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INFINITELY MANY SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS WITH NONLINEAR NEUMANN BOUNDARY CONDITIONS

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Abstract. In this article, we study a Kirchhoff type problem with nonlinear Neumann boundary conditions on a bounded domain. By using variational methods, we prove the existence of infinitely many solutions.

1. Introduction

In this work, we study the multiplicity of solutions for the elliptic problem

$$\begin{gathered} -\Big[M\Big(\int_\Omega |\nabla u|^p dx\Big)\Big]^{p-1}\Delta_p u = f(x,u), \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda k(u) + \mu g(u), \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $M(t) = a + bt$, $p > N$, $a > 0, b \geq 0$, Ω is a nonempty bounded open subset of $\mathbb{R}^N$ with a boundary of class C^1, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, λ, μ are positive real parameters, the functions f, k, g satisfy hypotheses stated as follows:

(H1) $k, g \in C(\mathbb{R},\mathbb{R})$ and there exist two positive constants ρ, ρ^* such that
$$|K(u)| + |G(u)| \leq \rho^*(|u|^\rho + 1)$$
for all $u \in \mathbb{R}$, where $K(u) = \int_0^u k(s)ds$, $G(u) = \int_0^u g(s)ds$.

(H2) $f \in C(\overline{\Omega} \times \mathbb{R},\mathbb{R})$ and there exist two positive constants a_1, a_2 such that
$$a_1|u|^p \leq -F(x,u) \leq a_2|u|^p$$
for all $(x,u) \in \Omega \times \mathbb{R}$, where $F(x,u) = \int_0^u f(x,s)ds$.

(H3) $f \in C(\overline{\Omega} \times \mathbb{R},\mathbb{R})$ and there exist two positive constants ϱ, ϱ^* such that
$$|F(x,u)| \leq \varrho^*(|u|^\varrho + 1)$$
for all $(x,u) \in \Omega \times \mathbb{R}$.

(H4) there exist two positive constants b_1, b_2 such that
$$b_1|u|^p \leq -G(u) \leq b_2|u|^p$$
for all $u \in \mathbb{R}$.

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Problem (1.1) is the nonlocal problem, which is related to the model introduced by Kirchhoff [15],

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Interest of the mathematicians on the nonlocal problems has increased because they represent a variety of relevant physical and engineering situations[4, 12]. Many interesting results for Kirchhoff type problems were obtained and we refer to[1, 2, 3, 8, 9, 11, 13, 14, 17, 18, 19] and references therein for an overview on these subjects. Relatively speaking, Kirchhoff type problems with nonlinear boundary conditions have rarely been considered. In addition, when involving the existence of infinitely many solutions, most results assume that nonlinear term is odd in order to apply some variant of the classical Lusternik-Schnirelmann theory and only a few papers deal with nonlinearities having no symmetry properties[5, 6, 16].

The main purpose of this article is to establish the existence of infinitely many solutions for (1.1) without the assumption of symmetry property, by adopting the framework of Bonanno and Molica Bisci [6].

2. Preliminaries

Let X be a reflexive real Banach space and $\mathcal{I}_\lambda : X \to \mathbb{R}$ a functional satisfying the structure hypothesis:

(H5) $\mathcal{I}_\lambda(u) = \Psi(u) - \lambda\Phi(u)$ for all $u \in X$, where $\Psi, \Phi : X \to \mathbb{R}$ are two functions of class C^1 on X with Ψ coercive, i.e. $\lim_{\|u\|\to+\infty} \Psi(u) = +\infty$, and λ is a real parameter.

Provided that $\inf_X \Psi < r$, put

$$\phi_{\mathcal{I}_\lambda}(r) := \inf_{u\in\Psi^{-1}(]-\infty,r[)} \frac{\big(\sup_{u\in(\Psi^{-1}]-\infty,r[)} \Phi(u)\big) - \Phi(u)}{r - \Psi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \phi_{\mathcal{I}_\lambda}(r), \quad \delta := \liminf_{r\to(\inf_X \Psi)^+} \phi_{\mathcal{I}_\lambda}(r).$$

When $\gamma = 0$(or $\delta = 0$) we agree to read $\frac{1}{\gamma}$ (or $\frac{1}{\delta}$) as $+\infty$. Our main tool is a smooth version of critical point theorem which are recalled below, see [6].

