Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 193, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NONLOCAL PROBLEMS FOR HYPERBOLIC EQUATIONS WITH DEGENERATE INTEGRAL CONDITIONS

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Abstract. In this article, we consider a problem for hyperbolic equation with standard initial data and nonlocal condition. A distinct feature of this problem is that the nonlocal second kind integral condition degenerates and turns into a first kind. This has an important bearing on the method of the study of solvability. All methods worked out earlier for this purpose break down in the case under consideration. We propose a new approach which enables to prove a unique solvability of the nonlocal problem with degenerate integral condition.

1. Introduction

In this article, we consider the hyperbolic equation

$$\mathcal{L}u \equiv u_{tt} - (a(x,t)u_x)_x + c(x,t)u = f(x,t) \tag{1.1}$$

The nonlocal problem consists of finding a solution to (1.1) in $Q_T = (0,l) \times (0,T)$, $l, T < \infty$, satisfying the initial condition

$$u(x,0) = 0, \quad u_t(x,0) = 0, \tag{1.2}$$

boundary condition

$$u_x(0,t) = 0 \tag{1.3}$$

and the nonlocal condition

$$\alpha(t)u(l,t) + \int_0^l K(x)u(x,t)\,dx = 0. \tag{1.4}$$

A feature of the nonlocal condition (1.4) is that coefficient $\alpha(t)$ may vanish at some points.

Recently, there has been considerable interest in nonlocal problems for differential equations. There are at least two main reasons of this. One of them is that nonlocal problems form a new and important division of differential equation theory that generates a need in developing some new methods of research [26]. The second reason is that various phenomena of modern natural science lead to nonlocal problems on mathematical modeling; furthermore, nonlocal models turn out to

2010 *Mathematics Subject Classification.* 35L10, 35L20, 35L99.
Key words and phrases. Nonlocal problem; hyperbolic equation; integral condition; dynamical boundary condition; degenerate integral condition.

Submitted May 12, 2016. Published July 15, 2016.

be often more precise [6]. Finally, we note that certain subclass of nonlocal problems, namely, problems with nonlocal conditions, are related to inverse problems for partial differential equations [9, 10, 15, 16].

Nowadays various nonlocal problems for partial differential equations have been actively studied and one can find a lot of papers dealing with them (see [3, 8, 13, 14, 18, 19, 27] and references therein). We focus our attention on nonlocal problems with integral conditions for hyperbolic equations; see also [2, 5, 7, 8, 12, 17, 22, 23, 24, 25, 28].

It is well known that the classical methods used widely to prove solvability of initial-boundary problems break down when applied to nonlocal problems. Nowadays some methods have been advanced for overcoming difficulties arising from nonlocal conditions. These methods are different and the choice of a concrete one depends on a form of a nonlocal condition. In this article, we focus on spatial nonlocal integral conditions, of which we give three examples:

$$\int_0^l K(x,t)u(x,t)\,dx = 0, \tag{1.5}$$

$$u_x(l,t) + \int_0^l K(x,t)u(x,t)\,dx = 0, \tag{1.6}$$

$$\alpha(t)u(l,t) + \int_0^l K(x,t)u(x,t)\,dx = 0. \tag{1.7}$$

. Condition (1.5) is a nonlocal first kind condition, (1.6) and (1.7) are second kind nonlocal conditions. The kind of a nonlocal integral condition depends on the presence or lack of a term containing a trace of the required solution or its derivative outside the integral. Problems with nonlocal conditions of the forms (1.5) and (1.6) are investigated in [7, 23, 25, 28]. We pay attention on the third one, (1.7). If $\alpha(t) = 1$, to show solvability of the problem with this integral condition we can use the method initiated in [17], developed for multidimensional hyperbolic equation. Namely, we introduce an operator

$$Bu \equiv u(x,t) + \int_0^l K(x,t)u(x,t)\,dx$$

and reduce the nonlocal problem to a standard initial-boundary problem for a loaded equation with respect to a new unknown function $v(x,t) = Bu$. This method works provided that B is invertible.

It is easy to see that an attempt to apply this method when $\alpha(t)$ is not constand, and may vanish, leads to the third kind operator equation with all ensuing consequences. Motivated by this, we suggest a new approach to problem (1.1)–(1.4). This approach enables us to obtain a priori estimates in Sobolev spaces and to prove the solvability of the problem. Furthermore, this technique shows that nonlocal integral conditions are closely connected with dynamical boundary conditions [1, 11, 20, 29] and extend them.

