g}, \tag{4.9}$$
$$\check{\xi}(t_*)=0, \tag{4.10}$$

respectively, where $\check{f}=Q_\lambda f$ and $\check{g}=Q_\lambda g$. Let us put

$$v^\omega(t)=\overline{v}(t)+\check{v}(t),\quad t\geqslant t_*, \tag{4.11}$$
$$\xi^\omega(t)=\overline{\xi}(t)+\check{\xi}(t),\quad t\geqslant t_*. \tag{4.12}$$

Our goal is to show first that there exists some $\tau>0$ such that

$$v^\omega(t),\quad \xi^\omega(t)\in G^2_\tau.$$

Since $\overline{v}(t)$ and $\overline{\xi}(t)$ are just trigonometric polynomials, and in particular, are analytic, we need to show that we can choose λ large enough, such that $\check{v}(t)$ and $\check{\xi}(t)$ belong to G^2_τ for some $\tau>0$. Finally, we will show that $(v^\omega(x,t),\xi^\omega(x,t))$ is indeed the asymptotic approximation of $(u(x,t),b(x,t))$.

We want to point out that in order to prove that the solutions of (4.7)-(4.8) and (4.9)-(4.10) belong to a Gevrey class of real analytic functions, we consider the Galerkin procedure to (4.7) and (4.9), respectively. However, we omit this standard procedure, and obtain formal a priori estimates on the solutions in relevant Gevrey space norm. Taking formally the inner product of (4.7) with $\check{v}$, and (4.9) with $\check{\xi}$ in G^1_τ, respectively, we obtain the following inequalities

$$\begin{aligned}
&\frac{1}{2}(|\check{v}|^2_{1,\tau}+\alpha^2|\check{v}|^2_{2,\tau})+\nu|\check{v}|^2_{2,\tau}\\
&\leqslant|(\check{f},\check{v})_{1,\tau}|+|(B(\overline{v},\overline{v}),\check{v})_{1,\tau}|+|(B(\overline{v},\check{v}),\check{v})_{1,\tau}|+|(B(\check{v},\overline{v}),\check{v})_{1,\tau}|\\
&\quad+|(B(\check{v},\check{v}),\check{v})_{1,\tau}|+|(B(\overline{\xi},\overline{\xi}),\check{v})_{1,\tau}|+|(B(\overline{\xi},\check{\xi}),\check{v})_{1,\tau}|\\
&\quad+|(B(\check{\xi},\overline{\xi}),\check{v})_{1,\tau}|+|(B(\check{\xi},\check{\xi}),\check{v})_{1,\tau}|
\end{aligned}\tag{4.13}$$

and

$$\begin{aligned} &\frac{1}{2}(|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)+\mu|\check{\xi}|_{2,\tau}^2 \\ &\leqslant |(\check{g},\check{\xi})_{1,\tau}|+|(B(\overline{v},\overline{\xi}),\check{\xi})_{1,\tau}|+|(B(\overline{v},\check{\xi}),\check{\xi})_{1,\tau}|+|(B(\check{v},\overline{\xi}),\check{\xi})_{1,\tau}| \\ &\quad +|(B(\check{v},\check{\xi}),\check{\xi})_{1,\tau}|+|(B(\overline{\xi},\overline{v}),\check{\xi})_{1,\tau}|+|(B(\overline{\xi},\check{v}),\check{\xi})_{1,\tau}| \\ &\quad +|(B(\check{\xi},\overline{v}),\check{\xi})_{1,\tau}|+|(B(\check{\xi},\check{v}),\check{v})_{1,\tau}| \end{aligned} \tag{4.14}$$

accordingly. We next estimate the terms on the right-hand side of above (4.13) and (4.14).

Firstly, using subsequently the Cauchy-Schwarz and Young inequalities, as well as Lemma 2.3, we obtain assuming $\tau\leqslant\tau_0$ that

$$|(\check{f},\check{v})_{1,\tau}|\leqslant|\check{f}|_{1,\tau}|\check{v}|_{1,\tau}\leqslant\frac{2}{\nu\lambda}|\check{f}|_{1,\tau}^2+\frac{\nu}{8}|\check{v}|_{2,\tau}^2, \tag{4.15}$$

$$|(\check{g},\check{\xi})_{1,\tau}|\leqslant|\check{g}|_{1,\tau}|\check{\xi}|_{1,\tau}\leqslant\frac{2}{\mu\lambda}|\check{g}|_{1,\tau}^2+\frac{\mu}{8}|\check{\xi}|_{2,\tau}^2. \tag{4.16}$$

