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EXISTENCE OF NONNEGATIVE SOLUTIONS FOR SINGULAR ELLIPTIC PROBLEMS

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Abstract. We prove the existence of nonnegative nontrivial weak solutions to the problem

$$\begin{gathered} -\Delta u = au^{-\alpha}\chi_{\{u>0\}} - bu^p \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^n$. A sufficient condition for the existence of a continuous and strictly positive weak solution is also given, and the uniqueness of such a solution is proved. We also prove a maximality property for solutions that are positive a.e. in Ω.

1. Introduction and statement of the problem

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, let a and b be nonnegative functions on Ω, and let α and p be positive real numbers. Consider the following singular elliptic problem

$$\begin{gathered} -\Delta u = au^{-\alpha} - bu^p \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega \end{gathered} \tag{1.1}$$

Problems like (1.1) appear in chemical catalysts process, non-Newtonian fluids, and in models for the temperature of electrical devices (see e.g., [10, 7, 16, 19]).

Several works can be found concerning the existence of positive solutions to (1.1) for the case $b = 0$, i.e., for the problem $-\Delta u = au^{-\alpha}$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω; let us mention a few: Classical solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $u(x) > 0$ for all $x \in \Omega$ were obtained by Crandall, Rabinowitz and Tartar [11] under the following hypothesis: $a \in C^1(\overline{\Omega})$ and $\min_{\overline{\Omega}} a > 0$. Lazer and McKenna [24] proved the existence of positive weak solutions $u \in H_0^1(\Omega)$ to (1.1) assuming that $a \in C^\gamma(\overline{\Omega})$, $\gamma \in (0,1)$, and, again, a strictly positive on $\overline{\Omega}$. The case $0 \le a \in L^\infty(\Omega)$, $a \not\equiv 0$ (that is: $|\{x \in \Omega : a(x) > 0\}| > 0$) was studied by Del Pino [12]. Situations where a is singular on the boundary $\partial\Omega$ were considered by Bougherara, Giacomoni and Hernández [5].

The existence of classical solutions to problem (1.1) was proved by Coclite and Palmieri [9] for a and b in $C^1(\overline{\Omega})$, $0 < p < 1$, and a strictly positive on $\overline{\Omega}$ (see [9,

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Theorem 1]). Related singular elliptic problems were treated by Shi and Yao [29], and by Aranda and Godoy [3], [2]. Elliptic problems with singular terms and free boundaries were considered by Dávila and Montenegro [13], [14].

Ghergu and Rădulescu [22] studied multi-parameter singular bifurcation problems of the form $-\Delta u = g(u) + \lambda|\nabla u|^p + \mu f(.,u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $\lambda, \mu \geq 0$, $0 < p \leq 2$, $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function such that $f(.,s)$ is nondecreasing with respect to s, and $g : (0,\infty) \to (0,\infty)$ is a nonincreasing Hölder continuous function such that $\lim_{s\to 0^+} g(s) = \infty$. When $g(s)$ behaves like $s^{-\alpha}$ near the origin, with $0 < \alpha < 1$, the asymptotic behavior of the solution around the bifurcation point is established.

Dupaigne, Ghergu and Rădulescu [18] obtained various existence and nonexistence results for Lane–Emden–Fowler equations with convection and singular potential of the form $-\Delta u \pm p(d_\Omega(x))g(u) = \lambda f(x,u) + \mu|\nabla u|^\beta$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, $\lambda > 0$, $\mu \in \mathbb{R}$, $0 < \beta \leq 2$, $p(d_\Omega(x))$ is a positive weight possibly singular at $\partial\Omega$, $g \in C^1(0,\infty)$ is a positive decreasing function such that $\lim_{s\to 0^+} g(s) = \infty$, $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is positive on $\Omega \times (0,\infty)$ and satisfies that $s \to f(x,s)$ is nondecreasing and also that $f(x,s)$ is either linear or sublinear with respect to s.

Rădulescu [28] states existence, nonexistence and uniqueness results for blow-up boundary solutions of logistic equations and for Lane-Emden-Fowler equations with singular nonlinearities and subquadratic convection term.

Existence and nonexistence results for solutions to the inequality $Lu \geq K(x)u^p$ in Ω, $u > 0$ in Ω were obtained by Ghergu, Liskevich and Sobol [20] for the case where Ω is a punctured ball $B_R(0)\backslash\{0\}$, $p \in \mathbb{R}$, $K \in L^\infty_{\text{loc}}(B_R(0)\backslash\{0\})$, $ess\inf K > 0$, and $Lu := \sum_{1\leq i,j\leq n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{1\leq j\leq n} b_j(x)\frac{\partial u}{\partial x_j}$, where the matrix $\mathbf{a} = \{a_{ij}(x)\}_{1\leq i,j\leq n}$ is symmetric, uniformly elliptic on Ω, with each $a_{ij} \in L^\infty(B_R(0))$, and each b_j is a measurable function and satisfies $ess\sup_{x\in B_R(0)\backslash\{0\}} |x|b_j(x) < \infty$.

Existence and uniqueness results were obtained by Bougherara and Giacomoni [4] for mild solutions to singular initial value parabolic problems involving the p-Laplacian operator of the form $u_t - \Delta_p u = u^{-\alpha} + f(x,u)$ in $Q_T := (0,T) \times \Omega$, $u = 0$ on $(0,T) \times \partial\Omega$, $u > 0$ in Q_T, $u(0,x) = u_0(x)$ in Ω where Ω is a regular bounded domain in $\mathbb{R}^n$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded below Carathéodory function and nonincreasing with respect to the second variable, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, $\alpha > 0$, $T > 0$, and u_0 in a suitable functional space.

Singularly perturbed elliptic problems on an annulus whose solutions concentrate in a circle were studied by Manna and Srikanth [27].

Let us mention also that Loc and Schmitt [26], [25], extended the method of sub and supersolutions to deal with singular elliptic problems. A comprehensive treatment of the subject can be found in Ghergu and Rădulescu's book [21] (see also [28]), and in the survey article [15], by Díaz and Hernández.

Let us state the problem that we will consider from now on: Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, $\alpha \in (0,1)$, and $p \in (0, 2^* - 1)$, where 2^* is defined by $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$ if $n > 2$ and $2^* = \infty$ if $n \leq 2$. Let a and b be nonnegative functions such that a belongs to $L^\infty(\Omega)$, $a \not\equiv 0$, and b is in $L^r(\Omega)$, with $r = \frac{2}{1-p}$ if $p < 1$, and $r = \infty$ otherwise.

We are concerned with weak solutions to the problem

$$
\begin{gathered}
-\Delta u = au^{-\alpha}\chi_{\{u>0\}} - bu^p \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
u \geq 0 \quad \text{in } \Omega
\end{gathered} \tag{1.2}
$$

where $au^{-\alpha}\chi_{\{u>0\}}$ stands for the function defined by $au^{-\alpha}\chi_{\{u>0\}}(x) = a(x)u(x)^{-\alpha}$ if $u(x) \neq 0$, and $au^{-\alpha}\chi_{\{u>0\}}(x) = 0$ if $u(x) = 0$.

By a weak solution to (1.2) we mean a nonnegative function $u \in H_0^1(\Omega)$ such that, for all φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, $(au^{-\alpha}\chi_{\{u>0\}} - bu^p)\varphi \in L^1(\Omega)$, and the following holds

$$
\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega (au^{-\alpha}\chi_{\{u>0\}} - bu^p)\varphi. \tag{1.3}
$$

The main aim of this work is to prove the existence of at least one nonnegative weak solution $u \not\equiv 0$ to the stated problem (see Theorem 3.1). Additionally, we give a condition on a, b that guarantees the existence of a strictly positive weak solution to (1.2) (see Theorem 3.5). In Theorem 3.8 we prove that there is at most one solution that is positive a.e. in Ω, and give a maximality property for such a solution. Examples of non-existence of strictly positive solutions, and of non-uniqueness of the nonnegative solutions, are also provided.

To prove Theorem 3.1, we show that the energy functional J associated with (1.2) attains its minimum at some nonnegative nontrivial $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Note that J may fail to be Gateaux differentiable at u; despite this fact, we manage to prove that the said minimizer is indeed a weak solution of problem (1.2). Theorem 3.5 is proved using the sub and supersolutions method for singular elliptic problems developed in [26].

2. Preliminary lemmas

Let $J : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional associated with (1.2),

$$
J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{1-\alpha}\int_\Omega a|u|^{1-\alpha} + \frac{1}{1+p}\int_\Omega b|u|^{1+p}. \tag{2.1}
$$

Let us start with the following lemma.

