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EXISTENCE OF TRAVELING WAVEFRONTS FOR INTEGRODIFFERENCE EQUATIONS WITH BILATERAL EXPONENTIAL KERNEL

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Abstract. In this article, we study the existence of traveling wavefronts for integrodifference equation with a bilateral exponential kernel, namely, the Laplacian kernel. The existence of traveling wavefronts is proved by combining the monotone iteration technique with the upper and lower solution method. The minimal spreading speed c^* is given, which can be figured out exactly when all parameters are given explicitly.

1. Introduction

In 1937, a model for the spatial spread of an advantageous gene in a population living in a homogeneous one dimensional habitat was proposed by Fisher [6]. In this model, the time evolution of the fraction $u(x,t)$ of the advantageous gene in the population at the point x and at the time t is governed by a partial differential equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.1}$$

where $f \in C^1[0,1]$ and $f(0) = f(1) = 0$. In the same year, Kolmogorov, Petrovskii and Piskunov [8] studied the same system, where $f \in C^2[0,1]$ and $f(0) = f(1) = 0$.

In previous few decades, there have been extensive investigations on traveling wave solutions and asymptotic behaviors in terms of spreading speeds for various evolution systems. Traveling waves were studied for nonlinear reaction-diffusion equations modeling physical and biological phenomena [16, 17, 25], for lattice differential equation [1, 4, 28, 30] and for time-delayed reaction-diffusion equations [22, 23, 24, 29].

Since the observation is often discontinuous, many discrete-time models are derived from different fields, such as difference equations [9] and integrodifference equations [26, 27]. As for the references mentioned above, much attention has been paid to the discrete-time model

$$u_{n+1}(x) = Q[u_n](x), \tag{1.2}$$

where $x \in \mathbb{H} \subseteq \mathbb{R}$, $\mathbb{H}$ is a habitat and Q is a continuous mapping with respect to a proper topology. When we consider an organism with synchronous nonoverlapping

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generations, $u_n(x)$ can be viewed as the population density of the species at the point $x \in \mathbb{H}$ in population dynamics. System (1.2) implies that the evolution of the current individuals only depends on the individuals at the previous unit time or generation.

When the life cycle of an organism consists of distinct growth and dispersal stages, and if these stages are synchronized within a population, the discrete-time models may be more accurate representations than continuous-time equation. Many plants, insect and migrating bird species in temperate climates fall into this category. Assume that there are two distinct stages that define the life cycle of these organisms, a sedentary stage and a dispersal stage. All growth occurs during the sedentary stage and all movement occurs during the dispersal stage.

To formulate an integrodifference equation, when the population is continuously distributed, we denote the density of the population at time or generation n at location x as $u_n(x)$. The sedentary stage is described by some non-negative function $f(u)$, e.g. the Beverton-Holt Stock-recruitment curve [2] or the Ricker curve [21], and the dispersal stage by a dispersal kernel, $k(x, y)$, where the product $k(x, y)dy$ is the probability that an individual who will move from the interval $(y, y + dy]$ to the point x [19]. The population density in the next generation is obtained by tallying arrivals at location x from all possible locations y, or mathematically as the integral operator

$$u_{n+1}(x) = \int_\Omega k(x, y)f(u_n(y))dy, \tag{1.3}$$

where Ω is the habitat of the organism. If the environment is isotropic, one may hope that the kernels $k(x, y)$ is symmetric in x and y, $k(x, y) = k(y, x)$. Dispersal tends to depend only on distance between source and destination, so the kernel may depend on absolute location or on relation distance. Let $k(x - y)$ be the spatial dispersal probability function of the species jumping from y to x, then we obtain the following model

$$u_{n+1}(x) = \int_\Omega k(x - y)f(u_n(y))dy. \tag{1.4}$$

In the past three decades, the traveling wave solutions of (1.4) have been widely studied, we refer to Hsu and Zhao [7], Kot [10], Liang and Zhao [11], Neubert and Caswell [18], Weinberger [26, 27]. In these papers, the monotonicity of the function f plays a very important role. Recently, Lin and Li [15] and Lin et al [14] considered the existence of traveling wave solutions of a competitive system by a cross iteration scheme. In population dynamics, one typical integrodifference equation describing the age-structure and the birth function is (locally) monotone, then the traveling wave solutions and asymptotic spreading were studied by Lin and Li [13] and Pan and Lin [20].

In 1992, Kot [10] studied the discrete-time traveling waves, when the integrodifference equation with the kernel being the bilateral exponential distribution

$$k(x, y) = \frac{1}{2}\alpha \exp(-\alpha|x - y|),$$

for a scalar equation with compensatory growth and two kinds of special recruitment cures. His research observed only simple traveling waves. However, in different biological systems there are kinds of recruitment cures, we cannot get the traveling waves following [10]. Motivated by the studies in [3] and in [12], in this paper, we investigate the existence of traveling wavefronts to the following integrodifference equation

$$u_{n+1}(x)=\frac{\alpha}{2}\int_{\Omega}\exp(-\alpha|x-y|)f(u_n(y))dy. \tag{1.5}$$

For convenience, we only study the case that the environment Ω is $\mathbb{R}$. Thus equation (1.5) become

$$u_{n+1}(x)=\frac{\alpha}{2}\int_{\mathbb{R}}\exp(-\alpha|x-y|)f(u_n(y))dy. \tag{1.6}$$

The rest of this paper is organized as follows. In section 2, we obtain the existence of traveling wavefronts by using upper and lower solution method. In section 3, some numerical simulations are given to illustrate our main results. A brief conclusion will also be given in this section.