Secondly, using Lemma 2.4, Young inequality and the Poincaré-type inequalities of Lemma 2.3, we obtain the following series of estimates for all $t\geqslant t_*$:

$$\begin{aligned} |(B(\overline{v},\overline{v}),\check{v})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\overline{v}|_{1,\tau}^{3/2}|\overline{v}|_{2,\tau}^{1/2}|\check{v}|_{2,\tau} \\ &\leqslant \frac{c(\Omega)e^{4\tau\lambda^{1/2}}(M_1^{(1)})^3M_2^{(1)}}{\nu\lambda_1^{3/2}}+\frac{\nu}{8}|\check{v}|_{2,\tau}^2, \end{aligned} \tag{4.17}$$

$$\begin{aligned} |(B(\overline{v},\check{v}),\check{v})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\overline{v}|_{1,\tau}^{1/2}|\overline{v}|_{2,\tau}^{1/2}|\check{v}|_{1,\tau}|\check{v}|_{2,\tau} \\ &\leqslant \frac{c(\Omega)e^{\tau\lambda^{1/2}}(M_1^{(1)}M_2^{(1)})^{1/2}}{\lambda^{1/2}\lambda_1^{3/4}}|\check{v}|_{2,\tau}^2, \end{aligned} \tag{4.18}$$

$$|(B(\check{v},\overline{v}),\check{v})_{1,\tau}|\leqslant c(\Omega)\lambda_1^{-3/4}|\check{v}|_{1,\tau}^{1/2}|\check{v}|_{2,\tau}^{3/2}|\overline{v}|_{1,\tau}\leqslant\frac{c(\Omega)e^{\tau\lambda^{1/2}}M_1^{(1)}}{\lambda^{1/4}\lambda_1^{3/4}}|\check{v}|_{2,\tau}^2, \tag{4.19}$$

$$\begin{aligned} |(B(\check{v},\check{v}),\check{v})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\check{v}|_{1,\tau}^{3/2}|\check{v}|_{2,\tau}^{3/2}\leqslant c(\Omega)\lambda^{-3/4}\lambda_1^{-3/4}|\check{v}|_{2,\tau}^3 \\ &\leqslant \frac{c(\Omega)}{\lambda^{3/4}\lambda_1^{3/4}\alpha^3}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2)^{3/2}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} |(B(\overline{\xi},\overline{\xi}),\check{v})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\overline{\xi}|_{1,\tau}^{3/2}|\overline{\xi}|_{2,\tau}^{1/2}|\check{v}|_{2,\tau} \\ &\leqslant \frac{c(\Omega)e^{4\tau\lambda^{1/2}}(M_1^{(2)})^3M_2^{(2)}}{\nu\lambda_1^{3/2}}+\frac{\nu}{8}|\check{v}|_{2,\tau}^2, \end{aligned} \tag{4.21}$$

$$\begin{aligned} |(B(\overline{\xi},\check{\xi}),\check{v})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-4/3}|\overline{\xi}|_{1,\tau}^{1/2}|\overline{\xi}|_{2,\tau}^{1/2}|\check{\xi}|_{1,\tau}|\check{v}|_{2,\tau} \\ &\leqslant \frac{c(\Omega)e^{\tau\lambda^{1/2}}(M_1^{(2)}M_2^{(2)})^{1/2}}{\lambda^{1/2}\lambda_1^{3/4}}(|\check{v}|_{2,\tau}^2+|\check{\xi}|_{2,\tau}^2), \end{aligned} \tag{4.22}$$

$$\begin{aligned} |(B(\check{\xi},\overline{\xi}),\check{v})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\check{\xi}|_{1,\tau}^{1/2}|\check{\xi}|_{2,\tau}^{1/2}|\overline{\xi}|_{1,\tau}|\check{v}|_{2,\tau} \\ &\leqslant c(\Omega)\lambda_1^{-3/4}\lambda^{-1/4}|\check{\xi}|_{2,\tau}|\overline{\xi}|_{1,\tau}|\check{v}|_{2,\tau} \\ &\leqslant \frac{c(\Omega)e^{\tau\lambda^{1/2}}M_1^{(2)}}{\lambda_1^{3/4}\lambda^{1/4}}(|\check{v}|_{2,\tau}^2+|\check{\xi}|_{2,\tau}^2), \end{aligned} \tag{4.23}$$