Lemma 2.1. *The following statements hold:*

(i) *J is coercive on $H_0^1(\Omega)$.*

(ii) *$\inf_{u\in H_0^1(\Omega)} J(u) > -\infty$.*

((iii) *$\inf_{u\in H_0^1(\Omega)} J(u)$ is achieved at some $u \in H_0^1(\Omega)$.*

Proof. Let $u \in H_0^1(\Omega)$. Since $0 < 1-\alpha < 1$, the Hölder's and Poincare's inequalities give

$$
\frac{1}{1-\alpha}\int_\Omega a|u|^{1-\alpha} \leq c\|\nabla u\|_2^{1-\alpha}
$$

for some positive constant c independent of u, and so $J(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - c\|\nabla u\|_2^{1-\alpha}$, which clearly implies (i) and (ii).

To prove (iii), let $\beta = \inf_{u\in H_0^1(\Omega)} J(u)$, and consider a sequence $\{u_j\}_{j\in\mathbb{N}} \subset H_0^1(\Omega)$ such that $\lim_{j\to\infty} J(u_j) = \beta$. Then, by i), $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Let q be in $(p+1, 2^*)$. Since the inclusion $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is a compact map, we can assume (taking a subsequence if necessary) that $\{u_j\}_{j\in\mathbb{N}}$ converges strongly to some $u \in L^q(\Omega)$. Since $\{u_j\}_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, there exists $v \in H_0^1(\Omega)$, and a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$, such that the subsequence converges strongly to v in $L^2(\Omega)$, and $\{\nabla u_{j_k}\}_{k\in\mathbb{N}}$ converges weakly to ∇v in $L^2(\Omega,\mathbb{R}^n)$. Thus $v=u$, $\{u_{j_k}\}_{k\in\mathbb{N}}$ converges to u in $L^q(\Omega)$, and

$$\|\nabla u\|_2 \le \liminf_{k\to\infty} \|\nabla u_{j_k}\|_2. \tag{2.2}$$

On the other hand, the Nemytskii operators $f(u):=|u|^{1-\alpha}$ and $g(u):=|u|^{1+p}$ are continuous from $L^2(\Omega)$ into $L^{\frac{2}{1-\alpha}}(\Omega)$, and from $L^q(\Omega)$ into $L^{\frac{q}{1+p}}(\Omega)$, respectively [1, Theorem 1.2.1] and so, since $a\in L^\infty(\Omega)$ and $b\in L^r(\Omega)$,

$$\begin{aligned}&\lim_{j\to\infty}\int_\Omega \Big(\frac{1}{1-\alpha}a|u_{j_k}|^{1-\alpha}-\frac{1}{1+p}b|u_{j_k}|^{1+p}\Big)\\ &=\int_\Omega (\frac{1}{1-\alpha}a|u|^{1-\alpha}-\frac{1}{1+p}b|u|^{1+p})\end{aligned} \tag{2.3}$$

which, combined with (2.2), gives $J(u)\le \liminf_{k\to\infty} J(u_{j_k})=\beta$, therefore (iii) holds (since $\beta\le J(u)$). □

Corollary 2.2. $\inf_{u\in H_0^1(\Omega)} J(u)$ *is achieved at some nonnegative* $u\in H_0^1(\Omega)$.

Proof. Lemma 2.1 states that J attains its minimum at some $u\in H_0^1(\Omega)$. Since $J(u)=J(|u|)$, a nonnegative minimizer exists. □

For the rest of this article, we fix a nonnegative minimizer for J on $H_0^1(\Omega)$, and denote it by $\mathbf{u}$.

Lemma 2.3. *The equality*

$$\int_\Omega \langle\nabla\mathbf{u},\nabla(\mathbf{u}\varphi)\rangle=\int_\Omega (a\mathbf{u}^{1-\alpha}-b\mathbf{u}^{1+p})\varphi \tag{2.4}$$

holds for any $\varphi\in H^1(\Omega)\cap L^\infty(\Omega)$ *such that* $\varphi\mathbf{u}\in H_0^1(\Omega)$.

Proof. Let $\varphi\in H^1(\Omega)\cap L^\infty(\Omega)$ be such that $\varphi\mathbf{u}\in H_0^1(\Omega)$; satisfying, in addition, $\|\varphi\|_\infty\le\frac12$. Let $\tau\in R$ such that $|\tau|<1$. Then $\mathbf{u}+\tau\mathbf{u}\varphi\ge 0$, and $J(\mathbf{u})\le J(\mathbf{u}+\tau\mathbf{u}\varphi)$. A computation shows that this inequality can be written as

$$\begin{aligned}&\tau\int_\Omega\langle\nabla\mathbf{u},\nabla(\mathbf{u}\varphi)\rangle\\ &\ge\frac{1}{1-\alpha}\int_\Omega a\mathbf{u}^{1-\alpha}\big((1+\tau\varphi)^{1-\alpha}-1\big)-\frac{1}{1+p}\int_\Omega b\mathbf{u}^{1+p}\big((1+\tau\varphi)^{1+p}-1\big)\\ &\quad-\frac{\tau^2}{2}\int_\Omega \mathbf{u}^2|\nabla\varphi|^2-\frac{\tau^2}{2}\int_\Omega\varphi^2|\nabla\mathbf{u}|^2-\tau^2\int_\Omega\mathbf{u}\varphi\langle\nabla\mathbf{u},\nabla\varphi\rangle.\end{aligned} \tag{2.5}$$

Note that, for $\gamma>0$, the second-order Taylor expansion of the function $h(t)=(1+t)^\gamma-1$ gives

$$(1+\tau\varphi)^\gamma-1=\gamma\tau\varphi-\frac{\tau^2}{2}\gamma(\gamma-1)(1+\zeta_{\tau,\gamma})^{\gamma-2}\varphi^2 \tag{2.6}$$

for some measurable function $\zeta_{\tau,\gamma}:\Omega\to\mathbb{R}$ satisfying $|\zeta_{\tau,\gamma}|\le|\tau\varphi|\le\frac12$. Inserting (2.6) (used with $\gamma=1-\alpha$ and $\gamma=1+p$) in (2.5), we obtain

$$\begin{aligned}
&\tau\int_\Omega\langle\nabla\mathbf{u},\nabla(\mathbf{u}\varphi)\rangle\\
&\ge\tau\int_\Omega a\mathbf{u}^{1-\alpha}\varphi-\frac{\tau^2}{2}\alpha\int_\Omega a\mathbf{u}^{1-\alpha}(1+\zeta_{\tau,1-\alpha})^{-\alpha-1}\varphi^2\\
&\quad-\Big(\tau\int_\Omega b\mathbf{u}^{1+p}\varphi+\frac{\tau^2}{2}p\int_\Omega b\mathbf{u}^{1+p}(1+\zeta_{\tau,1+p})^{p-1}\varphi^2\Big)\\
&\quad-\frac{\tau^2}{2}\int_\Omega\mathbf{u}^2|\nabla\varphi|^2-\frac{\tau^2}{2}\int_\Omega\varphi^2|\nabla\mathbf{u}|^2-\tau^2\int_\Omega\mathbf{u}\varphi\langle\nabla\mathbf{u},\nabla\varphi\rangle.
\end{aligned}\tag{2.7}$$

Also, $1+\zeta_{\tau,1-\alpha}\ge\frac12$ and $1+\zeta_{\tau,1+p}\ge\frac12$, and thus

$$\begin{gathered}
|\int_\Omega a\mathbf{u}^{1-\alpha}(1+\zeta_{\tau,1-\alpha})^{-\alpha-1}\varphi^2|\le c,\\
|\int_\Omega b\mathbf{u}^{1+p}(1+\zeta_{\tau,1+p})^{p-1}\varphi^2|\le c
\end{gathered}$$

for some positive constant c independent of τ. Now we take τ positive in (2.7). Dividing by τ, and then letting $\tau\to0^+$, from (2.7) we obtain

$$\int_\Omega\langle\nabla\mathbf{u},\nabla(\mathbf{u}\varphi)\rangle\ge\int_\Omega a\mathbf{u}^{1-\alpha}\varphi-\int_\Omega b\mathbf{u}^{1+p}\varphi.$$

We note that this inequality holds if we put $-\varphi$ instead of φ; therefore we obtain also the reverse inequality, and we conclude that (2.4) is valid for $\|\varphi\|_\infty\le\frac12$. Finally, since both sides in (2.4) are linear on φ, the assumption $\|\varphi\|_\infty\le\frac12$ can be removed. □