2. Existence of traveling wavefronts

In this section, we shall establish the existence of traveling wavefronts of (1.6) by combining the monotone iteration technique with the upper and lower solutions method. Let

$$C(\mathbb{R},\mathbb{R})=\{u|u:\mathbb{R}\to\mathbb{R}\text{ is uniformly continuous and bounded}\}.$$

Then $C(\mathbb{R},\mathbb{R})$ is a Banach space equipped with supremum norm $|\cdot|$. If $a,b\in\mathbb{R}$ with $a<b$, then we denote

$$C_{[a,b]}=\{u\in C(\mathbb{R},\mathbb{R}):a\le u(x)\le b\text{ for all }x\in\mathbb{R}\}.$$

Throughout the remainder of this paper, we assume that

- (H1) $f(0)=0$, $f(1)=1$, and $f(u)>u$ for any $u\in(0,1)$.
- (H2) f is a C^2 function and $0<f'(u)\le f'(0)$ for $u\in[0,1)$.

By (H2), there exists a constant $L>0$ such that $|f''(u)|<f'(0)L$ for any $u\in[0,1]$.

Definition 2.1. A traveling wave solution of (1.6) is a special solution with the form $u_n(x)=\phi(x+cn)$, with $c>0$ is the wave speed that the wave profile $\phi\in C(\mathbb{R},\mathbb{R})$ spreads in $\mathbb{R}$. In particular, if $\phi(\xi)$ is monotone in $\xi\in\mathbb{R}$, then it is called a traveling wavefront.

By Definition 2.1, the traveling wavefront $\phi(\xi)$ of (1.6) must satisfy the integral equation

$$\begin{aligned}\phi(\xi+c)&=\frac{\alpha}{2}\int_{\mathbb{R}}\exp(-\alpha|x-y|)f(\phi(y+cn))dy\\&=\frac{\alpha}{2}\int_{\mathbb{R}}\exp(-\alpha|x-y|)f(\phi(\xi-x+y))dy\\&=\frac{\alpha}{2}\int_{\mathbb{R}}\exp(-\alpha|X|)f(\phi(\xi-X))dX,\end{aligned} \tag{2.1}$$

where $\xi=x+cn$, $X=x-y$.

Because of the background of traveling wavefronts [10, 18], we also require that ϕ satisfies the asymptotic boundary value conditions,

$$\lim_{\xi\to-\infty}\phi(\xi)=0,\quad \lim_{\xi\to\infty}\phi(\xi)=1. \tag{2.2}$$

Thus, our intention is to prove the existence of a monotone solution of (2.1) with boundary value conditions (2.2). For this purpose, we rewrite (2.1) as

$$\phi(\xi) = \frac{\alpha}{2}\int_{\mathbb{R}} \exp(-\alpha|x|) f(\phi(\xi - x - c))dx, \quad \xi \in \mathbb{R}. \tag{2.3}$$

The linearization of (2.3) in the neighborhood of $\phi = 0$ is

$$\begin{aligned}\phi(\xi) &= \frac{\alpha}{2}\int_{\mathbb{R}} \exp(-\alpha|x|) f'(0)\phi(\xi - x - c)dx \\ &= \frac{f'(0)\alpha}{2}\int_{\mathbb{R}} \exp(-\alpha|x|)\phi(\xi - x - c)dx.\end{aligned} \tag{2.4}$$

One may attempt to find a solution of the form

$$\phi(\xi) = e^{\lambda\xi}, \tag{2.5}$$

where λ is a positive number. Then

$$e^{\lambda\xi} = \frac{f'(0)\alpha}{2}\int_{\mathbb{R}} e^{-\alpha|x|}e^{\lambda(\xi - x - c)}dx.$$

Thus,

$$1 = \frac{f'(0)\alpha}{2}\int_{\mathbb{R}} e^{-\alpha|x|}e^{-\lambda(x+c)}dx.$$

We define

$$\Delta(\lambda, c) = \frac{f'(0)\alpha}{2}\int_{\mathbb{R}} e^{-\alpha|x|}e^{-\lambda(x+c)}dx \tag{2.6}$$

for any $\lambda \in (0, \alpha), c \in (0, \infty)$. Then $\Delta(\lambda, c)$ is well defined and the following result holds.

Lemma 2.2. *There exists a constant $c^* > 0$ such that $\Delta(\lambda, c) = 1$ has exactly two positive roots if $c > c^*$ while $\Delta(\lambda, c) = 1$ has no real root if $c < c^*$. Moreover, if $c > c^*$ holds and $\lambda_1(c)$ is the smaller root, $\lambda_2(c)$ is the other root, then for any $\eta \in (1, \frac{\lambda_2(c)}{\lambda_1(c)})$, $\Delta(\eta\lambda_1(c), c) < 1$ holds.*

Proof. For any $\lambda \in (0, \alpha)$,

$$\begin{aligned}\Delta(\lambda, c) &= \frac{f'(0)\alpha}{2}\int_{-\infty}^{+\infty} e^{-\alpha|x|}e^{-\lambda(x+c)}dx \\ &= \frac{f'(0)\alpha}{2}\int_{-\infty}^{0} e^{\alpha x}e^{-\lambda(x+c)}dx + \frac{f'(0)\alpha}{2}\int_{0}^{+\infty} e^{-\alpha x}e^{-\lambda(x+c)}dx \\ &= \frac{f'(0)\alpha}{2e^{\lambda c}}\int_{-\infty}^{0} e^{(\alpha-\lambda)x}dx + \frac{f'(0)\alpha}{2e^{\lambda c}}\int_{0}^{+\infty} e^{-(\alpha+\lambda)x}dx \\ &= \frac{f'(0)\alpha}{2e^{\lambda c}}\,\frac{1}{\alpha - \lambda} + \frac{f'(0)\alpha}{2e^{\lambda c}}\,\frac{1}{\alpha + \lambda} \\ &= \frac{f'(0)\alpha^2}{e^{\lambda c}(\alpha - \lambda)(\alpha + \lambda)}.\end{aligned}$$

