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BOUNDED SOLVABILITY OF MIXED-TYPE FUNCTIONAL DIFFERENTIAL OPERATORS FOR FIRST ORDER

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Abstract. In this article, we study all boundedly solvable extensions generated by linear mixed-type (forward-backward) functional differential-operator expressions of first order in the Hilbert space of vector-functions on a finite interval. Our main tools are methods of operator theory. Also we study the structure of the spectrum of these extensions.

1. Introduction

Mixed-type (forward-backward) differential equations are a large class of functional differential equations in which time derivative may depend both on past and future values of the argument. The fundamental interest and analysis to such equations are motivated by applications in different fields, for example in control theory, population genetics, population growth, epidemiology, nerve conduction theory, economy, physics, electrodynamics, observer theory, spatial lattice and etc.(see [3] and references therein). The qualitative analysis of mixed-type functional differential equations is quite complicated. On the other hand the analysis of considered boundary or initial value problem for mixed-type functional differential equations for even order is really complicated.

Mallet-Paret [6] established an existence theory for such equations using a Fredholm theory and the implicit function theory. Some spectral investigations of such equations can be found in [6, 7, 9], and references therein. The numerical approach to these problems can be seen in Ford, Lumb, Teodoro, Lima et al [1].

Since analytical expression of solutions, eigenvalues and corresponding eigenfunctions is very difficult (they are ill-posed), then the methods of numerical analysis play significant role in this theory. It is known that an operator

$$A : D(A) \subset H \to H$$

in a Hilbert space H is called boundedly solvable, if A is one-to-one,

$$AD(A) = H \quad \text{and} \quad A^{-1} \in L(H).$$

Firstly using methods of operator theory, we describe all boundedly solvable extensions of minimal operator generated by some mixed-type differential operator expression for first order in the Hilbert space of vector-functions on a finite interval.

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This is done in terms of the boundary values. Lastly, the structure of spectrum of these extensions is investigated.

2. Description of boundedly solvable extensions

Consider the simplest scalar mixed type functional differential equation in nonhomogeneous form

$$\begin{gathered}\dot{x}(t) = ax(t) + bx(t-\alpha) + cx(t+\tau) + h(t)\\ \alpha, \tau \geq 0, \quad a, b, c \in \mathbb{C},\ t \in [t_1, t_2]\\ x(t) = \varphi(t), \quad t \in [t_1-\alpha, t_1], \quad x(t) = \psi(t), \quad t \in [t_2, t_2+\tau]\end{gathered} \tag{2.1}$$

Note that without loss of generality it can be assumed $a = 0$. For this it is sufficient to use the substitution $v(t) = e^{at}u(t), t \in [a, b]$.

In addition to the above boundary problem functions φ and ψ can be chosen as

$$\begin{gathered}\varphi(t) = 0, t \in [t_1-\alpha, t_1],\\ \psi(t) = 0, t \in [t_2, t_2+\tau]\end{gathered}$$

For this it is sufficient to use the substitution

$$v(t) = \begin{cases} \varphi(t), & t_1-\alpha \leq t < t_1\\ u(t), & t_1 \leq t \leq t_2\\ \psi(t), & t_2 < t \leq t_2+\tau \end{cases} - \begin{cases} \varphi(t), & t_1-\alpha \leq t < t_1\\ 0, & t_1 \leq t \leq t_2\\ \psi(t), & t_2 < t \leq t_2+\tau \end{cases}$$

Hence in this situation it is sufficient to consider the following nonhomogeneous mixed type functional differential equation

$$u'(t) = bu(t-\alpha) + cu(t+\tau) + h(t), \quad t \in [t_1, t_2]$$

with boundary conditions

$$u(t) = 0, \quad t \in [t_1-\alpha, t_1) \cup (t_2, t_2+\tau]$$

Now consider the mixed-type differential-operator expression of the form

$$l(u) = u'(t) + A(t)u(t-\alpha) + B(t)u(t+\tau)$$

in the Hilbert space of vector-functions on a finite interval $L^2(H, (a, b))$, where

(1) $\alpha \geq 0, \tau \geq 0$;
(2) H is a separable Hilbert space with inner product $(.,.)_H$ and norm $\|\cdot\|_H$;
(3) the operator- functions $A : [a, b] \to L(H)$ and $B : [a, b] \to L(H)$ are continuous on the uniform operator topology.