Theorem 2.1. *Assume that* (H5) *holds. Then:*

(a) *For each $r > \inf_X \Psi$ and every $\lambda \in]0, 1/\phi_{\mathcal{I}_\lambda}(r)[$, the restriction of the functional $\mathcal{I}_\lambda$ to $\Psi^{-1}(]-\infty, r[)$ has a global minimum, which a s critical point (local minimum) of $\mathcal{I}_\lambda$ in X.*

(b) *If $\gamma < +\infty$, then, for each $\lambda \in]0, 1/\gamma[$, the following alternative holds:either*
   (1) *$\mathcal{I}_\lambda$ possess a global minimum, or*
   (2) *there is a sequence $\{u_n\}$ of critical points of $\mathcal{I}_\lambda$ such that $\lim_{n\to\infty} \Psi(u_n) = +\infty$.*

(c) *If $\delta < +\infty$, then, for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*
   (1) *there is a global minimum of Φ which is a local minimum of $\mathcal{I}_\lambda$, or*
   (2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points of $\mathcal{I}_\lambda$ such that $\lim_{n\to\infty} \Psi(u_n) = \inf_X \Psi$, which weakly converges to a global minimum of Φ.*

Let $W^{1,p}(\Omega)$ be the usual Sobolev space endowed with the norm

$$\|u\| := \Big(\int_\Omega (|\nabla u|^p + |u|^p)dx\Big)^{1/p}$$

or

$$\|u\|_* := \Big(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} |u|^p d\sigma\Big)^{1/p},$$

where $d\sigma$ is the measure on the boundary. Clearly, $\|\cdot\|$ is equivalent to $\|\cdot\|_*$. Let

$$\mathrm{k} := \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u(x)|}{\|u\|}, \quad \mathrm{k}_* := \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u(x)|}{\|u\|_*} \tag{2.1}$$

Since $p > N$, the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, and thus $0 < \mathrm{k}, \mathrm{k}_* < \infty$,

$$|u(x)| \le \mathrm{k}\|u\|, \quad |u(x)| \le \mathrm{k}_*\|u\|_* \quad \text{for } u \in W^{1,p}(\Omega). \tag{2.2}$$

A weak solution of problem (1.1), we mean that a function $u \in W^{1,p}(\Omega)$ satisfies

$$\begin{gathered}\Big[M\Big(\int_\Omega |\nabla u|^p dx\Big)\Big]^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx \\ - \int_\Omega f(x,u)v dx - \int_{\partial\Omega} (\lambda k(u) + \mu g(u))v d\sigma = 0\end{gathered}$$

for every $v \in W^{1,p}(\Omega)$.

Define the functionals on $W^{1,p}(\Omega)$ by

$$\Gamma(u) = \frac{1}{p}\int_0^{\int_\Omega |\nabla u|^p dx} M^{p-1}(s)ds = \begin{cases} \frac{1}{bp^2}\big[\big(a + b\int_\Omega |\nabla u|^p dx\big)^p - a^p\big], & b > 0,\\ \frac{a^{p-1}}{p}\int_\Omega |\nabla u|^p dx, & b = 0,\end{cases}$$

$$\psi(u) = \Gamma(u) - \int_\Omega F(x,u)dx, \quad \varphi(u) = \int_{\partial\Omega} \big(K(u) + \frac{\mu}{\lambda}G(u)\big)d\sigma,$$

$$\psi_*(u) = \Gamma(u) - \mu\int_{\partial\Omega} G(u)d\sigma, \quad \varphi_*(u) = \int_{\partial\Omega} K(u)d\sigma + \frac{1}{\lambda}\int_\Omega F(x,u)dx,$$

$$J_\lambda(u) = \psi(u) - \lambda\varphi(u), \quad I_\lambda(u) = \psi_*(u) - \lambda\varphi_*(u).$$

Conditions (H1) and (H2) (or (H1) and (H3)) and $p > N$ guarantee that ψ, φ (or ψ_*, φ_*) are well defined and of class C^1. Moreover,

$$\begin{aligned}\langle J_\lambda'(u), v\rangle &= \langle I_\lambda'(u), v\rangle \\ &= \Big[M\Big(\int_\Omega |\nabla u|^p dx\Big)\Big]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx \\ &\quad - \int_\Omega f(x,u)v dx - \int_{\partial\Omega}(\lambda k(u) + \mu g(u))v d\sigma.\end{aligned}$$

Hence, the critical points of J_λ or I_λ are the weak solutions of (1.1).