Lemma 2.4. *There exists $v\in H_0^1(\Omega)$ such that $J(v)<0$.*

Proof. It is sufficient to show that there exists a function $\Phi\in H_0^1(\Omega)$ such that $\int_\Omega a|\Phi|^{1-\alpha}>0$. Indeed, if such a Φ exists, then, for $t>0$, we have

$$\begin{aligned}
J(t\Phi)&=\frac{t^2}{2}\|\nabla\Phi\|_2^2-\frac{t^{1-\alpha}}{1-\alpha}\int_\Omega a|\Phi|^{1-\alpha}+\frac{t^{1+p}}{1+p}\int_\Omega b|\Phi|^{1+p}\\
&=t^{1-\alpha}(\frac{t^{1+\alpha}}{2}\|\nabla\Phi\|_2^2-\frac{1}{1-\alpha}\int_\Omega a|\Phi|^{1-\alpha}+\frac{t^{p+\alpha}}{1+p}\int_\Omega b|\Phi|^{1+p})
\end{aligned}$$

which gives that $J(t\Phi)$ is negative for t positive and small enough. Such a Φ can be constructed as follows: Let $h\in C_c^\infty(\mathbb{R}^n)$ be a nonnegative radial function with support in the unit ball $B=\{x\in\mathbb{R}^n:|x|<1\}$, and such that $\int_B h=1$. For $\varepsilon>0$ let $h_\varepsilon(x):=\frac{1}{\varepsilon^n}h(\frac{x}{\varepsilon})$. For $\delta>0$ let $\Omega_\delta:=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\delta\}$. Since $|\{x\in\Omega:a(x)>0\}|>0$, we have $|\{x\in\Omega:a(x)>0\}\cap\Omega_\delta|>0$ for δ positive and small enough. We fix such a δ, and set $E=\{x\in\Omega:a(x)>0\}\cap\Omega_\delta$. For $\varepsilon>0$ we define $\Phi_\varepsilon:=h_\varepsilon*\chi_E$. Then $\Phi_\varepsilon\in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\Phi_\varepsilon)\subset\Omega$ for $\varepsilon<\delta$. Thus $\Phi_\varepsilon\in C_c^\infty(\Omega)$ for $\varepsilon<\delta$. Also, $\lim_{\varepsilon\to0^+}\Phi_\varepsilon=\chi_E$ with convergence in $L^2(\Omega)$ (see [6, Theorem 4.22]). Then $\lim_{\varepsilon\to0^+}a\Phi_\varepsilon^{1-\alpha}=a\chi_E$ with convergence in $L^1(\Omega)$ (see [1, Theorem 1.2.1]), therefore

$$\lim_{\varepsilon\to0^+}\int_\Omega a\Phi_\varepsilon^{1-\alpha}=\int_\Omega a(\chi_E)^{1-\alpha}=\int_\Omega a\chi_E>0.$$

Then $\int_\Omega a|\Phi_\varepsilon|^{1-\alpha} > 0$ for ε small enough. □

Corollary 2.5. $\mathbf{u} \not\equiv 0$.

Remark 2.6. Let us observe that $\nabla(v^2) = 2v\nabla(v)$ for any (possibly unbounded) $v \in H^1(\Omega)$. Indeed, for $k \in \mathbb{N}$, let v_k be the truncation of v, defined by $v_k(x) = v(x)$ if $|v(x)| \le k$, and by $v_k(x) = k\ sign(v(x))$ otherwise. Then $\{v_k\}_{k\in\mathbb{N}}$ converges to v in $H^1(\Omega)$ as k tends to ∞, and, since each v_k is bounded, it follows from the chain rule (as stated e.g. in [23, Lemma 7.5]) that $\frac{\partial}{\partial x_i}(v_k^2) = 2v_k\frac{\partial v_k}{\partial x_i}$, $i = 1, 2, \dots, n$. Since $\{v_k\}_{k\in\mathbb{N}}$ converges to v in $L^2(\Omega)$, we have that $\{v_k^2\}_{k\in\mathbb{N}}$ converges to v^2 in $L^1(\Omega)$, and so also in $D'(\Omega)$. Then $\{\frac{\partial}{\partial x_i}(v_k^2)\}_{k\in\mathbb{N}}$ converges to $\frac{\partial}{\partial x_i}(v^2)$ in $D'(\Omega)$. Since $\{2v_k\frac{\partial v_k}{\partial x_i}\}_{k\in\mathbb{N}}$ converges to $2v\frac{\partial v}{\partial x_i}$ in $L^1(\Omega)$, and therefore in $D'(\Omega)$, we obtain that, for each i, $\frac{\partial}{\partial x_i}(v^2) = 2v\frac{\partial v}{\partial x_i}$.

Lemma 2.7. $\mathbf{u} \in L^\infty(\Omega)$.

Proof. Let Ω' be a bounded $C^{0,1}$ domain such that $\overline{\Omega} \subset \Omega'$, and let $\widetilde{\mathbf{u}}$, $\widetilde{a} : \mathbb{R}^n \to \mathbb{R}$ be the extensions by zero of $\mathbf{u}$ and a respectively. We consider first the case $n > 2$. Let $r = \frac{1-\alpha}{2}$, $\eta = \frac{2^*}{1-\alpha}$. Then $0 < r < 1$, $\eta > 1$, and $a\mathbf{u}^{2r} \in L^\eta(\Omega)$. Let $z \in W^{2,\eta}(\Omega') \cap W_0^{1,\eta}(\Omega')$ be the solution of

$$
\begin{gathered}
-\Delta z = 2\widetilde{a}\widetilde{\mathbf{u}}^{2r} \quad \text{in } \Omega', \\
z = 0 \quad \text{on } \partial\Omega'.
\end{gathered} \tag{2.8}
$$

Let $\widetilde{z} : \mathbb{R}^n \to \mathbb{R}$ be the extension by zero of z and let φ be a nonnegative function in $C_c^\infty(\Omega')$ By Remark 2.6 and Lemma 2.3 we have

$$
\begin{aligned}
\int_{\Omega'} \langle \nabla(\widetilde{\mathbf{u}}^2), \nabla\varphi\rangle &= \int_{\Omega'} \langle 2\widetilde{\mathbf{u}}\nabla\widetilde{\mathbf{u}}, \nabla\varphi\rangle \\
&= \int_\Omega 2\mathbf{u}\langle\nabla\mathbf{u}, \nabla\varphi\rangle \le \int_\Omega 2\langle\nabla(\mathbf{u}\varphi), \nabla\mathbf{u}\rangle \\
&= 2\int_\Omega (a\mathbf{u}^{1-\alpha} - b\mathbf{u}^{p+1})\varphi \\
&\le 2\int_{\Omega'} \widetilde{a}\widetilde{\mathbf{u}}^{2r}\varphi = \int_{\Omega'} \langle\nabla z, \nabla\varphi\rangle
\end{aligned} \tag{2.9}
$$

For $\varepsilon > 0$ let h_ε be the mollifiers defined as in the proof of Lemma 2.3. For ε small enough we have $0 \le \varphi * h_\varepsilon \in C_c^\infty(\Omega')$, and so, by (2.9),

$$
\begin{aligned}
\int_{\Omega'} \langle\nabla(h_\varepsilon * \widetilde{\mathbf{u}}^2), \nabla\varphi\rangle &= \int_{\Omega'} \langle\nabla(\widetilde{\mathbf{u}}^2), h_\varepsilon * \nabla\varphi\rangle \\
&= \int_{\Omega'} \langle\nabla(\widetilde{\mathbf{u}}^2), \nabla(h_\varepsilon * \varphi)\rangle \\
&\le \int_{\Omega'} \langle\nabla z, \nabla(h_\varepsilon * \varphi)\rangle
\end{aligned}
$$

where we have used that, since h_ε is an even function, the convolution operator with kernel h_ε is self-adjoint in $L^2(\mathbb{R}^n)$. Recall that $\widetilde{z} \in W^{1,\eta}(\mathbb{R}^n)$ and $\operatorname{supp}(\widetilde{z}) \subset \overline{\Omega'}$. Also, $\nabla\widetilde{z} = \nabla z$ a.e. in Ω', and $\nabla\widetilde{z} = 0$ a.e. in $\mathbb{R}^n - \Omega'$. Thus

$$
\int_{\Omega'} \langle\nabla z, \nabla(h_\varepsilon * \varphi)\rangle = \int_{\mathbb{R}^n} \langle\nabla\widetilde{z}, \nabla(h_\varepsilon * \varphi)\rangle
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \langle \nabla (h_\varepsilon * \widetilde{z}), \nabla \varphi \rangle \\
&= \int_{\Omega'} \langle \nabla (h_\varepsilon * \widetilde{z}), \nabla \varphi \rangle.
\end{aligned}
$$