We remark that when

(1) $\alpha > 0$ and $\tau = 0$,
(2) $\alpha = 0$ and $\tau > 0$,
(3) $\alpha > 0$ and $\tau > 0$

the differential expression $l(.)$ is expressed as a retarded, advanced and mixed-type delay differential expression in $L^2(H, (a, b))$.

Now let us introduce the special operators: $S_\alpha^- : L^2(H, (a, b)) \to L^2(H, (a, b))$,

$$S_\alpha^- u(t) = \begin{cases} 0 & \text{if } a < t < a+\alpha,\\ u(t-\alpha), & \text{if } a+\alpha < t < b \end{cases}$$

and $S_\tau^+ : L^2(H,(a,b)) \to L^2(H,(a,b))$,

$$S_\tau^+ u(t) = \begin{cases} u(t+\tau), & \text{if } a<t<b-\tau,\\ 0, & \text{if } b-\tau<t<b. \end{cases}$$

According to differential expression $l(\cdot)$ we will consider the differential-operator-expression in a direct sum

$$\mathcal{H} = L^2(H,(a-\alpha,a)) \oplus L^2(H,(a,b)) \oplus L^2(H,(b,b+\tau))$$

$$k(u) = (u_1, l(u_2), u_3), \quad u = (u_1,u_2,u_3) \tag{2.2}$$

where

$$l(u_2) = u_2'(t) + A(t)S_\alpha^- u_2(t) + B(t)S_\tau^+ u_2(t)$$

On the other hand here we will consider also the simply differential expression

$$m(v) = v'(t) \tag{2.3}$$

in $L^2(H,(a,b))$. By the standard way minimal M_0 and maximal M operators corresponding to differential expression (2.3) can be defined in $L^2(H,(a,b))$ (see [2]).

For the operators S_α^- and S_τ^+ we have

$$\begin{aligned} \|S_\alpha^- u\|^2_{L^2(H,(a,b))} &= \int_a^b (S_\alpha u(t), S_\alpha u(t))_H dt\\ &= \int_{a+\alpha}^b (u(t-\alpha), u(t-\alpha))_H dt\\ &= \int_a^{b-\alpha} (u(t),u(t))_H dt\\ &\le \int_a^b \|u(t)\|_H^2 dt = \|u\|^2_{L^2(H,(a,b))} \end{aligned}$$

and

$$\begin{aligned} \|S_\tau^+ u\|^2_{L^2(H,(a,b))} &= \int_a^b (S_\tau u(t), S_\tau u(t))_H dt\\ &= \int_a^{b-\tau} (u(t+\tau), u(t+\tau))_H dt\\ &= \int_{a+\tau}^b (u(t),u(t))_H dt\\ &\le \int_a^b \|u(t)\|_H^2 dt = \|u\|^2_{L^2(H,(a,b))} \end{aligned}$$

for all $u \in L^2(H,(a,b))$. Then $\|S_\alpha^-\| \le 1, \|S_\tau^+\| \le 1$. That is,

$$S_\alpha^-, S_\tau^+ \in L(L^2(H,(a,b))), \quad \alpha,\tau \ge 0\,.$$

In this work we define

$$\begin{gathered} C_{\alpha\tau}(t) = A(t)S_\alpha^- + B(t)S_\tau^+, \quad a<t<b;\\ L_0 := M_0 + C_{\alpha\tau}(t),\\ L_0 : \mathring{W}_2^1(H,(a,b)) \subset L^2(H,(a,b)) \to L^2(H,(a,b));\\ L := M + C_{\alpha\tau}(t), \end{gathered}$$

$$D(L) : W_2^1(H,(a,b)) \subset L^2(H,(a,b)) \to L^2(H,(a,b)).$$

Then the operators

$$K_0 := E_1 \oplus L_0 \oplus E_2,$$
$$K := E_1 \oplus L \oplus E_2$$

are called the minimal and maximal operators corresponding to differential expressions (2.2) respectively. Here E_1 and E_2 are identity operators in $L^2(H,(a-\alpha,a))$ and $L^2(H,(b,b+\tau))$ respectively.