3. Main results

Put

$$a_3 = \big\{\frac{a^{p-1}}{p}, a_1\big\}, \quad b_3 = \big\{\frac{a^{p-1}}{p}, b_1\big\}, \quad |\Omega| = \int_\Omega dx, \quad |\partial\Omega| = \int_{\partial\Omega} d\sigma.$$

Moreover, let

$$A_\infty := \liminf_{\xi\to+\infty} \frac{\max_{|t|\le\xi} K(t)}{\xi^p}, \quad A_0 := \liminf_{\xi\to 0^+} \frac{\max_{|t|\le\xi} K(t)}{\xi^p},$$

$$B_\infty := \limsup_{\xi\to+\infty} \frac{K(\xi)}{\xi^p}, \quad B_0 := \limsup_{\xi\to 0^+} \frac{K(\xi)}{\xi^p},$$
$$G_\infty := \limsup_{\xi\to+\infty} \frac{\max_{|t|\le\xi} G(t)}{\xi^p}, \quad G_0 := \limsup_{\xi\to 0^+} \frac{\max_{|t|\le\xi} G(t)}{\xi^p}.$$
$$F_\infty := \limsup_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}, \quad F_0 := \limsup_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}$$

We present our main results as follows.

Theorem 3.1. *Assume that* (H1), (H2) *hold and there exist two real sequences* $\{\alpha_n\},\{\beta_n\}$ *with* $\lim_{n\to\infty}\beta_n = +\infty$ *and a positive constant* $\rho > 0$ *such that*

$$|\alpha_n| < \frac{1}{\mathrm{k}}\Big(\frac{a_3}{a_2|\Omega|}\Big)^{1/p}\beta_n, \quad G(t)\ge 0, \quad \forall t\ge\rho,$$
$$\mathcal{A}_\infty := \lim_{n\to\infty} \frac{\max_{|t|\le\beta_n} K(t) - K(\alpha_n)}{\beta_n^p - \mathrm{k}^p a_2 a_3^{-1}|\Omega|\alpha_n^p} < \frac{a_3 B_\infty}{\mathrm{k}^p|\Omega|a_2},$$
$$\mathcal{G}_\infty := \lim_{n\to\infty} \frac{\max_{|t|\le\beta_n} G(t) - G(\alpha_n)}{\beta_n^p - \mathrm{k}^p a_2 a_3^{-1}|\Omega|\alpha_n^p} < +\infty.$$

Then for each $\lambda\in\Lambda :=]\lambda_1,\lambda_2[$*, where*

$$\lambda_1 = \frac{|\Omega|a_2}{|\partial\Omega|B_\infty}, \quad \lambda_2 = \frac{a_3}{\mathrm{k}^p|\partial\Omega|\mathcal{A}_\infty},$$

there exists $\mu_\lambda > 0$*, where*

$$\mu_\lambda = \frac{1}{\mathcal{G}_\infty}\Big(\frac{a_3}{\mathrm{k}^p|\partial\Omega|} - \lambda\mathcal{A}_\infty\Big),$$

such that for all $\mu\in[0,\mu_\lambda[$*,* (1.1) *has an unbounded sequence of weak solutions.*

Proof. First, we observe that owing to the condition $\mathcal{A}_\infty < a_3B_\infty/(\mathrm{k}^p|\Omega|a_2)$, the interval Λ is non-empty. Moreover, for each fixed $\bar\lambda\in\Lambda$ and taking into account that $\mathcal{A}_\infty\bar\lambda < a_3/\mathrm{k}^p|\partial\Omega|$, one has $0<\mu_{\bar\lambda}<\infty$. Our aim is to apply Theorem 2.1. For this end, we show $\gamma<+\infty$, where γ is defined in Theorem 2.1.