We see that $\Delta(\lambda, c)$ is continuous in $c > 0$, $\lambda \in (0, \alpha)$. For fixed $c > 0$, direct calculations show that

$$\begin{aligned}\frac{\partial}{\partial \lambda}\Delta(\lambda, c) &= \frac{-cf'(0)\alpha^2 e^{-\lambda c}(\alpha^2-\lambda^2) + 2f'(0)\lambda\alpha^2 e^{-\lambda c}}{(\alpha^2-\lambda^2)^2} \\ &= \frac{f'(0)\alpha^2[2\lambda - c(\alpha^2-\lambda^2)]}{e^{\lambda c}(\alpha^2-\lambda^2)^2}, \\ \frac{\partial^2}{\partial \lambda^2}\Delta(\lambda, c) &= \frac{f'(0)\alpha^2 e^{-\lambda c}[(c(\alpha^2-\lambda^2)-2\lambda)^2 + 4\lambda^2 + 2(\alpha^2-\lambda^2)]}{(\alpha^2-\lambda^2)^3} > 0.\end{aligned}$$

It is easy to see that $\Delta(\lambda, c)$ is convex in $\lambda \in (0, \alpha)$ for fixed $c > 0$. Let $\frac{\partial}{\partial\lambda}\Delta(\lambda, c) = 0$. Then $\lambda(c) = \frac{1}{c}(\sqrt{1+\alpha^2c^2} - 1)$ attains the mimimun of $\Delta(\lambda, c)$ for fixed $c > 0$. Also

$$\begin{gathered}\Delta(\lambda(c), c) = \min_{\lambda\in(0,\alpha)} \Delta(\lambda, c) = \frac{f'(0)}{2}\exp[1-\sqrt{1+\alpha^2c^2}](\sqrt{1+\alpha^2c^2}+1). \\ \frac{d}{dc}\Delta(\lambda(c), c) = -\frac{c\alpha^2 f'(0)}{2}\exp[1-\sqrt{1+\alpha^2c^2}] < 0.\end{gathered}$$

It means that $\Delta(\lambda(c), c)$ is strictly decreasing in c. Since

$$\lim_{c\to 0+}\Delta(\lambda(c), c) = f'(0) > 1, \quad \lim_{c\to+\infty}\Delta(\lambda(c), c) = 0,$$

the continuity of $\Delta(\lambda(c), c)$ in c implies that there exists unique c^* such that $\Delta(\lambda(c^*), c^*) = 1$. For any $c < c^*$, $\Delta(\lambda(c), c) > 1$ for all $\lambda \in (0, \alpha)$, therefore, $\Delta(\lambda, c) = 1$ has no real root. For $c > c^*$, $\Delta(\lambda(c), c) < 1$. Since

$$\lim_{\lambda\to 0+}\Delta(\lambda, c) = f'(0) > 1, \quad \lim_{\lambda\to\alpha-0}\Delta(\lambda, c) = +\infty,$$

and $\Delta(\lambda, c)$ is strictly deceasing in $\lambda \in (0, \lambda(c))$ and strictly increasing in $\lambda \in (\lambda(c), \alpha)$, then $\Delta(\lambda, c) = 1$ has exactly two positive roots $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_1(c) \in (0, \lambda(c))$, $\lambda_2(c) \in (\lambda(c), \alpha)$ and $\Delta(\lambda, c) < 1$ for any $\lambda \in (\lambda_1(c), \lambda_2(c))$. This completes the proof. □

Remark 2.3. From the proof of Lemma 2.2, we know that c^* can be formulated explicitly as follows.

$$c^* = \frac{\sqrt{(z^*)^2 + 2z^*}}{\alpha},$$

where z^* is the unique positive solution to the equation

$$\frac{1}{2}f'(0)(z+2) = e^z.$$

Definition 2.4. A continuous function $\phi(\xi) \in C_{[0,1]}$ is called an upper solution of (2.3), if it satisfies

$$\phi(\xi) \geq \frac{\alpha}{2}\int_{\mathbb{R}} \exp(-\alpha|x|)f(\phi(\xi - x - c))dx, \quad \xi \in \mathbb{R}.$$

Similarly, a continuous function $\phi(\xi) \in C_{[0,1]}$ is called a lower solution of (2.3), if it satisfies

$$\phi(\xi) \leq \frac{\alpha}{2}\int_{\mathbb{R}} \exp(-\alpha|x|)f(\phi(\xi - x - c))dx, \quad \xi \in \mathbb{R}.$$

For fixed $c > c^*$, let $q > 1$, $\eta \in (1, \frac{\lambda_1(c)}{\lambda_2(c)})$ be given constants. We define continuous functions $\overline{\phi}(t)$ and $\underline{\phi}(t)$ as follows.

$$\begin{aligned}\overline{\phi}(t) &= \min\{1, e^{\lambda_1(c)t} + qe^{\eta\lambda_1(c)t}\},\\ \underline{\phi}(t) &= \max\{0, e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}\}.\end{aligned}$$

It is easy to see that both $\overline{\phi}(t)$ and $\underline{\phi}(t)$ are continuous functions with $0 < \overline{\phi}(t) \le 1$ and $0 \le \underline{\phi}(t) < 1$ for any $t \in (-\infty, +\infty)$. Clearly, there exists a constant $t^* < 0$ such that $\overline{\phi}(t)$ is strictly increasing for $t < t^*$ and $\overline{\phi}(t) = 1$ for $t \ge t^*$. Also there exists a constant $t_* < 0$ such that $\underline{\phi}(t) > 0$ for $t < t_*$ and $\underline{\phi}(t) = 0$ for $t \ge t_*$.

Proposition 2.5. *The function $\overline{\phi}(t)$ is an upper solution of* (2.3)*.*

Proof. By the definition of $\overline{\phi}$, $0 < \overline{\phi}(y) \le 1$ for all $y \in \mathbb{R}$. If $\overline{\phi}(t) = 1$ for some t, by (H1) and (H2), we have

$$\frac{\alpha}{2}\int_{\mathbb{R}} \exp(-\alpha|x|) f(\overline{\phi}(t-x-c))dx \le \frac{\alpha}{2}\int_{\mathbb{R}} \exp(-\alpha|x|)dx = 1 = \overline{\phi}(t).$$

Thus the result holds.