It is important to note that the solvability of boundary value problem (2.1) is equivalent to a solvability of operator equation

$$k(u) = H$$

where $u = (u_1, u_2, u_3)$, $H = (\varphi, h, \psi)$ in the direct sum of Hilbert spaces

$$L^2(t_1-\alpha,t_1) \oplus L^2(t_1,t_2) \oplus L^2(t_2,t_2+\tau)$$

In this paper the solvability of problem (2.1) will be investigated from this point of view in more general case of equation and space.

Now let $U(t,s)$, $t,s \in [a,b]$, be the family of evolution operators corresponding to the homogeneous differential equation

$$U_t'(t,s)f + C_{\alpha\tau}(t)U(t,s)f = 0, \quad t,s \in (a,b)$$
$$U(s,s)f = f, \quad f \in H$$

The operator $U(t,s), t,s \in [a,b]$ is a linear continuous boundedly invertible in H and

$$U^{-1}(t,s) = U(s,t), s,t \in [a,b]$$

(for more detail analysis of this concept see [5]).

Let us introduce the operator

$$Uz(t) := U(t,a)z(t),\ U : L^2(H,(a,b)) \to L^2(H,(a,b)).$$

In this case, it is easy to see that the following relation for the differentiable vector-function $z \in L^2(H,(a,b))$, $z : [a,b] \to H$ is valid

$$l(Uz) = Uz'(t) + (U_t' + C_{\alpha\tau}(t)U)z(t) = Um(z)$$

From this $U^{-1}lU(z) = m(z)$. Hence it is clear that if $\widetilde{L}$ is an extension of the minimal operator L_0; that is, $L_0 \subset \widetilde{L} \subset L$, then

$$U^{-1}L_0U = M_0, \quad M_0 \subset U^{-1}LU = \widetilde{M} \subset M, \quad U^{-1}LU = M.$$

For example, one can easily prove the validity of last relation. It is known that

$$D(M) = W_2^1(H(a,b), \quad D(M_0) = \overset{\circ}{W}{}_2^1(H(a,b)).$$

If $u \in D(M)$, then

$$l(Uz) = Um(z) \in L^2(H,(a,b));$$

that is, $Uu \in D(L)$. From last relation $M \subset U^{-1}LU$. Contrary, if a vector-function $u \in D(L)$, then

$$m(U^{-1}v) = U^{-1}l(v) \in L^2(H,(a,b));$$

that is, $U^{-1}v \in D(M)$. From last relation $U^{-1}L \subset MU$; that is $U^{-1}LU \subset M$. Hence $U^{-1}LU = M$.

Theorem 2.1. $\ker L_0 = \{0\}$ *and* $\overline{R}(L_0) \neq L^2(H,(0,1))$.

Theorem 2.2. *Each boundedly solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H,(a,b))$ is generated by the differential-operator expression $l(.)$ and boundary condition*

$$(B+E)u(a) = BU(a,b)u(b), \tag{2.4}$$

where $B \in L(H)$ and E is a identity operator in H. The operator B is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L} = L_B$.

On the contrary, the restriction of the maximal operator L_0 to the manifold of vector-functions satisfy the condition (2.4) for some bounded operator $B \in L(H)$ is a boundedly solvable extension of the minimal operator L_0 in the $L^2(H,(a,b))$.

Proof. Firstly, it is described all boundedly solvable extensions $\widetilde{M}$ of the minimal operator M_0 in $L^2(H,(a,b))$ in terms of boundary values.

Consider the following so-called Cauchy extension M_c,

$$M_c u = u'(t),$$
$$M_c : D(M_c) = \{u \in W_2^1 H(a,b) : u(a) = 0\} \subset L^2(H,(a,b)) \to L^2(H,(a,b))$$

of the minimal operator M_0. It is clear that M_c is a boundedly solvable extension of M_0 and

$$M_c^{-1} := L^2(H,(a,b)) \to L^2(H,(a,b)),\ M_c^{-1}f(t) = \int_a^t f(x)dx, \quad f \in L^2(H,(a,b)).$$