Then

$$\int_{\Omega'} \langle \nabla (h_\varepsilon * \widetilde{\mathbf{u}}^2), \nabla \varphi \rangle \le \int_{\Omega'} \langle \nabla (h_\varepsilon * \widetilde{z}), \nabla \varphi \rangle$$

and so the divergence theorem gives

$$-\int_{\Omega'} \varphi \Delta (h_\varepsilon * \widetilde{\mathbf{u}}^2) \le -\int_{\Omega'} \varphi \Delta (h_\varepsilon * \widetilde{z}).$$

Since this inequality holds for all nonnegative $\varphi \in C_c^\infty(\Omega')$ we obtain

$$-\Delta (h_\varepsilon * \widetilde{\mathbf{u}}^2) \le -\Delta (h_\varepsilon * \widetilde{z}) \text{ in } \Omega'.$$

We have also $h_\varepsilon * \widetilde{\mathbf{u}}^2 = 0 \le h_\varepsilon * \widetilde{z}$ on $\partial \Omega'$. Thus, the classical maximum principle gives $h_\varepsilon * \widetilde{\mathbf{u}}^2 \le h_\varepsilon * \widetilde{z}$ in Ω'. Now, $\widetilde{\mathbf{u}}^2$ and $\widetilde{z}$ belong to $L^{\frac{2^*}{2}}(\mathbb{R}^n)$, and so $\lim_{\varepsilon \to 0^+} (h_\varepsilon * \widetilde{\mathbf{u}}^2) = \widetilde{\mathbf{u}}^2$, and $\lim_{\varepsilon \to 0^+} (h_\varepsilon * \widetilde{z}) = \widetilde{z}$, in both cases with convergence in $L^{\frac{2^*}{2}}(\mathbb{R}^n)$. Then, $\lim_{\varepsilon \to 0^+} (h_\varepsilon * \widetilde{\mathbf{u}}^2)|_\Omega = \widetilde{\mathbf{u}}^2_{|\Omega}$; and $\lim_{\varepsilon \to 0^+} (h_\varepsilon * \widetilde{z})|_\Omega = \widetilde{z}|_\Omega$, in each case with convergence in $L^{\frac{2^*}{2}}(\Omega)$. Then $\mathbf{u}^2 \le z$ in Ω.

Now the lemma follows from the following standard bootstrap argument: Let $\{\eta_j\}_{j \in \mathbb{N}}$ be recursively defined by $\eta_1 = \eta^*$ and by $\eta_{j+1} = \eta_j^*$. We can see inductively that $\mathbf{u} \in L^{2\eta_j}(\Omega)$ for all j. Indeed, $z \in W^{2,\eta}(\Omega')$, and so $z \in L^{\eta^*}(\Omega')$. Then $\mathbf{u}^2 \in L^{\eta^*}(\Omega)$, and thus $\mathbf{u} \in L^{2\eta^*}(\Omega) = L^{2\eta_1}(\Omega)$. Suppose now that $\mathbf{u} \in L^{2\eta_j}(\Omega)$, then $2\widetilde{a}\widetilde{\mathbf{u}}^{2r} \in L^{\frac{\eta_j}{p}}(\Omega') \subset L^{\eta_j}(\Omega')$, and so $z \in W^{2,\eta_j}(\Omega') \subset L^{\eta_j^*}(\Omega') = L^{\eta_j^*}(\Omega')$, which gives $\mathbf{u} \in L^{2\eta_j^*}(\Omega) = L^{2\eta_{j+1}}(\Omega)$. Thus $\mathbf{u} \in L^{2\eta_j}(\Omega)$ for all j, and so, taking j large enough, we obtain $\mathbf{u} \in L^s(\Omega)$ for some $s > 2n$, then $2\widetilde{a}\widetilde{\mathbf{u}}^{2r} \in L^{\frac{s}{2r}}(\Omega') \subset L^{\frac{s}{2}}(\Omega')$. Thus $z \in W^{2,\frac{s}{2}}(\Omega') \subset L^\infty(\Omega')$. Since $\mathbf{u}^2 \le z$ in Ω, we obtain $\mathbf{u} \in L^\infty(\Omega)$.

Finally, if $n \le 2$, we have $\mathbf{u} \in L^s(\Omega)$ for all $s \in [1, \infty)$. We take $\eta > n$ and, for r, z, $\overline{z}$ and $\widetilde{\mathbf{u}}$ defined as above, we have $a\mathbf{u}^{2r} \in L^\eta(\Omega)$. Thus $\widetilde{z} \in W^{2,\eta}(\Omega') \subset C(\overline{\Omega'})$ and, as before, $\mathbf{u}^2 \le z$ in Ω. Then $\mathbf{u} \in L^\infty(\Omega)$ also in this case. □

Lemma 2.8.

$$\int_\Omega \langle \nabla \mathbf{u}, \nabla (\mathbf{u}\varphi) \rangle = \int_\Omega (a\mathbf{u}^{1-\alpha} - b\mathbf{u}^{1+p})\varphi \tag{2.10}$$

for all $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$. By Lemma 2.7 we have $\mathbf{u} \in L^\infty(\Omega)$ and so $\mathbf{u}\varphi \in H_0^1(\Omega)$. Thus Lemma 2.3 gives (2.10). □

3. Main results

Theorem 3.1. *There exists a nonnegative weak solution $0 \not\equiv u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem* (1.2).

Proof. Let $\mathbf{u}$ be the nonnegative minimizer of J considered in the previous section. Let ψ be a nonnegative function in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and let $\varepsilon > 0$. Note that $\frac{\psi}{\mathbf{u}+\varepsilon} \in H^1(\Omega) \cap L^\infty(\Omega)$, and that $\nabla(\mathbf{u}\frac{\psi}{\mathbf{u}+\varepsilon}) = \varepsilon \frac{\nabla \mathbf{u}}{(\mathbf{u}+\varepsilon)^2}\psi + \frac{\mathbf{u}}{\mathbf{u}+\varepsilon}\nabla\psi$, and so Lemma 2.8 gives

$$\varepsilon \int_\Omega \psi \frac{|\nabla \mathbf{u}|^2}{(\mathbf{u}+\varepsilon)^2} + \int_\Omega \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \langle \nabla \mathbf{u}, \nabla \psi \rangle = \int_\Omega (a\mathbf{u}^{1-\alpha} - b\mathbf{u}^{1+p}) \frac{1}{\mathbf{u}+\varepsilon} \psi. \tag{3.1}$$

Since $\nabla \mathbf{u} = 0$ a.e. on the set $\{x \in \Omega : \mathbf{u}(x) = 0\}$, and since $a\mathbf{u}^{1-\alpha} = b\mathbf{u}^{1+p} = 0$ on the same set, (3.1) can be written as

$$\begin{aligned} &\varepsilon \int_{\{\mathbf{u}>0\}} \psi \frac{|\nabla \mathbf{u}|^2}{(\mathbf{u}+\varepsilon)^2} + \int_{\{\mathbf{u}>0\}} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \langle \nabla \mathbf{u}, \nabla \psi \rangle + \int_{\{\mathbf{u}>0\}} b\mathbf{u}^p \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi \\ &= \int_{\{\mathbf{u}>0\}} a\mathbf{u}^{-\alpha} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi. \end{aligned} \tag{3.2}$$

Also

$$\lim_{\varepsilon \to 0^+} \Big(\frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \langle \nabla \mathbf{u}, \nabla \psi \rangle \Big) = \chi_{\{\mathbf{u}>0\}} \langle \nabla \mathbf{u}, \nabla \psi \rangle = \langle \nabla \mathbf{u}, \nabla \psi \rangle$$

a.e. in Ω, and $|\frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \langle \nabla \mathbf{u}, \nabla \psi \rangle| \le |\langle \nabla \mathbf{u}, \nabla \psi \rangle| \in L^1(\Omega)$, and so Lebesgue's dominated convergence theorem gives

$$\lim_{\varepsilon \to 0^+} \int_{\{\mathbf{u}>0\}} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \langle \nabla \mathbf{u}, \nabla \psi \rangle = \int_\Omega \langle \nabla \mathbf{u}, \nabla \psi \rangle. \tag{3.3}$$