If for some t, $\overline{\phi}(t) = e^{\lambda_1(c)t} + qe^{\eta\lambda_1(c)t}$, then by (H2) we have

$$\begin{aligned}&\frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)} f(\overline{\phi}(t-x-c))dx\\ &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)} f'(\theta\overline{\phi}(t-x-c))\overline{\phi}(t-x-c)dx\\ &\le \frac{f'(0)\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}\overline{\phi}(t-x-c)dx\\ &\le \frac{f'(0)\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}(e^{\lambda_1(c)(t-c-x)} + qe^{\eta\lambda_1(c)(t-c-x)})dx\\ &= e^{\lambda_1(c)t}\Delta(\lambda_1(c), c) + qe^{\eta\lambda_1(c)t}\Delta(\eta\lambda_1(c), c)\\ &\le e^{\lambda_1(c)t} + qe^{\eta\lambda_1(c)t} = \overline{\phi}(t).\end{aligned}$$

This completes the proof. □

Proposition 2.6. *The function $\underline{\phi}(t)$ is a lower solution of* (2.3) *for $1 < \eta < \min\{\frac{\lambda_2(c)}{\lambda_1(c)}, 2\}$ and $q > \frac{L\Delta(\eta\lambda_1(c),c)}{2(1-\Delta(\eta\lambda_1(c),c))} + 1$, where L satisfies $|f''(u)| \le f'(0)L$ for $u \in [0,1]$.*

Proof. If $\underline{\phi}(t) = 0$ for some t, then the result holds because $f(\underline{\phi}(t)) \ge 0$ for all $t \in \mathbb{R}$.

If $\underline{\phi}(t) = e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}$ for some t, then, by Taylor expansion with Lagrangian remainder, we have by (H1)

$$\begin{aligned}&\frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)} f(\underline{\phi}(t-x-c))dx\\ &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}\Big[f'(0)\underline{\phi}(t-x-c) + \frac{1}{2}f''(\theta\underline{\phi}(t-x-c))\underline{\phi}(t-x-c)^2\Big]dx,\end{aligned}$$

with $0 < \theta < 1$. Then,

$$\frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)} f(\underline{\phi}(t-x-c))dx$$

$$
\begin{aligned}
&\ge \frac{\alpha f'(0)}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}\underline{\phi}(t-x-c)dx - \frac{\alpha f'(0)L}{4}\int_{\mathbb{R}} e^{(-\alpha|x|)}\underline{\phi}(t-x-c)^2dx\\
&\ge \frac{\alpha f'(0)}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}[e^{\lambda_1(c)(t-x-c)} - qe^{\eta\lambda_1(c)(\xi-x-c)}]dx\\
&\quad - \frac{\alpha f'(0)L}{4}\int_{\mathbb{R}} e^{(-\alpha|x|)}\underline{\phi}(t-x-c)^\eta dx\\
&\ge e^{\lambda_1(c)t}\Delta(\lambda_1(c),c) - qe^{\eta\lambda_1(c)t}\Delta(\eta\lambda_1(c),c) - \frac{\alpha f'(0)L}{4}\int_{\mathbb{R}} e^{(-\alpha|x|)}e^{\eta\lambda_1(c)(t-x-c)}dx\\
&= e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}\Delta(\eta\lambda_1(c),c) - \frac{L}{2}e^{\eta\lambda_1(c)t}\Delta(\eta\lambda_1(c),c)\\
&\ge e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t} = \underline{\phi}(t).
\end{aligned}
$$

This completes the proof. □

Lemma 2.7. *Let $g(t) = e^{-\alpha|t|}$. Then g is uniformly continuous on $\mathbb{R}$*

Proof. Since $\lim_{|t|\to\infty} g(t) = 0$, for any $\varepsilon > 0$, there exists $K > 0$, such that $g(t) < \frac{\varepsilon}{2}$ for $|t| > K$. The uniform continuity of g on $[-K-1, K+1]$ means that there exists $\delta_1 > 0$, such that for any $t_1, t_2 \in [-K-1, K+1]$, $|g(t_1) - g(t_2)| < \varepsilon$. Let $\delta = \min\{1, \delta_1\}$. Then for any $t_1, t_2 \in \mathbb{R}$, $|t_1 - t_2| < \delta$, we have $|g(t_1) - g(t_2)| < \varepsilon$. Then g is uniformly continuous on $\mathbb{R}$. □

Theorem 2.8. *Assume that $c > c^*$ holds. Then* (2.3) *with* (2.2) *has a monotone solution $\phi(t)$ such that $\lim_{t\to-\infty}\phi(t)e^{-\lambda_1(c)t} = 1$.*

Proof. We now prove the result by standard iteration techniques [5, 24]. According to Definition 2.4 and Proposition 2.5, we define continuous functions $\overline{\phi}_1(t)$ and $\underline{\phi}_1(t)$ as follows.

$$
\begin{aligned}
\overline{\phi}_1(t) &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}f(\overline{\phi}(t-x-c))dx, \quad t\in\mathbb{R},\\
\underline{\phi}_1(t) &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}f(\underline{\phi}(t-x-c))dx, \quad t\in\mathbb{R}.
\end{aligned}
$$

Then $\overline{\phi}_1(t)$, $\underline{\phi}_1(t)$ are well defined and

$$
1 \ge \overline{\phi}(t) \ge \overline{\phi}_1(t) \ge \underline{\phi}_1(t) \ge \underline{\phi}(t) \ge 0, \quad t\in\mathbb{R}.
$$

Let

$$
\begin{aligned}
\overline{\phi}_{n+1}(t) &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}f(\overline{\phi}_n(t-x-c))dx, \quad t\in\mathbb{R},\\
\underline{\phi}_{n+1}(t) &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)}f(\underline{\phi}_n(t-x-c))dx, \quad t\in\mathbb{R},
\end{aligned}
$$

for $n = 1, 2, \dots$. By mathematical induction and $(H2)$, we have

$$
1 \ge \overline{\phi}_n(t) \ge \overline{\phi}_{n+1}(t) \ge \underline{\phi}_{n+1}(t) \ge \underline{\phi}_n(t) \ge 0, \quad t\in\mathbb{R}.
$$

We rewrite $\overline{\phi}_n(t)$ and $\underline{\phi}_n(t)$ as follows.

$$
\begin{aligned}
\overline{\phi}_n(t) &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|t-y|)}f(\overline{\phi}_{n-1}(y-c))dy,\\
\underline{\phi}_n(t) &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|t-y|)}f(\underline{\phi}_{n-1}(y-c))dy.
\end{aligned}
$$

The monotonicity of f and $\overline{\phi}$ means that $\overline{\phi}_n(t)$ is increasing in $t \in \mathbb{R}$ for each $n = 1, 2, \dots$. For any given finite interval $[-M, M]$, we now prove the uniform convergence of sequence $\overline{\phi}_n$ on $[-M, M]$.