$$
\begin{aligned}
&|(B(\check{\xi},\check{\xi}),\check{v})_{1,\tau}| \\
&\leqslant c(\Omega)\lambda_1^{-3/4}|\check{\xi}|_{1,\tau}^{3/2}|\check{\xi}|_{2,\tau}^{1/2}|\check{v}|_{2,\tau} \leqslant c(\Omega)\lambda^{-3/4}\lambda_1^{-3/4}|\check{\xi}|_{2,\tau}^{2}|\check{v}|_{2,\tau} \\
&\leqslant c(\Omega)\lambda^{-3/4}\lambda_1^{-3/4}\alpha^{-3}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2)^{3/2} \\
&\quad + c(\Omega)\lambda^{-3/4}\lambda_1^{-3/4}\beta^{-3}(|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)^{3/2} \\
&\leqslant c(\Omega,\alpha,\beta)\lambda^{-3/4}\lambda_1^{-3/4}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)^{3/2},
\end{aligned} \tag{4.24}
$$

$$
\begin{aligned}
|(B(\overline{v},\overline{\xi}),\check{\xi})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\overline{v}|_{1,\tau}^{1/2}|\overline{v}|_{2,\tau}^{1/2}|\overline{\xi}|_{1,\tau}|\check{\xi}|_{2,\tau} \\
&\leqslant \frac{c(\Omega)e^{4\tau\lambda^{1/2}}M_1^{(1)}M_2^{(1)}(M_1^{(2)})^2}{\mu\lambda_1^{3/2}} + \frac{\mu}{8}|\check{\xi}|_{2,\tau}^2,
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
|(B(\overline{v},\check{\xi}),\check{\xi})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\overline{v}|_{1,\tau}^{1/2}|\overline{v}|_{2,\tau}^{1/2}|\check{\xi}|_{1,\tau}|\check{\xi}|_{2,\tau} \\
&\leqslant \frac{c(\Omega)e^{\tau\lambda^{1/2}}(M_1^{(1)}M_2^{(1)})^{1/2}}{\lambda^{1/2}\lambda_1^{3/4}}|\check{\xi}|_{2,\tau}^2,
\end{aligned} \tag{4.26}
$$

$$
\begin{aligned}
|(B(\check{v},\overline{\xi}),\check{\xi})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\check{v}|_{1,\tau}^{1/2}|\check{v}|_{2,\tau}^{1/2}|\overline{\xi}|_{1,\tau}|\check{\xi}|_{2,\tau} \\
&\leqslant c(\Omega)\lambda_1^{-3/4}\lambda^{-1/4}M_1^{(2)}e^{\tau\lambda^{1/2}}(|\check{v}|_{2,\tau}^2+|\check{\xi}|_{2,\tau}^2),
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
&|(B(\check{v},\check{\xi}),\check{\xi})_{1,\tau}| \\
&\leqslant c(\Omega)\lambda_1^{-3/4}|\check{v}|_{1,\tau}^{1/2}|\check{v}|_{2,\tau}^{1/2}|\check{\xi}|_{1,\tau}|\check{\xi}|_{2,\tau} \\
&\leqslant c(\Omega)\lambda^{-3/4}\lambda_1^{-3/4}|\check{v}|_{2,\tau}|\check{\xi}|_{2,\tau}^2 \\
&\leqslant c(\Omega,\alpha,\beta)\lambda^{-3/4}\lambda_1^{-3/4}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)^{3/2},
\end{aligned} \tag{4.28}
$$

$$
\begin{aligned}
|(B(\overline{\xi},\overline{v}),\check{\xi})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\overline{\xi}|_{1,\tau}^{1/2}|\overline{\xi}|_{2,\tau}^{1/2}|\overline{v}|_{1,\tau}|\check{\xi}|_{2,\tau} \\
&\leqslant \frac{c(\Omega)e^{4\tau\lambda^{1/2}}M_1^{(2)}M_2^{(2)}(M_1^{(1)})^2}{\mu\lambda_1^{3/2}} + \frac{\mu}{8}|\check{\xi}|_{2,\tau}^2,
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
|(B(\overline{\xi},\check{v}),\check{\xi})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\overline{\xi}|_{1,\tau}^{1/2}|\overline{\xi}|_{2,\tau}^{1/2}|\check{v}|_{1,\tau}|\check{\xi}|_{2,\tau} \\
&\leqslant \frac{ce^{\tau\lambda^{1/2}}(M_1^{(2)}M_2^{(2)})^{1/2}}{\lambda^{1/2}\lambda_1^{3/4}}(|\check{v}|_{2,\tau}^2+|\check{\xi}|_{2,\tau}^2),
\end{aligned} \tag{4.30}
$$