By Assumption (H2), we have

$$a_3\|u\|^p \le \psi(u) \le \max\Big\{\frac{(2a)^{p-1}}{p}, a_2\Big\}\|u\|^p + \frac{(2b)^{p-1}}{p}\|u\|^{p^2}. \tag{3.1}$$

Put $r_n = \beta_n^p a_3/\mathrm{k}^p$ for all $n\in\mathbb{N}$, by (2.2) and (3.1), one has

$$\psi^{-1}(]-\infty,r_n]) \subseteq \{u\in W^{1,p}(\Omega) : \|u\|_\infty \le \beta_n\},$$
$$\psi(\alpha_n) = -\int_\Omega F(x,\alpha_n)dx \le a_2|\Omega|\alpha_n^p < r_n.$$

Hence,

$$\begin{aligned}
\phi_{J_\lambda}(r_n) &= \inf_{u\in\psi^{-1}(]-\infty,r_n[)} \frac{\big(\sup_{u\in(\psi^{-1}]-\infty,r_n[)}\varphi(u)\big) - \varphi(u)}{r_n - \psi(u)}\\
&\le \inf_{\psi(u)<r_n} \frac{|\partial\Omega|\max_{|t|\le\beta_n}[K(t)+\frac{\mu}{\lambda}G(t)] - \varphi(u)}{r_n-\psi(u)}\\
&\le \frac{|\partial\Omega|\max_{|t|\le\beta_n}[K(t)+\frac{\mu}{\lambda}G(t)] - \varphi(\alpha_n)}{r_n-\psi(\alpha_n)}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{|\partial\Omega| \max_{|t|\le\beta_n}[K(t)-K(\alpha_n)]}{r_n - a_2|\Omega|\alpha_n^p} + \frac{\mu}{\bar\lambda}\frac{|\partial\Omega| \max_{|t|\le\beta_n}[G(t)-G(\alpha_n)]}{r_n - a_2|\Omega|\alpha_n^p}\\
&\le \frac{\mathrm{k}^p|\partial\Omega|}{a_3}\Big(\mathcal{A}_\infty + \frac{\mu}{\bar\lambda}\mathcal{G}_\infty\Big) < +\infty.
\end{aligned}
$$

Since $\mu \in [0, \mu_{\bar\lambda}[$,

$$
\gamma \le \liminf_{n\to+\infty} \phi_{J_\lambda}(r_n) < \frac{\mathrm{k}^p|\partial\Omega|}{a_3}\Big(\mathcal{A}_\infty + \frac{\mu_{\bar\lambda}}{\bar\lambda}\mathcal{G}_\infty\Big) = \frac{1}{\bar\lambda} < +\infty;
$$

that is, $0 < \lambda_1 < \bar\lambda < 1/\gamma$. The condition (b) of Theorem 2.1 can be applied and either $J_{\bar\lambda}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1.1) such that $\|u_n\| \to \infty$ as $n \to \infty$.

Now, we verify that $J_{\bar\lambda}$ is unbounded from below. First, assume that $B_\infty = +\infty$. Accordingly, fixed C with $C > a_2|\Omega|/\bar\lambda|\partial\Omega|$ and $\{c_n\}$ be a sequence of positive numbers with $c_n \to +\infty$ as $n \to \infty$ such that

$$
K(c_n) > Cc_n^p, \quad n \text{ sufficiently large.}
$$

Taking the sequence $\{v_n\} \subseteq W^{1,p}(\Omega)$ such that $v_n(x) = c_n, x \in \bar\Omega$, for the sufficiently large n, one has

$$
\begin{aligned}
J_{\bar\lambda}(v_n) &= -\int_\Omega F(x, v_n)dx - \bar\lambda\int_{\partial\Omega} K(v_n)d\sigma - \mu\int_{\partial\Omega} G(v_n)d\sigma\\
&\le a_2|\Omega|c_n^p - \bar\lambda\int_{\partial\Omega} K(v_n)d\sigma \le (a_2|\Omega| - C\bar\lambda|\partial\Omega|)c_n^p;
\end{aligned}
$$

that is, $J_{\bar\lambda} \to -\infty$ as $n \to \infty$.