For any $\varepsilon > 0$, since $\frac{\alpha}{2}\int_{\mathbb{R}} e^{-\alpha|x|}dx = 1$, there exists $K_1 > 0$, such that

$$\frac{\alpha}{2}\int_{|x|>K_1} e^{-\alpha|x|}dx < \frac{1}{4}\varepsilon.$$

By Lemma 2.7, there exists $\delta > 0$, such that for any $t_1, t_2, |t_1 - t_2| < \delta$, we have $|g(t_1) - g(t_2)| < \frac{\varepsilon}{2\alpha(M+K_1)}$.

For any $t_1, t_2 \in [-M, M], |t_1 - t_2| < \delta$, we have

$$\begin{aligned}
&|\overline{\phi}_n(t_1) - \overline{\phi}_n(t_2)| \\
&= |\frac{\alpha}{2}\int_{\mathbb{R}} e^{-\alpha|t_1-y|} f(\overline{\phi}_n(y-c))dy - \frac{\alpha}{2}\int_{\mathbb{R}} e^{-\alpha|t_2-y|} f(\overline{\phi}_n(y-c))dy| \\
&\le \frac{\alpha}{2}\int_{\mathbb{R}} |(e^{-\alpha|t_1-y|} - e^{-\alpha|t_2-y|})| f(\overline{\phi}_n(y-c))dy \\
&\le \frac{\alpha}{2}\int_{|y|>M+K_1} |(e^{-\alpha|t_1-y|} - e^{-\alpha|t_2-y|})| f(\overline{\phi}_n(y-c))dy \\
&\quad + \frac{\alpha}{2}\int_{-M-K_1}^{M+K_1} |(e^{-\alpha|t_1-y|} - e^{-\alpha|t_2-y|})| f(\overline{\phi}_n(y-c))dy \\
&\le \alpha\int_{|x|>K_1} e^{-\alpha|x|}dx + \frac{\alpha}{2}\int_{-M-K_1}^{M+K_1} \frac{\varepsilon}{2\alpha(M+K_1)}dx \\
&= \varepsilon, \quad \text{for } n = 1, 2, \dots.
\end{aligned}$$

This implies that $\overline{\phi}_n(t)$ are equicontinuous for $n = 1, 2, \dots$ and $t \in [-M, M]$.

It is easy to know that $\overline{\phi}_n(t)$ converges to a nondecreasing continuous function uniformly on any compact subsets of $\mathbb{R}$. Let $\lim_{n\to\infty} \phi_n(t) = \phi(t)$. We claim that $\phi(t)$ satisfies

$$\phi(t) = \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)} f(\phi(t-x-c))dx.$$

Actually, for a given t and any $\varepsilon > 0$, there exists $K_1 > 0$ such that

$$\frac{\alpha}{2}\int_{|x|>K_1} e^{(-\alpha|x|)}dx < \frac{\varepsilon}{4}.$$

By uniform convergence of $\{\overline{\phi}_n\}$ on $[t-c-K_1, t-c+K_1]$, there exists $N > 0$ such that for any $n > N$ and $x \in [-K_1, K_1]$,

$$|f(\overline{\phi}_n(t-x-c)) - f(\phi(t-x-c))| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}
&|\frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)} f(\overline{\phi}_n(t-x-c))dx - \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)} f(\phi(t-x-c))dx| \\
&\le \frac{\alpha}{2}\int_{\mathbb{R}} e^{(-\alpha|x|)} |f(\overline{\phi}_n(t-x-c)) - f(\phi(t-x-c))|dx \\
&\le \alpha\int_{|x|>K_1} e^{(-\alpha|x|)}dx + \frac{\alpha}{2}\int_{-K_1}^{K_1} e^{(-\alpha|x|)} |f(\overline{\phi}_n(t-x-c)) - f(\phi(t-x-c))|dx \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } n > N.
\end{aligned}$$

Thus $\phi(t)$ is a monotone solution to (2.3).

To complete the proof of Theorem 2.8, we have to show that $\phi(t)$ satisfies (2.2). The iteration scheme shows that $\overline{\phi}(t) \geq \phi(t) \geq \underline{\phi}(t), t \in \mathbb{R}$. By the monotonicity of $\phi(t)$, we obtain

$$\lim_{t\to-\infty} \phi(t) \in [0, \inf_{t\in\mathbb{R}} \overline{\phi}(t)],$$

By the properties of $\overline{\phi}(t)$, we obtain $\lim_{t\to-\infty} \phi(t) = 0$. The monotonicity and the boundedness of ϕ imply the existence of $\lim_{t\to\infty} \phi(t)$. We claim that this limit is a constant solution of $\phi(t) = \frac{\alpha}{2} \int_{\mathbb{R}} e^{(-\alpha|x|)} f(\phi(t-x-c))dx$.