$$
\begin{aligned}
|(B(\check{\xi},\overline{v}),\check{\xi})_{1,\tau}| &\leqslant c(\Omega)\lambda_1^{-3/4}|\check{\xi}|_{1,\tau}^{1/2}|\check{\xi}|_{2,\tau}^{1/2}|\overline{v}|_{1,\tau}|\check{\xi}|_{2,\tau} \\
&\leqslant \frac{c(\Omega)e^{\tau\lambda^{1/2}}M_1^{(1)}}{\lambda^{1/4}\lambda_1^{3/4}}|\check{\xi}|_{2,\tau}^2,
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
&|(B(\check{\xi},\check{v}),\check{\xi})_{1,\tau}| \\
&\leqslant c(\Omega)\lambda_1^{-3/4}|\check{\xi}|_{1,\tau}^{1/2}|\check{\xi}|_{2,\tau}^{1/2}|\check{v}|_{1,\tau}|\check{\xi}|_{2,\tau} \\
&\leqslant c\lambda^{-1/4}\lambda_1^{-3/4}|\check{\xi}|_{2,\tau}^2|\check{v}|_{2,\tau} \\
&\leqslant c(\Omega,\alpha,\beta)\lambda^{-3/4}\lambda_1^{-3/4}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)^{3/2},
\end{aligned} \tag{4.32}
$$

Let us set $\tau := \min\{\lambda^{-1/2}, \tau_0\}$. Then we select λ large enough satisfying

$$\begin{aligned} &\max\Big\{\frac{c(\Omega)[(M_1^{(1)}M_2^{(1)})^{1/2}+M_1^{(2)}M_2^{(2)})^{1/2}]}{\lambda^{1/2}\lambda_1^{3/4}}, \frac{c(\Omega)(M_1^{(1)}+M_1^{(2)})}{\lambda^{1/4}\lambda_1^{3/4}}\Big\} \\ &\leqslant \frac{\min\{\nu,\mu\}}{8}. \end{aligned} \tag{4.33}$$

Taking (4.13)-(4.33) into account, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)+\frac{\nu}{8}|\check{v}|_{2,\tau}^2+\frac{\mu}{8}|\check{\xi}|_{2,\tau}^2 \\ &\leqslant \frac{c(\Omega,\alpha,\beta)}{\lambda^{3/4}\lambda_1^{3/4}}(1+\frac{1}{\alpha^3})(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)^{3/2} \\ &\quad+\frac{2}{\nu\lambda}|\check{f}|_{1,\tau}^2+\frac{2}{\mu\lambda}|\check{g}|_{1,\tau}^2+\frac{c(\Omega)(M_1^{(1)})^3M_2^{(1)}}{\nu\lambda_1^{3/2}}+\frac{c(\Omega)(M_1^{(2)})^3M_2^{(2)}}{\nu\lambda_1^{3/2}} \\ &\quad+\frac{c(\Omega)M_1^{(1)}M_2^{(1)}(M_1^{(2)})^2}{\mu\lambda_1^{3/2}}+\frac{c(\Omega)M_1^{(2)}M_2^{(2)}(M_1^{(1)})^2}{\mu\lambda_1^{3/2}}. \end{aligned} \tag{4.34}$$

Using Lemma 2.3 and setting

$$\delta_6 := \min\{\delta_1,\delta_2\} = \min\{(1/\lambda_1+\alpha^2)^{-1}, (1/\lambda_1+\beta^2)^{-1}\},$$

we can write

$$\frac{\nu}{8}|\check{v}|_{2,\tau}^2 \geqslant \frac{\nu\delta_6}{8}(\lambda^{-1}|\check{v}|_{2,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2) \geqslant \frac{\nu\delta_6}{8}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2), \tag{4.35}$$

$$\frac{\mu}{8}|\check{\xi}|_{2,\tau}^2 \geqslant \frac{\mu\delta_6}{8}(\lambda^{-1}|\check{\xi}|_{2,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2) \geqslant \frac{\mu\delta_6}{8}(|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2). \tag{4.36}$$