On the other hand, $\lim_{\varepsilon \to 0^+} a\mathbf{u}^{-\alpha} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi = a\mathbf{u}^{-\alpha}\psi$ on the set $\{x \in \Omega : \mathbf{u}(x) > 0\}$ and, since $a\mathbf{u}^{-\alpha} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi$ is non-increasing in ε, the monotone convergence theorem gives

$$\lim_{\varepsilon \to 0^+} \int_{\{\mathbf{u}>0\}} a\mathbf{u}^{-\alpha} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi = \int_{\{\mathbf{u}>0\}} a\mathbf{u}^{-\alpha}\psi = \int_\Omega a\mathbf{u}^{-\alpha} \chi_{\{\mathbf{u}>0\}} \psi \tag{3.4}$$

Also

$$\lim_{\varepsilon \to 0^+} \int_{\{\mathbf{u}>0\}} b\mathbf{u}^p \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi = \int_\Omega b\mathbf{u}^p \psi \tag{3.5}$$

Then, from (3.2), (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned} &\int_\Omega \langle \nabla \mathbf{u}, \nabla \psi \rangle + \int_\Omega b\mathbf{u}^p \psi \\ &= \lim_{\varepsilon \to 0^+} \Big(\int_{\{\mathbf{u}>0\}} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \langle \nabla \mathbf{u}, \nabla \psi \rangle + \int_{\{\mathbf{u}>0\}} b\mathbf{u}^p \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi \Big) \\ &\le \limsup_{\varepsilon \to 0^+} \Big(\int_{\{\mathbf{u}>0\}} \frac{\varepsilon \psi |\nabla \mathbf{u}|^2}{(\mathbf{u}+\varepsilon)^2} + \int_{\{\mathbf{u}>0\}} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \langle \nabla \mathbf{u}, \nabla \psi \rangle + \int_{\{\mathbf{u}>0\}} b\mathbf{u}^p \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi \Big) \\ &= \limsup_{\varepsilon \to 0^+} \int_{\{\mathbf{u}>0\}} a\mathbf{u}^{-\alpha} \frac{\mathbf{u}}{\mathbf{u}+\varepsilon} \psi \\ &= \int_\Omega a\mathbf{u}^{-\alpha} \chi_{\{\mathbf{u}>0\}} \psi. \end{aligned}$$

Thus

$$\int_\Omega \langle \nabla \mathbf{u}, \nabla \psi \rangle + \int_\Omega b\mathbf{u}^p \psi \le \int_\Omega a\mathbf{u}^{-\alpha} \chi_{\{\mathbf{u}>0\}} \psi. \tag{3.6}$$

Let us see that the reverse inequality in (3.6) holds: A computation gives, for $t > 0$,

$$\begin{aligned} 0 &\le \frac{1}{t} (J(\mathbf{u}+t\psi) - J(\mathbf{u})) \\ &= \int_\Omega \langle \nabla \mathbf{u}, \nabla \psi \rangle + \frac{t}{2} \int_\Omega |\nabla \psi|^2 - \frac{1}{(1-\alpha)t} \int_\Omega a((\mathbf{u}+t\psi)^{1-\alpha} - \mathbf{u}^{1-\alpha}) \\ &\quad + \frac{1}{(1+p)t} \int_\Omega b((\mathbf{u}+t\psi)^{1+p} - \mathbf{u}^{1+p}), \end{aligned}$$

and so

$$\begin{aligned} &\frac{1}{(1-\alpha)t}\int_\Omega a((\mathbf{u}+t\psi)^{1-\alpha}-\mathbf{u}^{1-\alpha}) \\ &\le \int_\Omega \langle \nabla \mathbf{u}, \nabla\psi\rangle + \frac{1}{(1+p)t}\int_\Omega b((\mathbf{u}+t\psi)^{1+p}-\mathbf{u}^{1+p}) + \frac{t}{2}\int_\Omega |\nabla\psi|^2. \end{aligned} \tag{3.7}$$

The mean value theorem gives $(\mathbf{u}+t\psi)^{1-\alpha}-\mathbf{u}^{1-\alpha} = (1-\alpha)(\mathbf{u}+\sigma_t)^{-\alpha}t\psi$ for some measurable function σ_t such that $0<\sigma_t<t\psi$. Thus

$$\begin{aligned} &\frac{1}{(1-\alpha)t}\int_\Omega a((\mathbf{u}+t\psi)^{1-\alpha}-\mathbf{u}^{1-\alpha}) \\ &= \frac{1}{(1-\alpha)t}\int_{\{a>0\}\cap\{\psi>0\}} a((\mathbf{u}+t\psi)^{1-\alpha}-\mathbf{u}^{1-\alpha}) \\ &= \int_{\{a>0\}\cap\{\psi>0\}} a(\mathbf{u}+\sigma_t)^{-\alpha}\psi \end{aligned}$$

Now, $\lim_{t\to 0^+} a(\mathbf{u}+\sigma_t)^{-\alpha}\psi = a\mathbf{u}^{-\alpha}\psi$ $a.e$ on the set $\{a>0\}\cap\{\psi>0\}$. Then, by Fatou's Lemma,

$$\begin{aligned} &\liminf_{t\to 0^+}\frac{1}{(1-\alpha)t}\int_\Omega a((\mathbf{u}+t\psi)^{1-\alpha}-\mathbf{u}^{1-\alpha}) \\ &= \liminf_{t\to 0^+}\int_{\{a>0\}\cap\{\psi>0\}} a(\mathbf{u}+\sigma_t)^{-\alpha}\psi \\ &\ge \int_{\{a>0\}\cap\{\psi>0\}} a\mathbf{u}^{-\alpha}\psi \ge \int_\Omega a\mathbf{u}^{-\alpha}\chi_{\{\mathbf{u}>0\}}\psi. \end{aligned} \tag{3.8}$$

Again by the mean value theorem, we have

$$\frac{1}{(1+p)t}\int_\Omega b((\mathbf{u}+t\psi)^{1+p}-\mathbf{u}^{1+p}) = \int_\Omega b(\mathbf{u}+\sigma_t)^p\psi.$$

Note that, for $0<t<1$, we have $0\le b(\mathbf{u}+\sigma_t)^p\psi \le b(\mathbf{u}+\psi)^{p+1}\in L^1(\Omega)$. Also, $\lim_{t\to 0^+} b(\mathbf{u}+\sigma_t)^p\psi = b\mathbf{u}^p\psi$ $a.e.$ in Ω. Thus, by Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0^+}\frac{1}{(1+p)t}\int_\Omega b((\mathbf{u}+t\psi)^{1+p}-\mathbf{u}^{1+p}) = \int_\Omega b\mathbf{u}^p\psi. \tag{3.9}$$

Now, from (3.7), (3.8), and (3.9), we obtain

$$\int_\Omega \langle\nabla\mathbf{u},\nabla\psi\rangle + \int_\Omega b\mathbf{u}^p\psi \ge \int_\Omega a\mathbf{u}^{-\alpha}\chi_{\{\mathbf{u}>0\}}\psi \tag{3.10}$$

Since $b\mathbf{u}^p\psi\in L^1(\Omega)$, (3.10) implies that $a\mathbf{u}^{-\alpha}\chi_{\{\mathbf{u}>0\}}\psi\in L^1(\Omega)$. We apply (3.10), combined with (3.6), to complete the proof. □

Remark 3.2. It is well known (see e.g., [17]) that, for $m\in L^\infty(\Omega)$ such that $|\{x\in\Omega : m(x)>0\}|>0$, there exists a unique $\lambda=\lambda_1(-\Delta,\Omega,m)$ such that the problem

$$\begin{gathered} -\Delta\varphi_1 = \lambda m\varphi_1 \quad \text{in } \Omega, \\ \varphi_1 = 0 \quad \text{on } \partial\Omega, \\ \varphi_1 > 0 \quad \text{in } \Omega \end{gathered}$$

has a solution $\varphi_1 \in H_0^1(\Omega)$. This solution is unique up to a multiplicative constant, belongs to $C^{1,\gamma}(\overline{\Omega})$ for some $0<\gamma<1$, satisfies that $|\nabla\varphi|(x)>0$ for all $x\in\partial\Omega$, and there are positive constants c_1, c_2 such that $c_1 d_\Omega \le \varphi \le c_2 d_\Omega$ in Ω, where $d_\Omega : \Omega \to \mathbb{R}$ is the function defined by

$$d_\Omega(x) = \operatorname{dist}(x, \partial\Omega).$$

λ_1 and φ_1 are called, respectively, the principal eigenvalue and a positive principal eigenfunction for $-\Delta$ in Ω, with Dirichlet boundary condition and weight m.