Now we assume that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L^2(H,(a,b))$. In this case it is known that domain of $\widetilde{M}$ can be written in direct sum in form

$$D(\widetilde{M}) = D(M_0) \oplus (M_c^{-1} + B)V,$$

where $V = \ker M = H$, $B \in L(H)$ (see [10]). Therefore for each $u(t) \in D(\widetilde{M})$ it holds

$$u(t) = u_0(t) + M_c^{-1}f + Bf, \quad u_0 \in D(M_0), \quad f \in H.$$

That is,

$$u(t) = u_0(t) + tf + Bf, \quad u_0 \in D(M_0), \quad f \in H.$$

Hence

$$u(a) = Bf, \quad u(b) = f + Bf = (B+E)f$$

and from these relations it follows that

$$(B+E)u(a) = Bu(b). \tag{2.5}$$

On the other hand uniqueness of operator $B \in L(H)$ is clear from the work [10]. Therefore $\widetilde{M} = M_B$. This completes the necessary part of this assertion.

On the contrary, if M_B is a operator generated by differential expression (2.3) and boundary condition (2.5), then M_B is boundedly invertible and

$$M_B^{-1} := L^2(H,(a,b)) \to L^2(H,(a,b)),$$
$$M_B^{-1}f(t) = \int_a^t f(x)dx + B\int_a^b f(x)dx, \quad f \in L^2(H,(a,b))$$

Consequently, all boundedly solvable extensions of the minimal operator M_0 in $L^2(H,(0,1))$ are generated by differential expression (2.3) and boundary condition (2.5) with any linear bounded operator B.

Now consider the general case. For the this in the $L^2(H,(a,b))$ introduce an operator $U : L^2(H,(a,b)) \to L^2(H,(a,b))$, by

$$(Uz)(t) := U(t,a)z(t), \quad z \in L^2(H,(a,b))$$

From the properties of family of evolution operators $U(t,s), t,s \in [a,b]$ imply that a operator U is a linear continuous boundedly invertible and

$$(U^{-1}z)(t) = U(a,t)z(t).$$

On the other hand from the relations

$$U^{-1}L_0U = M_0, \quad U^{-1}\widetilde{L}U = \widetilde{M}, \quad U^{-1}LU = M$$

it implies that an operator U is one-to-one between of sets of boundedly solvable extensions of minimal operators L_0 and M_0 in $L^2(H,(a,b))$.

Extension $\widetilde{L}$ of the minimal operator L_0 is boundedly solvable in $L^2(H,(a,b))$ if and only if the operator $\widetilde{M} = U^{-1}\widetilde{L}U$ is an extension of the minimal M_0 in $L^2(H,(a,b))$. Then $u \in D(\widetilde{L})$ if and only if

$$(B+E)U(a,a)u(a) = BU(a,b)u(b);$$

that is,

$$(B+E)u(a) = BU(a,b)u(b).$$

□

This proves the validity of the claims in theorem.

From the above theorem, we have following assertion.

Theorem 2.3. *Every boundedly solvable extension $\widetilde{K}$ of the minimal operator K_0 in $\mathcal{H}$ is generated by differential-operator expression* (2.2) *and boundary condition*

$$(B+E)u_2(a) = BU(a,b)u_2(b)$$

where $B \in L(H)$ and E are a identity operators in H. The operator B is determined uniquely by the extension $\widetilde{K}$, i.e. $\widetilde{K} = K_B$ And vice versa.

Corollary 2.4. *The resolvent operator $R_\lambda(K_B), \lambda \in \rho(K_B)$ of any boundedly solvable operator K_B of the minimal operator K_0, generated by differential expression* (2.2) *with boundary condition*

$$(B+E)u_2(a) = BU(a,b)u_2(b), \quad B \in L(H)$$

is of the form

$$R_\lambda(K_B) = (E_1, R_\lambda(L_B), E_2),$$

where $R_\lambda(K_B) : \mathcal{H} \to \mathcal{H}$,

$$\begin{aligned} R_\lambda(L_B)f(t) = U(t,a)(E + B(1-e^\lambda))^{-1}B\int_a^b e^{\lambda(b-s)}U(a,s)f(s)ds \\ + \int_a^t e^{\lambda(b-s)}U(a,s)f(s)ds, \quad f \in L^2(H,(a,b)). \end{aligned}$$