Substituting (4.35)-(4.36) into (4.34) gives

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2) \\ &\leqslant -\frac{\delta_4\delta_6}{8}(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2) \\ &\quad+\frac{c(\Omega,\alpha,\beta)}{\lambda^{3/4}\lambda_1^{3/4}}(1+\frac{1}{\alpha^3})(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)^{3/2} \\ &\quad+\frac{2}{\nu\lambda}|\check{f}|_{1,\tau}^2+\frac{2}{\mu\lambda}|\check{g}|_{1,\tau}^2+K, \end{aligned} \tag{4.37}$$

where the constant

$$\begin{aligned} K := \frac{c(\Omega)}{\lambda_1^{3/2}}\Big[&\frac{(M_1^{(1)})^3M_2^{(1)}+(M_1^{(2)})^3M_2^{(2)}}{\nu} \\ &+\frac{M_1^{(1)}M_2^{(1)}(M_1^{(2)})^2+M_1^{(2)}M_2^{(2)}(M_1^{(1)})^2}{\mu}\Big] \end{aligned}$$

is independent of λ.

To apply Lemma 2.6 to the function $(|\check{v}|_{1,\tau}^2+\alpha^2|\check{v}|_{2,\tau}^2+|\check{\xi}|_{1,\tau}^2+\beta^2|\check{\xi}|_{2,\tau}^2)$ satisfying inequality (4.37), we need check the condition (2.6). In fact, we can choose λ large enough, such that

$$\frac{c(\Omega,\alpha,\beta)(1+\frac{1}{\alpha^3})}{\lambda^{3/4}\lambda_1^{3/4}}\Big(\frac{2(\nu|\check{f}|_{1,\tau}^2+\mu|\check{g}|_{1,\tau}^2)}{\lambda}+K\Big)^{1/2} < (\frac{\delta_4\delta_6}{16})^{3/2}. \tag{4.38}$$

For such choice of λ, we conclude from Lemma 2.5 that $(|\check{v}|_{1,\tau}^2 + \alpha^2|\check{v}|_{2,\tau}^2 + |\check{\xi}|_{1,\tau}^2 + \beta^2|\check{\xi}|_{2,\tau}^2)$ is bounded for all $t \geqslant t_*$, and hence

$$v^\omega(t) \in L^\infty([t_*;+\infty);G_\tau^2) \quad \text{and} \quad \xi^\omega(t) \in L^\infty([t_*;+\infty);G_\tau^2),$$

that is (4.3) and (4.4) are proved.

We now are left to show that $(v^\omega(x,t),\xi^\omega(x,t))$ is the asymptotic approximation of the solution $(u(x,t),b(x,t))$ of the regularized MHD equations. Put

$$\begin{gathered} z = u - v^\omega, \quad \overline{z} = P_\lambda(u - v^{(2)}) = P_\lambda u - \overline{v}, \\ \zeta = b - \xi^\omega, \quad \overline{\zeta} = P_\lambda(b - \xi^{(2)}) = P_\lambda b - \overline{\xi}, \end{gathered}$$

and denote

$$\check{z} = Q_\lambda u - \check{v}, \tag{4.39}$$
$$\check{\zeta} = Q_\lambda b - \check{\xi}. \tag{4.40}$$

Relations (4.11)-(4.12) and (4.39)-(4.40) give

$$\begin{gathered} \overline{z} + \check{z} = u - v^\omega = z, \\ \overline{\zeta} + \check{\zeta} = b - \xi^\omega = \zeta. \end{gathered} \tag{4.41}$$

Obviously, by the construction and Theorem 3.1, we have

$$\lim_{t\to+\infty} |P_\lambda u(t) - \overline{v}(t)|_1 = \lim_{t\to+\infty} |\overline{z}|_1 = 0, \tag{4.42}$$
$$\lim_{t\to+\infty} |P_\lambda b(t) - \overline{\xi}(t)|_1 = \lim_{t\to+\infty} |\overline{\zeta}|_1 = 0. \tag{4.43}$$