Remark 3.3. It is well known that, under our assumptions on Ω, α, and a, the problem

$$\begin{gathered} -\Delta\theta = a\theta^{-\alpha} \text{ in } \Omega, \\ \theta = 0 \text{ on } \partial\Omega, \\ \theta > 0 \text{ in } \Omega \end{gathered}$$

has a unique weak solution $\theta \in H_0^1(\Omega)$. Moreover, θ is in $C(\overline{\Omega})$, and $\theta \ge c' d_\Omega$ for some positive constant c' (see [12, 3]). A computation shows that (in weak sense) $-\Delta(\theta^{\alpha+1}) = -(\alpha+1)\theta^\alpha\Delta\theta - (\alpha+1)\alpha\theta^{\alpha-2}|\nabla\theta|^2 \le (\alpha+1)\|a\|_\infty$ in Ω, and so we have $\theta \le c'' d_\Omega^{\frac{1}{\alpha+1}}$ in Ω, for some constant $c''>0$.

Remark 3.4. Following [26], we say that $w \in W^{1,2}_{\rm loc}(\Omega)$ is a subsolution (supersolution) to the problem

$$-\Delta z = az^{-\alpha} - bz^p \text{ in } \Omega \tag{3.11}$$

in the sense of distributions, if, and only if: $w>0$ a.e. in Ω, $aw^{-\alpha} - bw^p \in L^1_{\rm loc}(\Omega)$, and for all nonnegative $\varphi \in C_c^\infty(\Omega)$, it holds that

$$\int_\Omega \langle \nabla w, \nabla\varphi\rangle \le (\ge) \int_\Omega (aw^{-\alpha} - bw^p)\varphi.$$

We also say that $z \in W^{1,2}_{\rm loc}(\Omega)$ is a solution, in the sense of distributions, of (3.11) if, and only if, $z>0$ a.e. in Ω, and, for all $\varphi \in C_c^\infty(\Omega)$ it holds that

$$\int_\Omega \langle \nabla z, \nabla\varphi\rangle = \int_\Omega (az^{-\alpha} - bz^p)\varphi.$$

For subsolutions, supersolutions and solutions defined in the above sense, [26, Theorem 2.4] says that, if (3.11) has a subsolution $\underline{z}$ and a supersolution $\overline{z}$ (in the sense of distributions), both in $L^\infty_{\rm loc}(\Omega)$, and such such that $0<\underline{z}(x)\le\overline{z}(x)$ a.e. $x\in\Omega$, and if there exists $k \in L^\infty_{\rm loc}(\Omega)$ such that $|as^{-\alpha} - bs^p| \le k(x)$ a.e. $x\in\Omega$ for all $s \in [\underline{z}(x), \overline{z}(x)]$; then (3.11) has a solution z in the sense of distributions, and z satisfies $\underline{z} \le z \le \overline{z}$ a.e. in Ω.

Theorem 3.5. *Suppose that $a \ge \varepsilon b$ for some $\varepsilon>0$. Then there exists a weak solution $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ of (1.2) such that $v \ge c d_\Omega$ in Ω for some $c>0$, and $v \in C^1_{\rm loc}(\Omega)\cap C(\overline{\Omega})$.*

Proof. Suppose that $a \ge \varepsilon b$ for some $\varepsilon>0$. Let $\varphi_1 \in H_0^1(\Omega)$ be the positive principal eigenfunction associated to the weight function a, normalized by $\|\varphi_1\|_\infty = 1$ (see Remark 3.2). Note that (in weak sense), for t positive and small enough,

$$-\Delta(t\varphi_1) \le a(t\varphi_1)^{-\alpha} - b(t\varphi_1)^p \text{ in } \Omega. \tag{3.12}$$

Indeed, $-\Delta(t\varphi_1) = \lambda_1 a t\varphi_1$, and so (3.12) is equivalent to $(1-\lambda_1(t\varphi_1)^{1+\alpha})a \ge (t\varphi_1)^{p+\alpha}b$ in Ω. But, for t small enough, we have $b(t\varphi_1)^{p+\alpha} \le bt^{p+\alpha} \le \frac{1}{2}\varepsilon b \le$

$\frac{1}{2}a \le (1-\lambda_1(t\varphi_1)^{1+\alpha})a$ in Ω. Since $t\varphi_1 > 0$ in Ω, it follows that, for such a t, $t\varphi_1$ is a subsolution of (1.2), in the sense of Remark 3.4. On the other hand, let $\theta \in H_0^1(\Omega) \cap C(\overline{\Omega})$ be the solution of the problem $-\Delta\theta = a\theta^{-\alpha}$ in Ω, $\theta = 0$ on $\partial\Omega$. Since $\theta \ge c' d_\Omega$ in Ω for some $c' > 0$, we have that θ is strictly positive in Ω, and, by diminishing t if necessary, we can assume, that $t\varphi_1 \le \theta$. Clearly (in weak sense) $-\Delta\theta \ge a\theta^{-\alpha} - b\theta^p$ in Ω, and so θ is a supersolution of (1.2), again in the sense of Remark 3.4). Since $t\varphi_1 \ge c_1 t d_\Omega$ in Ω for some $c_1 > 0$, and since $\theta \le c'' d_\Omega^{\frac{1}{\alpha+1}}$ in Ω for some $c'' > 0$, we have $[t\varphi_1(x), \theta(x)] \subset [c_1 t d_\Omega(x), c'' d_\Omega^{\frac{1}{\alpha+1}}(x)]$ for $x \in \Omega$. Therefore a.e. $x \in \Omega$, for all $s \in [t\varphi_1(x), \theta(x)]$, the following holds

$$|as^{-\alpha} - bs^p| \le \|a\|_\infty (c_1 t)^{-\alpha} d_\Omega(x)^{-\alpha} + \|b\|_\infty (c'')^p d_\Omega^{\frac{p}{\alpha+1}}(x) := k(x).$$

Since $k \in L^\infty_{\rm loc}(\Omega)$, [26, Theorem 2.4] (see Remark 3.4), says that there exists $v \in W^{1,2}_{\rm loc}(\Omega)$ such that $t\varphi_1 \le v \le \theta$ in Ω, and such that, for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \langle \nabla v, \nabla\varphi\rangle = \int_\Omega (av^{-\alpha} - bv^p)\varphi. \tag{3.13}$$

Note that $v \in H_0^1(\Omega)$: Indeed, let Ω' be a subdomain of Ω such that $\overline{\Omega'} \subset \Omega$. Since $v \ge c'' d_\Omega$ in Ω for some $c'' > 0$, we have $av^{-\alpha} - bv^p \in L^\infty(\Omega')$. Therefore, from (3.13), a density argument, and Lebesgue's dominated convergence theorem give that, for any $\varphi \in H_0^1(\Omega')$, it holds

$$\int_{\Omega'} \langle \nabla v, \nabla\varphi\rangle = \int_{\Omega'} (av^{-\alpha} - bv^p)\varphi. \tag{3.14}$$

Let $\varepsilon > 0$. Since $v \le \theta \le c'' d_\Omega^{\frac{1}{1+\alpha}}$ for some $c'' > 0$, we have that $\operatorname{supp}(v-\varepsilon)^+ \subset \Omega'$ for some subdomain Ω' such that $\overline{\Omega'} \subset \Omega$. Also $(v-\varepsilon)^+ \in H^1(\Omega)$ and so $(v-\varepsilon)^+ \in H_0^1(\Omega)$. Thus, from (3.14), we obtain

$$\begin{aligned}
\int_\Omega \chi_{\{v>\varepsilon\}} \nabla v . \nabla v &= \int_{\Omega'} \nabla v . \nabla (v-\varepsilon)^+ \\
&= \int_{\Omega'} (av^{-\alpha} - bv^p)(v-\varepsilon)^+ \\
&= \int_\Omega (av^{-\alpha} - bv^p)(v-\varepsilon)\chi_{\{v>\varepsilon\}}.
\end{aligned} \tag{3.15}$$

The monotone convergence theorem gives

$$\lim_{\varepsilon\to 0^+} \int_\Omega \chi_{\{v>\varepsilon\}} \nabla v . \nabla v = \int_\Omega \nabla v . \nabla v$$

and, since $av^{-\alpha} - bv^p \in L^1(\Omega)$, and $v \in L^\infty(\Omega)$, Lebesgue's dominated convergence theorem gives

$$\lim_{\varepsilon\to 0^+} \int_\Omega (av^{-\alpha} - bv^p)(v-\varepsilon)\chi_{\{v>\varepsilon\}} = \int_\Omega (av^{1-\alpha} - bv^{1+p}).$$

Taking limits in (3.15), we obtain

$$\int_\Omega \nabla v . \nabla v = \int_\Omega (av^{1-\alpha} - bv^{1+p}) < \infty.$$

Thus $v \in H^1(\Omega)$ and, since $t\varphi_1 \le v \le \theta$, we have $v \in H_0^1(\Omega)$. Note also that $av^{-\alpha} - bv^p \in L^1(\Omega)$ and so, again by a density argument, and applying Lebesgue's dominated convergence theorem, we conclude that (3.13) holds for all φ in $H_0^1(\Omega)\cap L^\infty(\Omega)$.