Next, assume that $B_\infty < +\infty$. Since $\bar\lambda > \lambda_1 = a_2|\Omega|/|\partial\Omega|B_\infty$, we fix $0 < \varepsilon < B_\infty - \frac{a_2|\Omega|}{\bar\lambda|\partial\Omega|}$. Let $\{c_n\}$ be a sequence of positive numbers with $c_n \to +\infty$ as $n \to \infty$ such that

$$
(B_\infty - \varepsilon)c_n^p < K(c_n) < (B_\infty + \varepsilon)c_n^p, \quad n \text{ sufficiently large}
$$

Arguing as before and by choosing $v_n \equiv c_n$, for the sufficiently large n, one has

$$
\begin{aligned}
J_{\bar\lambda}(v_n) &= -\int_\Omega F(x, v_n)dx - \bar\lambda\int_{\partial\Omega} K(v_n)d\sigma - \mu\int_{\partial\Omega} G(v_n)d\sigma\\
&\le a_2|\Omega|c_n^p - (B_\infty - \varepsilon)\bar\lambda|\partial\Omega|c_n^p \to -\infty \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence, $J_{\bar\lambda}$ is unbounded from blew and the proof is complete. □

Corollary 3.2. *Assume that* (H1), (H2) *hold. Further suppose that* $G_\infty < +\infty$, $A_\infty < a_3B_\infty/(\mathrm{k}^p|\Omega|a_2)$ *and there exists* $\rho > 0$ *such that* $G(t) \ge 0$ *for all* $t \ge \rho$. *Then for each* $\lambda \in]\lambda_3, \lambda_4[$, *where*

$$
\lambda_3 = \frac{|\Omega|a_2}{|\partial\Omega|B_\infty}, \quad \lambda_4 = \frac{a_3}{\mathrm{k}^pA_\infty|\partial\Omega|},
$$

there exists $\tilde\mu_\lambda > 0$, *where*

$$
\tilde\mu_\lambda = \frac{1}{G_\infty}\Big(\frac{a_3}{\mathrm{k}^p|\partial\Omega|} - \lambda A_\infty\Big),
$$

such that for all $\mu \in [0, \tilde\mu_\lambda[$, (1.1) *has an unbounded sequence of weak solutions.*

Proof. Let $\{\beta_n\}$ be a sequence of positive numbers which approaches infinity such that

$$A_\infty = \lim_{\xi\to+\infty} \frac{\max_{|t|\le\beta_n} K(t)}{\beta_n^p}.$$

Taking $\alpha_n = 0$ for every $n \in \mathbb{N}$ and noting that

$$\mathcal{G}_\infty = \lim_{n\to+\infty} \frac{\max_{|t|\le\beta_n} G(t)}{\beta_n^p} \le G_\infty,$$

from Theorem 3.1, we obtain the conclusion. $\square$

Applying part (c) of Theorem 2.1, we get the following theorem.

Theorem 3.3. *Assume that* (H1), (H2) *hold and there exist two real sequences* $\{\alpha_n\}, \{\beta_n\}$ *with* $\lim_{n\to\infty}\beta_n = 0$ *and positive constant* $\rho > 0$ *such that*

$$|\alpha_n| < \frac{1}{\mathrm{k}}\Big(\frac{a_3}{a_2|\Omega|}\Big)^{1/p}\beta_n, \quad G(t) \ge 0, \quad \forall 0 \le t \le \rho,$$

$$\mathcal{A}_0 := \lim_{n\to\infty} \frac{\max_{|t|\le\beta_n} K(t) - K(\alpha_n)}{\beta_n^p - \mathrm{k}^p a_2 a_3^{-1}|\Omega|\alpha_n^p} < \frac{a_3 B_0}{\mathrm{k}^p|\Omega|a_2},$$

$$\mathcal{G}_0 := \lim_{n\to\infty} \frac{\max_{|t|\le\beta_n} G(t) - G(\alpha_n)}{\beta_n^p - \mathrm{k}^p a_2 a_3^{-1}|\Omega|\alpha_n^p} < +\infty.$$

Then for each $\lambda \in]\lambda_4, \lambda_5[$, *where*

$$\lambda_4 = \frac{|\Omega|a_2}{|\partial\Omega|B_0}, \quad \lambda_5 = \frac{a_3}{\mathrm{k}^p|\partial\Omega|\mathcal{A}_0},$$

there exists $\bar\mu_\lambda > 0$, *where*

$$\bar\mu_\lambda = \frac{1}{\mathcal{G}_0}\Big(\frac{a_3}{\mathrm{k}^p|\partial\Omega|} - \lambda\mathcal{A}_0\Big),$$

such that for all $\mu \in [0, \bar\mu_\lambda[$, (1.1) *has a sequence of weak solutions, which converges strongly to zero.*