Therefore, to prove (4.5) and (4.6), we need to show

$$\lim_{t\to+\infty} |Q_\lambda u(t) - \check{v}(t)|_1 = \lim_{t\to+\infty} |\check{z}(t)|_1 = 0, \tag{4.44}$$
$$\lim_{t\to+\infty} |Q_\lambda b(t) - \check{\xi}(t)|_1 = \lim_{t\to+\infty} |\check{\zeta}(t)|_1 = 0. \tag{4.45}$$

By the property of the operator $B(\cdot,\cdot)$ and relation (4.41), we have

$$\begin{aligned} B(u,u) - B(v^\omega,v^\omega) &= B(u - v^\omega,u) + B(v^\omega,u) - B(v^\omega,v^\omega) \\ &= B(z,u) + B(v^\omega,u - v^\omega) \\ &= B(z,u) + B(u - z,z) \\ &= B(z,u) + B(u,z) - B(z,z). \end{aligned} \tag{4.46}$$

Similarly,

$$B(b,b) - B(\xi^\omega,\xi^\omega) = B(b,\zeta) + B(\zeta,b) - B(\zeta,\zeta), \tag{4.47}$$
$$B(u,b) - B(v^\omega,\xi^\omega) = B(z,b) + B(u,\zeta) - B(z,\zeta), \tag{4.48}$$
$$B(b,u) - B(\xi^\omega,v^\omega) = B(\zeta,u) + B(b,z) - B(\zeta,z). \tag{4.49}$$

From (2.2), (4.7) and (4.46)-(4.47), we find that $\check{z}(t)$ satisfies

$$\begin{gathered} \check{z}_t + \nu A\check{z} + \alpha^2 A\check{z}_t + Q_\lambda(B(u,z) + B(z,u) - B(z,z)) \\ - Q_\lambda(B(b,\zeta) + B(\zeta,b) - B(\zeta,\zeta)) = 0, \quad t > t_*, \end{gathered} \tag{4.50}$$
$$\check{z}(t_*) = Q_\lambda u(t_*). \tag{4.51}$$

Also, by (2.3), (4.9) and (4.48)-(4.49), we see that $\check{\zeta}(t)$ satisfies

$$\begin{aligned} &\check{\zeta}_t + \mu A\check{\zeta} + \beta^2 A\check{\zeta}_t + Q_\lambda(B(u,\zeta) + B(z,b) - B(z,\zeta)) \\ &- Q_\lambda(B(b,z) + B(\zeta,u) - B(\zeta,z)) = 0, \quad t > t_*, \end{aligned} \tag{4.52}$$

$$\check{\zeta}(t_*) = Q_\lambda b(t_*). \tag{4.53}$$

Taking the inner product of equation (4.50) with $\check{z}$ in H, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\check{z}\|^2 + \alpha^2\|\check{z}\|_1^2) + \nu\|\check{z}\|_1^2 \\ &\leqslant |(B(\check{z},u),\check{z})| + |(Q_\lambda(B(u,\overline{z}) + B(\overline{z},u) - B(z,\overline{z})),\check{z})| \\ &\quad + |(Q_\lambda(B(b,\overline{\zeta}) + B(\overline{\zeta},b) - B(\overline{\zeta},\overline{\zeta}) - B(\check{\zeta},\overline{\zeta}) - B(\overline{\zeta},\check{\zeta})),\check{z})| \\ &\quad - (Q_\lambda B(b,\check{\zeta}),\check{z}) + |(B(\check{\zeta},b),\check{z})| + (Q_\lambda B(\check{\zeta},\check{\zeta}),\check{z}). \end{aligned} \tag{4.54}$$

At the same time, taking the inner product of equation (4.52) with $\check{\zeta}$ in H gives

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\check{\zeta}\|^2 + \beta^2\|\check{\zeta}\|_1^2) + \mu\|\check{\zeta}\|_1^2 \\ &\leqslant |(Q_\lambda(B(u,\overline{\zeta}) + B(\overline{z},b) - B(\overline{z},\overline{\zeta})),\check{z})| \\ &\quad + |(Q_\lambda(-B(b,\overline{z}) + B(\overline{\zeta},u) - B(\overline{\zeta},z)),\check{\zeta})| + |(B(\check{z},b),\check{\zeta})| \\ &\quad + |(B(\check{z},\overline{\zeta}),\check{\zeta})| - (Q_\lambda B(b,\check{z}),\check{\zeta}) + |(B(\check{\zeta},u),\check{z})| + (Q_\lambda B(\check{\zeta},\check{z}),\check{\zeta}). \end{aligned} \tag{4.55}$$