2. Hypotheses, notation and auxiliary assertions

In this article we use the assumptions:

(H1) $a, c \in C^1(\bar{Q}_T)$, $a_{xt} \in C(\bar{Q}_T)$;

(H2) $f, f_t, f_{tt} \in C(\bar{Q}_T)$, $\int_0^l K(x)f(x,0)\,dx = 0$;

(H3) $K \in C^2[0,l]$, $K(l) > 0$, $K'(0) = 0$;
(H4) $\alpha \in C^{IV}[0,T]$, $\alpha(t) > 0$, $t \in (0,T)$, $\alpha(0) = 0$, $\alpha'(0) = 0$;
(H5) $\int_0^l K(x)u(x,0)\,dx = 0$, $\int_0^l K(x)u_t(x,0)\,dx = 0$.

We denote $H(x,t) = (a(x,t)K'(x))_x - K(x)c(x,t)$.

Lemma 2.1. *Under assumptions* (H1)–(H5) *the nonlocal condition* (1.4) *is equivalent to the nonlocal dynamical condition*

$$\begin{aligned} &K(l)a(l,t)u_x(l,t) - K'(l)a(l,t)u(l,t) + (\alpha(t)u(l,t))_{tt} \\ &+ \int_0^l H(x,t)u(x,t)\,dx + \int_0^l K(x)f(x,t)\,dx = 0. \end{aligned} \tag{2.1}$$

Proof. Let $u(x,t)$ be a solution of (1.1) satisfying (1.3) and (1.4). Differentiating (1.4) with respect to t we obtain

$$(\alpha(t)u(l,t))_{tt} + \int_0^l K(x)u_{tt}(x,t)\,dx = 0. \tag{2.2}$$

Taking into account that $u_{tt} = f + (au_x)_x - cu$ and

$$\int_0^l K(x)(au_x)_x\,dx = K(l)a(l,t)u_x(l,t) - K'(l)a(l,t)u(l,t) + \int_0^l (aK'(x))_x u\,dx$$

as $u_x(0,t) = 0$, $K'(0) = 0$ we obtain (2.1). $\square$

The converse is also true. Indeed, let $u(x,t)$ be a solution of (1.1) and (2.1) hold. Integrating $\int_0^l (aK'(x))_x\,dx$ we easily arrive to (2.2). Now we integrate (2.2) with respect to t twice, use (H4), (H5) and obtain (1.4).

The conclusion of this Lemma allows us to pass to the nonlocal problem with dynamical condition (2.1). Note that this condition includes $u_x(l,t)$. This fact makes it possible to use a technique presented in the next section, namely, compactness method.

3. Main results

We consider the problem

$$\mathcal{L}u \equiv u_{tt} - (a(x,t)u_x)_x + c(x,t)u = f(x,t), \quad (x,t) \in Q_T, \tag{3.1}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0, \tag{3.2}$$

$$u_x(0,t) = 0, \tag{3.3}$$

$$\begin{aligned} &K(l)a(l,t)u_x(l,t) - K'(l)a(l,t)u(l,t) + (\alpha(t)u(l,t))_{tt} \\ &+ \int_0^l Hu\,dx + \int_0^l Kf\,dx = 0. \end{aligned} \tag{3.4}$$

We denote

$$\begin{gathered} W(Q_T) = \{u : u \in W_2^1(Q_T), u_{tt} \in L_2(Q_T \cup \Gamma_l), \}, \\ \hat{W}(Q_T) = \{v : v \in W(Q_T), v(x,T) = 0\} \end{gathered}$$

where $W_2^1(Q_T)$ is the Sobolev space, and

$$\Gamma_l = \{(x,t) : x = l, t \in [0,T]\}.$$

Using a standard method [21, p. 92] and taking into account (3.4) we derive an equality

$$
\begin{aligned}
&K(l)\int_0^T\int_0^l (u_{tt}v + au_xv_x + cuv)\,dx\,dt + \int_0^T v(l,t)\int_0^l Hu\,dx\,dt \\
&- K'(l)\int_0^T v(l,t)a(l,t)u(l,t)\,dt - \int_0^T (\alpha(t)u(l,t))_t v_t(l,t)\,dt \\
&= K(l)\int_0^T\int_0^l fv\,dx\,dt - \int_0^T v(l,t)\int_0^l Kf\,dx\,dt.
\end{aligned}
\tag{3.5}
$$

Definition 3.1. A function $u \in W(Q_T)$ is said to be a generalized solution to the problem (3.1)–(3.4) if $u(x,0) = u_t(x,0) = 0$ and for every $v \in \hat{W}(Q_T)$ the identity (3.5) holds.