Let Ω' be an arbitrary subdomain of Ω such that $\overline{\Omega'}\subset\Omega$, and let Ω'' be such that $\overline{\Omega'}\subset\Omega''\subset\overline{\Omega''}\subset\Omega$. Since $v\in L^\infty(\Omega'')$ and $(av^{-\alpha}-bv^p)|_{\Omega''}\in L^\infty(\Omega'')$, we have $v|_{\Omega'}\in W^{2,s}(\Omega')$ for all $s\in[1,\infty)$ (see e.g., Proposition 4.1.2 in [8]) and so $v|_{\Omega'}\in C^1(\overline{\Omega'})$. Thus $v\in C^1_{\rm loc}(\Omega)$ and, since $t\varphi_1\le v\le\theta$, v is continuous on $\partial\Omega$. $\square$

Example 3.6. Let $\Omega=(0,2\pi)$, $\alpha=1/3$, and $p\in(0,1/5)$. Let a and b be the functions defined on Ω by $a=2(1-\cos(2x))\sqrt[3]{\sin^2(x)}$, $b(x)=2|\sin^2(x)|^{-p}$. Then $a\ge0$, $b\ge0$, $0\not\equiv a\in L^\infty(\Omega)$ and $b\in L^{\frac{2}{1-p}}(\Omega)$. Consider now the following three functions in $C^1(\overline{\Omega})$: $u(x)=\sin^2(x)\chi_{(0,\pi)}$, $v(x)=\sin^2(x)\chi_{(0,2\pi)}$, and $w(x)=\sin^2(x)\chi_{(\pi,2\pi)}$. A computation shows that u, v, and w are all weak solutions of (1.2) (v is in fact a classical solution). Therefore (without additional assumptions on a and b) uniqueness is not to be expected for nonnegative nontrivial weak solutions of (1.2). Notice that $w\equiv0$ on $(0,\pi)$. Note also that $v(x)>0$ for $x\in\Omega-\{\pi\}$ and $v(\pi)=0$, therefore, by Theorem 3.8 below, there is no continuous and strictly positive solution to (1.2).

Example 3.7. *Let $\Omega=(0,2)$, let $\alpha\in(0,1)$, $p\in(0,1)$, let $b:=\chi_{(0,1)}$ and let $a:=\chi_{(1,1+\delta)}$, with*
$$0<\delta\le\big(\frac{1-\alpha}{2}\big)^{\frac{1}{1-\alpha}}\Big(\big(\frac{2}{p+1}\big)^{\frac{1}{1-p}}\big(\frac{1-p}{2}\big)^{\frac{1+p}{1-p}}\Big)^{\frac{1+\alpha}{1-\alpha}}.$$
Let us show that the problem
$$\begin{gathered}-u''=au^{-\alpha}-bu^p\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega\end{gathered}\tag{3.16}$$
has no weak solution $u\in H_0^1(\Omega)$ such that $u>0$ a.e. in Ω. Let us suppose, for the sake of contradiction, that u is a weak solution such that $u>0$ a.e. in Ω. Since $H_0^1(\Omega)\subset C^\gamma(\overline{\Omega})$ for some $\gamma\in(0,1)$, we have $u\in C^\gamma(\overline{\Omega})$ for such a γ. Throughout this example, unless there is risk of confusion, the restrictions of u to $(0,1)$, $(1,1+\delta)$, and $(1+\delta,2)$, will be still denoted by u. Since u belongs to $C^\gamma([0,1])$, and $|u^p(x)-u^p(y)|\le|u(x)-u(y)|^p$ for any $x,y\in[0,1]$, we have $u^p\in C^{\cdot\gamma p}([0,1])$. Let $A=u(1)$. Since
$$\begin{gathered}-u''=-u^p\quad\text{in }(0,1),\\ u(0)=0,\\ u(1)=A\end{gathered}\tag{3.17}$$
we have that u is a classical solution of (3.17) that belongs to $C^2([0,1])\cap C([0,1])$ and so $-u''=-u^p$ in $[0,1]$. (see, e.g., [23, Theorem 6.14]). Note also that
$$u(x)\ge\big(\frac{1-p}{2}\big)^{\frac{2}{1-p}}\big(\frac{2}{p+1}\big)^{\frac{1}{1-p}}x^{\frac{2}{1-p}}\quad\text{for all }x\in[0,1].\tag{3.18}$$
Indeed, multiplying (3.17) by u' we obtain $\frac12((u')^2)'=\frac{1}{p+1}(u^{p+1})'$ on $[0,1]$, and so $\frac12(u'(x))^2-\frac{1}{p+1}u(x)^{p+1}=\frac12(u'(0))^2\ge0$ for all $x\in[0,1]$. Thus
$$(u')^2\ge\frac{2}{p+1}u^{p+1}\quad\text{in }[0,1].\tag{3.19}$$

As $u \geq 0$ *on* $[0,1]$ *and* $u(0)=0$, *we have* $u'(0) \geq 0$. *Observe also that* (3.17) *implies* $u'' \geq 0$ *on* $[0,1]$, *and so* u *is a convex function on* $[0,1]$. *Thus* u' *is nondecreasing on* $[0,1]$ *and, since* $u'(0) \geq 0$, *we have* $u' \geq 0$ *in* $[0,1]$, *and so, from* (3.19), *we conclude*

$$u' \geq (\frac{2}{p+1})^{1/2} u^{\frac{p+1}{2}} \quad \text{in } [0,1]. \tag{3.20}$$

If $u(\overline{x})=0$ *for some* $\overline{x} \in (0,1)$ *we would have* $u(x)=0$ *for all* $x \in (0,\overline{x})$, *which contradicts the assumption that* $u>0$ *a.e. in* Ω. *Thus* $u(x)>0$ *for all* $x \in [0,1]$, *therefore* (3.20) *can be rewritten as* $u^{-\frac{p+1}{2}} u' \geq (\frac{2}{p+1})^{1/2}$ *on* $[0,1]$. *By integrating this inequality over* $(0,x)$ *we obtain* $\frac{2}{1-p}(u(x))^{\frac{1-p}{2}} \geq (\frac{2}{p+1})^{1/2} x$ *for all* $x \in [0,1]$, *and so* (3.18) *holds. In particular we have*

$$u(1) \geq (\frac{1-p}{2})^{\frac{2}{1-p}} (\frac{2}{p+1})^{\frac{1}{1-p}} x^{\frac{2}{1-p}} \tag{3.21}$$

and then, by (3.20),

$$u'(1) \geq (\frac{2}{p+1})^{\frac{1}{1-p}} (\frac{1-p}{2})^{\frac{1+p}{1-p}}. \tag{3.22}$$

Consider now the restriction of u *to* $(1,1+\delta)$; $u \in H^1(1,1+\delta) \subset C([1,1+\delta])$, *and solves*

$$\begin{gathered} -u'' = u^{-\alpha} \text{ in } (1,1+\delta) \\ u(1) \geq 0, \ u(1+\delta) \geq 0. \end{gathered}$$

Let $\zeta \in H_0^1(1,1+\delta) \subset C([1,1+\delta])$ *be the solution to the problem*

$$\begin{gathered} -\zeta'' = \zeta^{-\alpha} \quad \text{in } (1,1+\delta) \\ \zeta > 0 \quad \text{in } (1,1+\delta) \\ \zeta(1)=0, \quad \zeta(1+\delta)=0. \end{gathered}$$