By the Poincaré inequality, we have

$$\nu|\check{z}|_1^2 \geqslant \frac{\nu}{2}(\lambda_1\|\check{z}\|^2 + |\check{z}|_1^2) \geqslant \frac{\nu}{2(\lambda_1^{-1} + \alpha^2)}(\|\check{z}\|^2 + \alpha^2|\check{z}|_1^2), \tag{4.56}$$

$$\mu|\check{\zeta}|_1^2 \geqslant \frac{\mu}{2}(\lambda_1\|\check{\zeta}\|^2 + |\check{\zeta}|_1^2) \geqslant \frac{\mu}{2(\lambda_1^{-1} + \beta^2)}(\|\check{\zeta}\|^2 + \beta^2|\check{\zeta}|_1^2). \tag{4.57}$$

It then follows from (4.54)-(4.57) that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\check{z}\|^2 + \alpha^2\|\check{z}\|_1^2 + \|\check{\zeta}\|^2 + \beta^2\|\check{\zeta}\|_1^2) \\ &+ \frac{\delta_4\delta_6}{2}(\|\check{z}\|^2 + \alpha^2\|\check{z}\|_1^2 + \|\check{\zeta}\|^2 + \beta^2\|\check{\zeta}\|_1^2) \\ &\leqslant |(B(\check{z},u),\check{z})| + |(B(\check{\zeta},b),\check{z})| + |(B(\check{z},b),\check{\zeta})| + |(B(\check{z},\overline{\zeta}),\check{\zeta})| \\ &\quad + |(B(\check{\zeta},u),\check{z})| + \Psi(t), \end{aligned} \tag{4.58}$$

where

$$\begin{aligned} \Psi(t) = &|(Q_\lambda(B(u,\overline{z}) + B(\overline{z},u) - B(z,\overline{z})),\check{z})| \\ &+ |(Q_\lambda(B(b,\overline{\zeta}) + B(\overline{\zeta},b) - B(\overline{\zeta},\overline{\zeta}) - B(\check{\zeta},\overline{\zeta}) - B(\overline{\zeta},\check{\zeta})),\check{z})| \\ &+ |(Q_\lambda(B(u,\overline{\zeta}) + B(\overline{z},b) - B(\overline{z},\overline{\zeta})),\check{z})| \\ &+ |(Q_\lambda(-B(b,\overline{z}) + B(\overline{\zeta},u) - B(\overline{\zeta},z)),\check{\zeta})|. \end{aligned} \tag{4.59}$$

Now the first five terms on the right-hand side of (4.58) can be estimated as follows. Using (2.1), Lemma 2.3 and (4.1)-(4.2), we obtain for $t \geqslant t_*$,

$$|(B(\check{z}(t),u(t)),\check{z}(t))| \leqslant \frac{c(\Omega)}{\lambda_1^{3/4}}|u(t)|_1\|\check{z}(t)\|^{1/2}|\check{z}(t)|_1^{3/2} \leqslant \frac{c(\Omega)M_1}{\alpha\lambda_1^{3/4}\lambda^{1/4}}|\check{z}(t)|_1^2, \tag{4.60}$$

$$
\begin{aligned}
|(B(\check{z}(t),b(t)),\check{\zeta}(t))| &\leqslant \frac{c(\Omega)}{\lambda_1^{3/4}}|b(t)|_1\|\check{z}(t)\|^{1/2}|\check{z}(t)|_1^{1/2}|\check{\zeta}(t)|_1 \\
&\leqslant \frac{c(\Omega)M_1}{\beta\lambda_1^{3/4}\lambda^{1/4}}|\check{z}(t)|_1|\check{\zeta}(t)|_1 \\
&\leqslant \frac{c(\Omega)M_1}{\beta\lambda_1^{3/4}\lambda^{1/4}}(|\check{z}(t)|_1^2+|\check{\zeta}(t)|_1^2),
\end{aligned} \tag{4.61}
$$

$$
|(B(\check{\zeta}(t),b(t)),\check{z}(t))| \leqslant \frac{c(\Omega)M_1}{\beta\lambda_1^{3/4}\lambda^{1/4}}(|\check{z}(t)|_1^2+|\check{\zeta}(t)|_1^2), \tag{4.62}
$$