Actually, let $\lim_{t\to\infty} \phi(t) = \phi^*$. Then by the continuity of f, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(u) - f(\phi^*)| < \frac{\varepsilon}{2}$ for $|u - \phi^*| < \delta$. Since $\lim_{t\to\infty} \phi(t) = \phi^*$, there exists $T > 0$, such that $|\phi(t) - \phi^*| < \delta$ for $t > T$. Similar to the argument as above, there exists $K_1 > 0$, satisfies

$$\frac{\alpha}{2} \int_{x>K_1} e^{(-\alpha|x|)} dx < \frac{\varepsilon}{4}.$$

Then by the monotonicity of ϕ and f, for any $t > T + K_1 + c$,

$$\begin{aligned}
&\Big|\frac{\alpha}{2} \int_{\mathbb{R}} e^{(-\alpha|x|)} f(\phi(t-x-c))dx - \frac{\alpha}{2} \int_{\mathbb{R}} e^{(-\alpha|x|)} f(\phi^*)dx\Big| \\
&\leq \frac{\alpha}{2} \int_{K_1}^{\infty} e^{(-\alpha|x|)} |f(\phi(t-x-c)) - f(\phi^*)|dx \\
&\quad + \frac{\alpha}{2} \int_{-\infty}^{K_1} e^{(-\alpha|x|)} |f(\phi(t-x-c)) - f(\phi^*)|dx \\
&\leq \alpha \int_{K_1}^{\infty} e^{(-\alpha|x|)} dx + \frac{\alpha}{2} \int_{K_1}^{\infty} e^{(-\alpha|x|)} |f(\phi(t-K_1-c)) - f(\phi^*)|dx \leq \varepsilon.
\end{aligned}$$

Thus we see that

$$\phi^* = \frac{\alpha}{2} \int_{\mathbb{R}} e^{(-\alpha|x|)} f(\phi^*)dx = f(\phi^*).$$

Clearly, $\phi^* = 1$.

From the definition of $\overline{\phi}(t)$ and $\underline{\phi}(t)$, we have

$$\overline{\phi}(t)e^{-\lambda_1(c)t} = \begin{cases} 1 + qe^{\lambda_1(c)t(\eta-1)} & t < t^* < 0, \\ e^{-\lambda_1(c)t} & t \geq t^*, \end{cases}$$

$$\underline{\phi}(t)e^{-\lambda_1(c)t} = \begin{cases} 1 - qe^{\lambda_1(c)t(\eta-1)} & t < t_* < 0, \\ 0 & t \geq t_*. \end{cases}$$

Therefore,

$$\begin{aligned}
\lim_{t\to-\infty} \overline{\phi}(t)e^{-\lambda_1(c)t} &= 1 + \lim_{t\to-\infty} qe^{\lambda_1(c)t(\eta-1)} = 1, \\
\lim_{t\to-\infty} \underline{\phi}(t)e^{-\lambda_1(c)t} &= 1 - \lim_{t\to-\infty} qe^{\lambda_1(c)t(\eta-1)} = 1.
\end{aligned}$$

Consequently,

$$\lim_{t\to-\infty} \phi(t)e^{-\lambda_1(c)t} = 1.$$

The proof of Theorem 2.8 is complete. □

Corollary 2.9. *Let $\phi(t)$ be as obtained in Theorem 2.8. Then $\phi \in C^1(\mathbb{R}, \mathbb{R})$ and $\lim_{t\to-\infty} \phi'(t)e^{-\lambda_1(c)t} = \lambda_1(c)$.*

Proof. Recall that $\phi(t)$ is a continuous monotone solution to (2.3), it satisfies

$$\phi(t) = \frac{\alpha}{2}\int_{\mathbb{R}} e^{-\alpha|x|} f(\phi(t-x-c))dx.$$

We rewrite $\phi(t)$ as

$$\begin{aligned}
\phi(t) &= \frac{\alpha}{2}\int_{\mathbb{R}} e^{-\alpha|t-y|} f(\phi(y-c))dy \\
&= \frac{\alpha}{2}\int_t^{\infty} e^{-\alpha(y-t)} f(\phi(y-c))dy + \frac{\alpha}{2}\int_{-\infty}^{t} e^{-\alpha(t-y)} f(\phi(y-c))dy \\
&= \frac{\alpha e^{\alpha t}}{2}\int_t^{\infty} e^{-\alpha y} f(\phi(y-c))dy + \frac{\alpha e^{-\alpha t}}{2}\int_{-\infty}^{t} e^{\alpha y} f(\phi(y-c))dy \\
&= e^{\alpha t} h_1(t) + e^{-\alpha t} h_2(t),
\end{aligned}$$

where $h_1(t) = \frac{\alpha}{2}\int_t^{\infty} e^{-\alpha y} f(\phi(y-c))dy$, $h_2(t) = \frac{\alpha}{2}\int_{-\infty}^{t} e^{\alpha y} f(\phi(y-c))dy$. By the continuity of ϕ and f, both h_1 and h_2 are differentiable, and

$$h_1'(t) = -\frac{\alpha}{2} e^{-\alpha t} f(\phi(t-c)), \quad h_2'(t) = \frac{\alpha}{2} e^{\alpha t} f(\phi(t-c)).$$

Hence ϕ is differentiable, and

$$\phi'(t) = \alpha e^{\alpha t} h_1(t) - \alpha e^{-\alpha t} h_2(t).$$

Clearly, $\phi \in C^1(\mathbb{R},\mathbb{R})$. Also

$$\begin{aligned}
&\lim_{t\to-\infty} \phi'(t) e^{-\lambda_1(c)t} \\
&= \lim_{t\to-\infty} \frac{\alpha h_1(t)}{e^{(\lambda_1(c)-\alpha)t}} - \lim_{t\to-\infty} \frac{\alpha h_2(t)}{e^{(\lambda_1(c)+\alpha)t}} \\
&= -\frac{\alpha^2}{2(\lambda_1(c)-\alpha)} \lim_{t\to-\infty} \frac{f(\phi(t-c))}{e^{\lambda_1(c)t}} - \frac{\alpha^2}{2(\lambda_1(c)+\alpha)} \lim_{t\to-\infty} \frac{f(\phi(t-c))}{e^{\lambda_1(c)t}} \\
&= \frac{\lambda_1(c)\alpha^2}{e^{\lambda_1(c)c}(\alpha^2-\lambda_1^2(c))} \lim_{t\to-\infty} \frac{f(\phi(t-c))}{\phi(t-c)} \frac{\phi(t-c)}{e^{\lambda_1(c)(t-c)}}.
\end{aligned}$$

Since

$$\lim_{t\to-\infty} \phi(t) = 0, \quad \lim_{u\to 0} \frac{f(u)}{u} = f'(0),$$

and by Theorem 2.8,

$$\lim_{t\to-\infty} \frac{\phi(t)}{e^{\lambda_1(c)t}} = 1,$$

we obtain

$$\lim_{t\to-\infty} \phi'(t) e^{-\lambda_1(c)t} = \frac{\lambda_1(c)\alpha^2 f'(0)}{e^{\lambda_1(c)c}(\alpha^2-\lambda_1^2(c))} = \lambda_1(c).$$

This completes the proof. $\square$

When we take the same parameters but with different wave speeds, the traveling wavefronts have different wave profiles. The numerical simulation can be observed Figure 4.