Corollary 3.4. *Assume that* (H1), (H2) *hold. Further suppose that* $G_0 < +\infty$, $A_0 < a_3 B_0/(\mathrm{k}^p|\Omega|a_2)$ *and there exists* $\rho > 0$ *such that* $G(t) \ge 0$ *for all* $0 < t \le \rho$. *Then for each* $\lambda \in]\lambda_7, \lambda_8[$, *where*

$$\lambda_7 = \frac{|\Omega|a_2}{|\partial\Omega|B_0}, \quad \lambda_8 = \frac{a_3}{\mathrm{k}^p A_0|\partial\Omega|},$$

there exists $\hat\mu_\lambda > 0$, *where*

$$\hat\mu_\lambda = \frac{1}{G_0}\Big(\frac{a_3}{\mathrm{k}^p|\partial\Omega|} - \lambda A_0\Big),$$

such that for all $\mu \in [0, \hat\mu_\lambda[$, (1.1) *has a sequence of weak solutions, which converges strongly to zero.*

Now, we consider the case when (H1), (H3), (H4) hold.

Theorem 3.5. *Assume that* (H1), (H3), (H4) *hold and* $F(x,u) \ge 0$ *for* $x \in \overline{\Omega}, u \ge r > 0$. *If there exists the positive constant* $\bar\lambda$ *such that*

$$\bar\lambda B_\infty > b_2, \quad F_\infty < \frac{b_3}{\mathrm{k}_*^p} - \bar\lambda|\partial\Omega|A_\infty,$$

then (1.1) *has an unbounded sequence of weak solutions for* $\lambda = \bar\lambda$, $\mu = 1$.

Proof. Let $\lambda = \bar\lambda, \mu = 1$ and $\{\beta_n\}$ be a sequence of positive numbers with $\beta_n \to +\infty$ as $n \to \infty$ such that
$$A_\infty = \lim_{n\to+\infty} \frac{\max_{|t|\le\beta_n} K(t)}{\beta_n^p}.$$
By Assumption (H4), we have
$$b_3\|u\|_*^p \le \psi_*(u) \le \max\Big\{\frac{(2a)^{p-1}}{p}, b_2\Big\}\|u\|_*^p + \frac{(2b)^{p-1}}{p}\|u\|_*^{p^2}. \tag{3.2}$$
Put $r_n = \beta_n^p b_3/\mathrm{k}_*^p$ for all $n \in \mathbb{N}$, by (2.2) and (3.2), one has
$$\psi_*^{-1}(]-\infty, r_n]) \subseteq \{u \in W^{1,p}(\Omega) : \|u\|_\infty \le \beta_n\}.$$
Hence,
$$\begin{aligned}
\phi_{I_\lambda}(r_n) &= \inf_{u\in\psi_*^{-1}(]-\infty,r_n[)} \frac{\big(\sup_{u\in(\psi_*^{-1}]-\infty,r_n[)}\varphi_*(u)\big) - \varphi_*(u)}{r_n - \psi_*(u)}\\
&\le \frac{\sup_{u\in(\psi_*^{-1}]-\infty,r_n[)}\varphi_*(u)}{r_n}\\
&\le \frac{\max_{\{u\in X:\|u\|_\infty\le\beta_n\}}\varphi_*(u)}{r_n}\\
&\le \frac{\mathrm{k}_*^p}{b_3}\Big(\frac{|\partial\Omega|\max_{|t|\le\beta_n}K(t)}{\beta_n^p} + \frac{1}{\bar\lambda}\frac{\int_\Omega \max_{|t|\le\beta_n}F(x,t)dx}{\beta_n^p}\Big)\\
&\le \frac{\mathrm{k}_*^p}{b_3}\Big(|\partial\Omega|A_\infty + \frac{1}{\bar\lambda}F_\infty\Big) < \frac{1}{\bar\lambda}.
\end{aligned}$$
The rest proof is similar to that of Theorem 3.1 and we omit it. □

Theorem 3.6. *Assume that* (H1), (H3), (H4) *hold and* $F(x,u) \ge 0$ *for* $x \in \bar\Omega, 0 \le u \le r (r > 0)$. *If there exists the positive constant* $\bar\lambda$ *such that*
$$\bar\lambda B_0 > b_2, \quad F_0 < \frac{b_3}{\mathrm{k}_*^p} - \bar\lambda|\partial\Omega|A_0,$$
then (1.1) *has a sequence of weak solutions for* $\lambda = \bar\lambda, \mu = 1$, *which converges strongly to zero.*

4. Examples

In this section, we present the two examples which provide the problems that admit infinitely many solutions.