$$
\begin{aligned}
|(B(\check{z},\overline{\zeta}),\check{\zeta})| &\leqslant \frac{c(\Omega)}{\lambda_1^{3/4}}|\overline{\zeta}(t)|_1\|\check{z}(t)\|^{1/2}|\check{z}(t)|_1^{1/2}|\check{\zeta}(t)|_1 \\
&\leqslant \frac{c(\Omega)(M_1/\beta+M_2^{(2)})}{\lambda_1^{3/4}\lambda^{1/4}}|\check{z}(t)|_1|\check{\zeta}(t)|_1 \\
&\leqslant \frac{c(\Omega)(M_1/\beta+M_2^{(2)})}{\lambda_1^{3/4}\lambda^{1/4}}(|\check{z}(t)|_1^2+|\check{\zeta}(t)|_1^2),
\end{aligned} \tag{4.63}
$$

$$
\begin{aligned}
|(B(\check{\zeta}(t),u(t)),\check{z}(t))| &\leqslant \frac{c}{\lambda_1^{3/4}}|u(t)|_1\|\check{\zeta}(t)\|^{1/2}|\check{\zeta}(t)|_1^{1/2}|\check{z}(t)|_1 \\
&\leqslant \frac{c(\Omega)M_1}{\alpha\lambda_1^{3/4}\lambda^{1/4}}|\check{z}(t)|_1|\check{\zeta}(t)|_1 \\
&\leqslant \frac{c(\Omega)M_1}{\alpha\lambda_1^{3/4}\lambda^{1/4}}(|\check{z}(t)|_1^2+|\check{\zeta}(t)|_1^2).
\end{aligned} \tag{4.64}
$$

Inequalities (4.60)-(4.64) yield

$$
\begin{aligned}
&|(B(\check{z},u),\check{z})|+|(B(\check{\zeta},b),\check{z})|+|(B(\check{z},b),\check{\zeta})|+|(B(\check{z},\overline{\zeta}),\check{\zeta})|+|(B(\check{\zeta},u),\check{z})| \\
&\leqslant \frac{c(\Omega,\alpha,\beta,M_1,M_2^{(2)})}{\lambda_1^{3/4}\lambda^{1/4}}(\|\check{z}(t)\|^2+\alpha^2|\check{z}(t)|_1^2+\|\check{\zeta}(t)\|^2+\beta^2|\check{\zeta}(t)|_1^2).
\end{aligned} \tag{4.65}
$$

Inserting (4.65) into (4.58) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\check{z}\|^2+\alpha^2\|\check{z}\|_1^2+\|\check{\zeta}\|^2+\beta^2\|\check{\zeta}\|_1^2\big) \\
&+\Big(\frac{\delta_4\delta_6}{2}-\frac{c(\Omega,\alpha,\beta,M_1,M_2^{(2)})}{\lambda_1^{3/4}\lambda^{1/4}}\Big)(\|\check{z}\|^2+\alpha^2|\check{z}|_1^2+\|\check{\zeta}\|^2+\beta^2|\check{\zeta}|_1^2) \\
&\leqslant \Psi(t).
\end{aligned} \tag{4.66}
$$

Note that we can choose λ large enough such that

$$
\frac{\delta_4\delta_6}{2}-\frac{c(\Omega,\alpha,\beta,M_1,M_2^{(2)})}{\lambda_1^{3/4}\lambda^{1/4}}>0. \tag{4.67}
$$

At the same time, employing the relations (4.42)-(4.43) and the fact that u and b are bounded in the V norm, we can conclude from (4.59) that

$$
\lim_{t\to+\infty}\Psi(t)=0. \tag{4.68}
$$

It then follows from (4.66)-(4.68) and the Gronwall inequality that

$$\lim_{t\to+\infty}(\|\check{z}\|^2+\alpha^2|\check{z}|_1^2+\|\check{\zeta}\|^2+\beta^2|\check{\zeta}|_1^2)=0,$$

for λ large enough, satisfying

$$\lambda > \Big(\frac{c(\Omega,\alpha,\beta,M_1,M_2^{(2)})}{\delta_4\delta_6}\Big)^4\lambda_1^{-3}. \tag{4.69}$$

Therefore, (4.44) and (4.45) are proved for λ large enough, satisfying relations (4.33), (4.38) and (4.69). The proof of Theorem 4.2 is complete. □

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