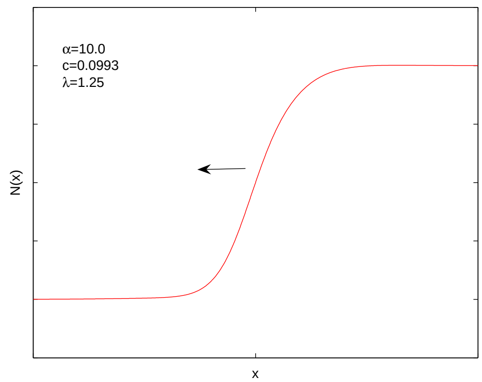

FIGURE 1. Traveling wave for a compensatory integrodiffence equation at a small speed. $\alpha = 10.0$, $\lambda = 1.25$, $K = 2$, $c = 0.0993$.

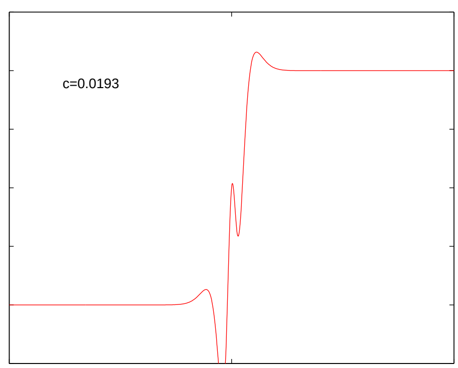

FIGURE 2. No traveling wavefronts exist at a small wave speed. $\alpha = 10.0$, $\lambda = 1.25$, $K = 2$, $c = 0.0193$.

3. Numerical simulations

In this section, we present some numerical simulations on traveling waves of the recursion (1.6) with Laplace kernel

$$k(x, y) = \frac{1}{2}\alpha \exp(-\alpha|x - y|),$$

and the Beverton Holt growth recruitment function

$$f(u) = \frac{\lambda u}{1 + \frac{(\lambda-1)u}{K}}.$$

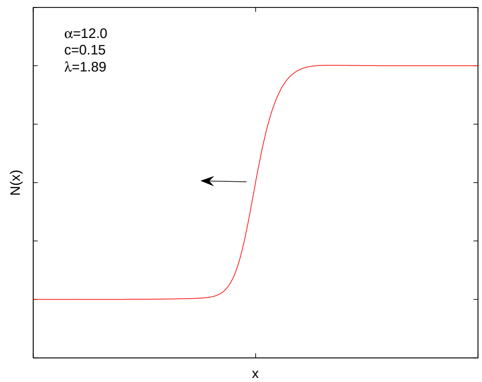

FIGURE 3. Traveling wavefronts with parameters $\alpha = 12.0$, $\lambda = 1.89$, $K = 2$, $c = 0.15$.

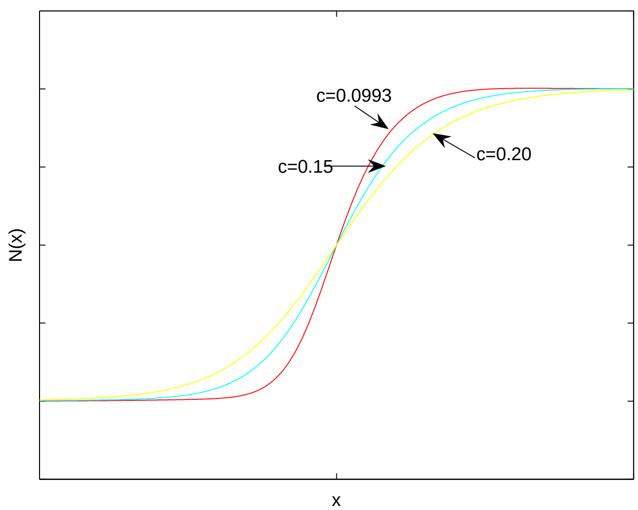

FIGURE 4. Different profile of wavefronts with parameters $\alpha = 10.0$, $\lambda = 1.25$, $K = 2$ and $c_1 = 0.15$, $c_2 = 0.0993$, $c_3 = 0.20$.

It is clear that the Beverton Holt growth recruitment function satisfies all our assumptions. In this case, the model under consideration is

$$u_{n+1}(x) = \frac{\alpha}{2}\int_\Omega \exp(-\alpha|x-y|)\frac{\lambda u_n(x)}{1+\frac{(\lambda-1)u_n(x)}{K}}dy. \tag{3.1}$$

Let $y_n(x) = u_n(x)/K$. Then we have

$$y_{n+1}(x) = \frac{\alpha}{2}\int_\Omega \exp(-\alpha|x-y|)\frac{\lambda y_n(x)}{1+(\lambda-1)y_n(x)}dy. \tag{3.2}$$

Firstly, we take the same parameters but with different wave speeds. When the wave speed larger than certain number, the traveling wavefronts exists and the numerical simulation can be observed in Figure 1. If the wave speed small than this number, there is no traveling wavefronts and the numerical simulation can be observed Figure 2.

When we take different parameters, there exists different wave speed, the numerical simulation can be observed Figure 3.