Example 2.5. Consider the forward-backward differential equation

$$\begin{gathered} u = (u_1,u_2,u_3) \in L^2(t_1-1,t_1) \oplus L^2(t_1,t_2) \oplus L^2(t_2,t_2+1) \\ u_2'(t) = au_2(t) + bu_2(t-1) + cu_2(t+1), \quad t \in [t_1,t_2],\ a,b,c \in \mathbb{C} \end{gathered}$$

with boundary conditions

$$u_2(t) = \varphi(t), \quad t_1 - 1 \le t \le t_1,$$
$$u_2(t) = \psi(t), \quad t_2 \le t \le t_2 + 1,$$

where $\varphi \in C[t_1 - 1, t_1]$ and $\psi \in C[t_2, t_2 + 1]$ (see [1]).

It is clear that from Theorem 2.2, that all L^2-boundedly solvable boundary value problem in this case can be written in the form

$$u_2'(t) = au_2(t) + bu_2(t-1) + cu_2(t+1) + f_2(t)$$
$$(\gamma + 1)U_2(t_1) = \gamma U(t_2, t_1)u_2(t_2)$$

and in this case solutions have the form

$$u = (\varphi, u_2, \psi), u_2(t) = U(t, t_1)\gamma \int_{t_1}^{t_2} U(t_1, s)f_2(s)ds + \int_{t_1}^{t} U(t_1, s)f_2(s)ds,$$

where: $\gamma \in \mathbb{C}$, $U(t,s) = \exp(aE + bS_1^- + cS_1^+)(t - s)$, $t_1 \le t$, $s \le t_2$, and $f_2 \in L^2(t_1, t_2)$.

3. Spectrum of boundedly solvable extensions

In this section we study the spectrum structure of boundedly solvable extension $K_B = E_1 \oplus L_B \oplus E_2$ of the minimal operator $K_0 = E_1 \oplus L_0 \oplus E_2$ in Hilbert space

$$\mathcal{H} = L^2(H, (a - \alpha, a)) \oplus L^2(H, (a, b)) \oplus L^2(H, (b, b + \tau)).$$

Firstly, note that as in [4] for the spectrum $\sigma(L_B)$ of any boundedly solvable extension L_B of L_0 the following assertion can be proved.

Theorem 3.1. *If L_B is a boundedly solvable extension of the minimal operator L_0 in the Hilbert space $L^2(H, (a, b))$, then spectrum set of L_B has the form*

$$\sigma(L_B) = \big\{\lambda \in \mathbb{C} : \lambda = \ln|\frac{\mu + 1}{\mu}| + i\arg(\frac{\mu + 1}{\mu}) + 2n\pi i,$$
$$\mu \in \sigma(B)\backslash\{0, -1\},\ n \in \mathbb{Z}\big\}$$

The following assertion follows from a result in [8].

Theorem 3.2. *If $\alpha, \tau \ge 0$, $\alpha + \tau > 0$ and $K_B = E_1 \oplus L_B \oplus E_2$ is any boundedly solvable extension on of the minimal operator $K_0 = E_1 \oplus L_0 \oplus E_2$ in $\mathcal{H}$, then*

$$\sigma_p(K_B) = \sigma_p(L_B) \cup \{1\},$$
$$\sigma_c(K_B) = \big[\big((\sigma_p(L_B))^c \cap \sigma_c(L_B) \cap \sigma_r(L_B)\big)^c\big]\backslash\{1\},$$
$$\sigma_r(K_B) = [(\sigma_p(L_B))^c \cap \sigma_r(L_B)]\backslash\{1\},$$
$$\rho(K_B) = \rho(L_B)\backslash\{1\},$$

where: $\sigma_p(.), \sigma_c(.), \sigma_r(.)$, and $\rho(.)$ denote point, continuous, residual and resolvent sets of an operator respectively.

We remark that when $\alpha\tau = 0$, i.e.

(1) $\alpha = 0$ and $\tau > 0$, advanced type,
(2) $\alpha > 0$ and $\tau = 0$, retarded type,
(3) $\alpha = 0$ and $\tau = 0$, ordinary type

the differential expression spectrum of boundedly solvable extensions is easy to be investigated as in the above theorem.

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