Example 4.1. Consider the differential equation
$$\begin{gathered}
-\Big[1 + b\int_0^1 |u'|^2 dx\Big]u''(x) + u(u\cos u + 2\sin u + 4) = 0, \quad x \in (0,1),\\
-u'(0) = \lambda k(u(0)) + \mu g(u(0)), \quad u'(1) = \lambda k(u(1)) + \mu g(u(1)),
\end{gathered} \tag{4.1}$$
where $b \ge 0$,
$$k(u) = \begin{cases} u^2\sin(\ln u), & u > 0,\\ 0, & u \le 0,\end{cases} \qquad g(u) = u - \sin u.$$
Then
$$f(u) = -u(u\cos u + 2\sin u + 4), \quad -F(u) = u^2(2 + \sin u),$$

$$K(u)=\begin{cases}\frac{u^3}{8}[3\sin(\ln u)-\cos(\ln u)], & u>0,\\ 0, & u\le 0,\end{cases}$$

$$G(u)=\frac{1}{2}u^2+\cos u-1,\quad |\Omega|=|\partial\Omega|=1,\quad a_1=1,\quad a_2=3,\quad a_3=\frac{1}{2}.$$

According to [7, Remark 1], one has the estimate $\mathrm{k}\le\sqrt{2}$. Taking $\alpha_n=e^{(2n+1)\pi}$, $\beta_n=e^{2(n+1)\pi}$, we easily obtain that

$$\mathcal{A}_\infty=0,\quad B_\infty=+\infty,\quad \mathcal{G}_\infty=\frac{e^{2\pi}-1}{2(e^{2\pi}-6\mathrm{k}^2)}.$$

By Theorem 3.1, (4.1) has an unbounded sequence of weak solutions for

$$\lambda>0,\quad 0\le\mu<\frac{e^{2\pi}-6\mathrm{k}^2}{\mathrm{k}^2(e^{2\pi}-1)}.$$

Example 4.2. Consider the differential equation

$$\begin{gathered}-\Big[M\Big(\int_\Omega|\nabla u|^p dx\Big)\Big]^{p-1}\Delta_p u=c(x)|u|^{\rho-1}u,\quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}=\lambda k(u)-|u|^{p-2}u,\quad \text{on } \partial\Omega,\end{gathered}\tag{4.2}$$

where $M(t)=a+bt$, $p>N$, $a>0, b\ge 0$, $1<\rho<p$, $c\in C(\overline{\Omega})$ and $c(x)\ge 0$, Ω is a nonempty bounded open subset of $\mathbb{R}^N$ with a boundary of class C^1,

$$k_1=2,\quad k_{n+1}=k_n^{12},\quad l_n=k_n^{10}, n\in\mathbb{N},$$

$$k(t)=\begin{cases}(l_n^{p-0.5}-k_n^{p+0.5})(1-|l_n-t|), & l_n-1\le t\le l_n+1, n\ge 1,\\ (k_{n+1}^{p+0.5}-l_n^{p-0.5})(1-|k_{n+1}-t|), & k_{n+1}-1\le t\le k_{n+1}+1, n\ge 1,\\ 0, & \text{otherwise.}\end{cases}$$

Noting that

$$K(k_n+1)=k_n^{p+0.5}-k_1^{p+0.5},\quad K(l_n+1)=l_n^{p-0.5}-k_1^{p+0.5},\quad n\ge 2,$$

we have

$$\lim_{n\to\infty}\frac{K(k_n+1)}{(k_n+1)^p}=+\infty,\quad \lim_{n\to\infty}\frac{K(l_n+1)}{(l_n+1)^p}=0.$$

Hence, $A_\infty=0$, $B_\infty=+\infty$. It is easy to check that $F_\infty=0$. By Theorem 3.5, (4.2) has an unbounded sequence of weak solutions for all $\lambda>0$.

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