In what follows, we give a brief conclusion. In this paper, we are concerned with an integro-difference equation with bilateral exponential kernel. Under certain conditions on growth function, we establish the existence of traveling wavefronts by using upper and lower solution method and monotone iteration techniques. Generally speaking, Laplacian kernel or Gaussian kernel can be used as the dispersal kernel. If Gaussian kernel is used, one can easily obtain the minimal spreading speed c^*. But for Laplacian kernel, there is no similar results can be found in the literature. In present paper, we use the Laplacian kernel as the dispersal kernel and get the exact expression for minimal spreading speed c^*. By Remark 2.3, we know that c^* can be numerically computed provided all parameters are given explicitly.

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References

[1] P. W. Bates, X. Chen, A. J. J. Chmaj; Traveling waves of bistable dynamics on a lattice, *SIAM, J. Math. Anal.*, 35 (2003), no. 2, 520-546.
[2] R. J. H. Beverton, S. J. Holt; On the dynamics of exploited fish populations, *Fish. Invest. Minist. Argic. Fish. Food (London)* Ser II, 19 (1957), 1-533.
[3] X. Chen, J. S. Guo; Existence and asymptotic stability of traveling wave of discrete quasilinear monostable equations, *J. Differential Equations*, 184 (2002), 549-569.
[4] S. N. Chow, J. Mallet-Paret, W. Shen; Traveling waves in lattice dynamical systems, *J. Differential Equations*, 149(1998), no. 2, 248-291.
[5] O. Diekmann; Thresholds and traveling waves for the geographical spread of infection, *J. Math. Biol.*, 6(1978), 109-130.
[6] R. A. Fisher; The advance of advantageous genes, *Ann. of Eugenetics.*, 7 (1937), 335-369.
[7] S. B. Hsu, X. Q. Zhao; Spreading speeds and traveling waves for nomonotone integrodifference equations, *SIAM. J. Math. Anal.*, 40 (2008), 776-789.
[8] A. N. Kolmogorov, I. G. Petrovskii, N. S. Piskunov; A study of the diffusion equation with increase in the quantity of matter and its application to a biological problem, *Bull. Moscou Univ. Ser. Int. Sect. A, Math.*, 1(1937), 1-25.
[9] V. L. Kocic, G. Ladas; Global Behavior of Nonlinear Difference Equations of Higher Order with Application, *Kluwer Acadamic Publishers*, Dordrecht, 1993.
[10] M. Kot; Discrete-time traveing waves: Ecological examples, *J. Math. Biol.*, 30(1992), 413-436.
[11] X. Liang, X. Q. Zhao; Asymptotic speeds and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.* , 60(2007), 1-40.
[12] G. Lin, W. T. Li, S. Ruan; Asymptotic stability of monostable wavefronts in discrete-time integral recursions, *Sci. China. Ser. A.*, 52(2009), 679-688.
[13] G. Lin, W. T. Li; Spreading speeds and traveling wavefronts for second order integrodifference equations, *J. Math. Anal. Appl.*, 361 (2010), 520-532.
[14] G. Lin, W. T. Li, S. Ruan; Spreading speeds and traveling waves in competitive recursion systems, *J. Math. Biol.*, 62 (2011), 162-201.
[15] G. Lin, W. T. Li; Traveling wave solutions of a competitive recursion, *Discrete Contin. Dyn. Syst. Ser. B.*, 17 (2012), 173-189.

[16] J. D. Murray; *Mathematical Biology, I*, An introduction, 3rd ed., Interdisciplinary Applied Mathematicals, 17, Springer, New York, 2002.
[17] J. D. Murray *Mathematical Biology, II*, Spatial Model and Biomedical Applications, 3rd ed., Interdisciplinary Applied Mathematicals, 18, Springer, New York, 2003.
[18] M. G. Neubert, H. Caswell; *Demography and dispersal: Calculation and sensitivity analysis of invasion speed for structured populations*, *Ecology.* 81(2000), 1613-1628.
[19] M. G. Neubert, M. Kot, M. Lewis; Dispersal and pattern formation in a discrere-time predatory-prey model, *Theor Popul. Biol.*, 48(1995), 7-43.
[20] S. Pan, G. Lin; Propagation of second order integrodifference equations with local monotonicity, *Nonlinear Anal. Real World Appl.*, 12 (2011), 535-544.
[21] W. E. Ricker; Stock and recruitment, *J. Fish. Res. Board. Canad.*, 19 (1954), 559-623.
[22] K. W. Schaaf; Asymptotic behavior and traveling wave solutions for parabolic functional-differential equations, *Trans. Amer. Math. Soc.*, 302 (1987), no.2, 587-615.
[23] H. L. Smith, X. Q. Zhao; Global asymptotic stability of traveling wave in delayed reaction-diffusion equations, *SIAM. J. Math. Anal.*, 31 (2000), no.3, 514-534.
[24] H. R. Thieme, X. Q. Zhao; Asymptotic speeds of spread and traveling wave for integral equations and delayed reaction-diffusion models, *J. Differential Equations*, 195 (2003), no.2, 430-470.
[25] A. I. Volpert, V. A. Volpert; Traveling Wave Solutions of Parabolic System, Translations of Mathematical Monographs, 140, American Mathematical Society, Providence, R. I., 1994.
[26] H. F. Weinberger; Long-time behavior of a class of biological model, *SIAM. J. Math. Anal.*, 13 (1982), 353-396.
[27] H. F. Weinberger; On spreading speed and traveling wave for growth and migration models in a periodic habitat, *J. Math. Biol.*, 45 (2002), 511-548.
[28] J. Wu, X. Zou; Asymptotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations, *J. Differential Equations*, 135 (1997), no. 2, 315-357.
[29] J. Wu, X. Zou; Traveling wave front of reaction-diffusion systems with delay, *J. Dynam. Differential Equations*, 13(2001), no.3, 651-687.
[30] B. Zinner, G. Harris, W. Hudson; Traveling wavefronts for the discrete Fishers equation, *J. Differential Equations,* 105 (1993), no. 1, 46-62.

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