Observe that $u \geq \zeta$ *on* $(1,1+\delta)$. *To prove this, suppose, for the sake of contradiction, that* $\{x \in (1,1+\delta) : u(x) < \zeta(x)\} \neq \emptyset$, *and let* U *be one of its connected components. Note that* U *is an open interval, since* u *and* ζ *are continuous on* $(1,1+\delta)$. *Since* $-\zeta'' = \zeta^{-\alpha} \leq u^{-\alpha} = -u''$ *on* U, *and* $\zeta = u$ *on* ∂U, *the maximum principle gives* $\zeta \leq u$ *on* U, *which is a contradiction. Thus* $u \geq \zeta$ *on* $(1,1+\delta)$ *as claimed.*

Recall that there exists $c>0$ *such that* $\zeta \geq cd$ *on* $(1,1+\delta)$, *where* $d(x) = \operatorname{dist}(x, \partial(1,1+\delta))$ *for all* $x \in (1,1+\delta)$ *(see Remark 3.3); therefore* $u \geq cd$ *on* $(1,1+\delta)$. *Note also that* $u(1+\delta)>0$. *If not, since* $u(2)=0$ *and* $u''=0$ *in* $(1,2)$, *we would have* $u=0$ *in* $(1,2)$; *which would contradict* $u>0$ *a.e. in* Ω. *Since* $u(1)>0$, $u(1+\delta)>0$, *and* $u \geq cd$ *on* $(1,1+\delta)$, *it follows that* $u(x)>0$ *for any* $x \in [1,1+\delta]$ *and, since* u *is continuous on* $[1,1+\delta]$, *we have* $u \geq const > 0$ *on* $[1,1+\delta]$. *Now*

$$\begin{aligned} |u^{-\alpha}(x) - u^{-\alpha}(y)| &= (u(x)u(y))^{-\alpha} |u(x)^{\alpha} - u(y)^{\alpha}| \\ &\leq (u(x)u(y))^{-\alpha} |u(x)-u(y)|^{\alpha} \end{aligned}$$

and so, since $u \in C^{\gamma}(\overline{\Omega})$, *we have* $u^{-\alpha} \in C^{\alpha\gamma}([1,1+\delta])$. *Let* $A = u(1)$, $B = u(1+\delta)$. *Since* u *solves*

$$\begin{gathered} -u'' = u^{-\alpha} \quad \text{in } (1,1+\delta) \\ u(1)=A, \quad u(1+\delta)=B, \end{gathered} \tag{3.23}$$

it follows that u is a classical solution of (3.23) *that belongs to* $C^2([1,1+\delta])\cap C([1,1+\delta])$ *(see* [23, Theorem 6.14]*).*

On the other hand, since $u''=0$ on $(1+\delta,2)$ and $u(2)=0$, we have

$$u(x)=\frac{u(1+\delta)}{1-\delta}(2-x)\quad \text{for all } x\in(1+\delta,2) \tag{3.24}$$

Since $u^{-\alpha}\in C^{\alpha\gamma}([1,1+\delta])$ and $u\in H_0^1(\Omega)\subset C(\overline{\Omega})$, we have $au^{-\alpha}-bu^p\in L^2(\Omega)$, and thus, from (3.16)*, it follows that $u\in W^{2,2}(\Omega)\subset C^1(\overline{\Omega})$. Multiplying* (3.23) *by u' we obtain*

$$\Big(\frac{1}{2}(u')^2\Big)'=-\frac{1}{1-\alpha}(u^{1-\alpha})'\quad \text{on } (1,1+\delta) \tag{3.25}$$

and so $\frac{1}{2}(u')^2+\frac{1}{1-\alpha}u^{1-\alpha}=\text{const}=\frac{1}{2}(u'(1))^2+\frac{1}{1-\alpha}u(1)^{1-\alpha}$. Therefore, for $x\in(1,1+\delta)$: $u'(x)=0$ if, and only if, $\frac{1}{1-\alpha}u^{1-\alpha}(x)=\frac{1}{2}(u'(1))^2+\frac{1}{1-\alpha}u(1)^{1-\alpha}$. If there were no x in $(1,1+\delta)$ such that $\frac{1}{1-\alpha}u^{1-\alpha}(x)=\frac{1}{2}(u'(1))^2+\frac{1}{1-\alpha}u(1)^{1-\alpha}$, we would have $u'(x)\neq 0$ for all $x\in(1,1+\delta)$; which would imply that $u'(x)>0$ for all $x\in(1,1+\delta)$ (since u' is continuous on $[1,1+\delta]$, and since $u'(1)>0$). Thus $u'(1+\delta)\geq 0$, but, by (3.24)*, $u'(1+\delta)=-\frac{u(1+\delta)}{1-\delta}<0$, which is a contradiction. Therefore $\{x\in(1,1+\delta):\frac{1}{1-\alpha}u^{1-\alpha}(x)=\frac{1}{2}(u'(1))^2+\frac{1}{1-\alpha}u(1)^{1-\alpha}\}\neq\emptyset$; let x_1 be its infimum. Since u is continuous, x_1 is a minimum, therefore we have $u(x_1)=(\frac{1-\alpha}{2}(u'(1))^2+u(1)^{1-\alpha})^{\frac{1}{1-\alpha}}$. Note that $u'(x)>0$ for all $x\in[1,x_1)$. Moreover,* (3.23) *gives that u is concave on $[1,1+\delta]$, and so $\frac{u(x_1)-u(1)}{x_1-1}\leq u'(1)$. Then, recalling* (3.22)*,*

$$\begin{aligned}
x_1-1 &\geq \frac{u(x_1)-u(1)}{u'(1)}=\frac{\big(\frac{1-\alpha}{2}(u'(1))^2+u(1)^{1-\alpha}\big)^{\frac{1}{1-\alpha}}-u(1)}{u'(1)}\\
&\geq \frac{\big(\frac{1-\alpha}{2}(u'(1))^2\big)^{\frac{1}{1-\alpha}}+(u(1)^{1-\alpha})^{\frac{1}{1-\alpha}}-u(1)}{u'(1)}\\
&= \frac{(\frac{1-\alpha}{2}(u'(1))^2)^{\frac{1}{1-\alpha}}}{u'(1)}=\big(\frac{1-\alpha}{2}\big)^{\frac{1}{1-\alpha}}\big(u'(1)\big)^{\frac{1+\alpha}{1-\alpha}}\\
&\geq \big(\frac{1-\alpha}{2}\big)^{\frac{1}{1-\alpha}}\Big(\big(\frac{2}{p+1}\big)^{\frac{1}{1-p}}\big(\frac{1-p}{2}\big)^{\frac{1+p}{1-p}}\Big)^{\frac{1+\alpha}{1-\alpha}}\geq\delta,
\end{aligned}$$

which contradicts $x_1<1+\delta$.

Theorem 3.8. *There is at most one weak solution $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$ of* (1.2) *such that $v(x)>0$ a.e. in Ω; and, if it exists, it satisfies $v\geq u$ for any other nonnegative weak solution $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$ of* (1.2)*.*

Proof. Since $s\to f(s):=as^{-\alpha}-bs^p$ is nondecreasing, the uniqueness assertion of the theorem follows from a standard argument: If w is another solution which is positive a.e. in Ω, take $\varphi:=v-w$ as a test function in the weak form of the equation

$$\begin{gathered}
-\Delta(v-w)=f(v)-f(w)\quad \text{in } \Omega,\\
v-w=0\quad \text{on } \partial\Omega
\end{gathered}$$

to obtain $\int_\Omega|\nabla(v-w)|^2=\int_\Omega(f(v)-f(w))(v-w)\leq 0$, which implies $v=w$.

Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a nonnegative solution of 1.2. Therefore, for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned} &\int_\Omega \langle \nabla(u-v), \nabla\varphi\rangle \\ &= \int_\Omega (au^{-\alpha}\chi_{\{u>0\}} - bu^p - (av^{-\alpha} - bv^p))\varphi \\ &= \int_{\{u>0\}} (f(u) - f(v))\varphi + \int_{\{u=0\}} (-av^{-\alpha} + bv^p)\varphi. \end{aligned} \tag{3.26}$$

Now, we take $\varphi = (u-v)^+$. Since $v > 0$ a.e. in Ω, we have

$$\int_{\{u=0\}} (-av^{-\alpha} + bv^p)(u-v)^+ = 0.$$

Thus, from (3.26), we obtain $\int_\Omega |\nabla(u-v)^+|^2 \le 0$, and so $u \le v$ in Ω. □

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