Theorem 3.2. *Under Hypotheses* (H1)–(H5)*, there exists a unique generalized solution to the problem* (3.1)–(3.4)*.*

Proof. We prove the existence part in several steps. First, we construct approximations of the generalized solution by the Faedo-Galerkin method. Second, we obtain a priori estimates to garantee weak convergence of approximations. Finally, we show that the limit of approximations is the required solution. Then we prove the uniqueness.

1: Approximate solutions. Let $w_k(x) \in C^2[0,l]$ be a basis in $W_2^1(\Omega)$. We define the approximations

$$
u^m(x,t) = \sum_{k=1}^m c_k(t)w_k(x), \tag{3.6}
$$

where $c_k(t)$ are solutions to the Cauchy problem

$$
\begin{aligned}
&K(l)\int_0^l (u_{tt}^m w_j + au_{x_i}^m w_j' + cu^m w_j)\,dx \\
&+ w_j(l)\int_0^l Hu^m\,dx - K'(l)a(l,t)u^m(l,t)w_j(l) + w_j(l)(\alpha(t)u^m(l,t))_{tt} \\
&= K(l)\int_0^l fw_j\,dx - w_j(l)\int_0^l Kf\,dx,
\end{aligned}
\tag{3.7}
$$

$$
c_k(0) = 0, \quad c_k'(0) = 0. \tag{3.8}
$$

Equation (3.7) can be rewritten after little manipulation as

$$
\sum_{k=1}^m [A_{kj}(t)c_k''(t) + B_{kj}(t)c_k'(t) + D_{kj}c_k(t) = f_j(t), \tag{3.9}
$$

where

$$
\begin{gathered}
A_{kj} = \int_0^l w_kw_j\,dx + \frac{1}{K(l)}\alpha(t)w_k(l)w_j(l), \quad B_{kj}(t) = \frac{2}{K(l)}\alpha'(t)w_k(l)w_j(l), \\
D_{kj}(t) = \int_0^l (aw_k'w_j' + cw_kw_j)\,dx + \frac{1}{K(l)}\big(\alpha''(t) - K'(l)a(l,t)\big)w_k(l)w_j(l) \\
+ \frac{1}{K(l)}w_j(l)\int_0^l Hw_k\,dx,
\end{gathered}
$$

$$f_j(t)=\int_0^l f(x,t)w_j(x)\,dx-\frac{1}{K(l)}w_j(l)\int_0^l Kf\,dx.$$

To show that (3.9) is solvable with respect to $c_k''(t)$, we consider a quadratic form $q=\sum_{i,j=1}^m A_{kj}\xi_k\xi_j$ and denote $\sum_{k=1}^m \xi_k w_k=\eta$. After substituting A_{kj} in q we obtain

$$q=\sum_{k,j=1}^m\int_0^l w_kw_j\,dx\xi_k\xi_j+\frac{\alpha(t)}{K(l)}\sum_{k,j=1}^m w_k(l)w_j(l)\xi_k\xi_j=\int_0^l|\eta|^2\,dx+\frac{\alpha(t)}{K(l)}|\eta(l)|^2\geq 0.$$

As $q=0$ if and only if $\eta=0$ and $\{w_k\}$ is linearly independent then $\xi_k=0$ for $k=1,\dots,m$; therefore q is positive definite. Hence (3.9) is solvable with respect to $c_k''(t)$. Thus, we can state that under (H1)–(H5) Cauchy problem (3.7)–(3.8) has a solution for every m and $\{u^m\}$ is constructed.

2: A priori estimates. To derive the first estimate we multiply (3.7) by $c_j'(t)$, sum over $j=1,\dots,m$ and integrate over $(0,\tau)$, where $\tau\in[0,T]$ is arbitrary:

$$\begin{aligned}
&K(l)\int_0^\tau\int_0^l(u_{tt}^mu_t^m+au_x^mu_{xt}^m+cu^mu_t^m)\,dx\,dt+\int_0^\tau u_t^m(l,t)\int_0^l Hu^m\,dx\,dt\\
&-K'(l)\int_0^\tau a(l,t)u^m(l,t)u_t^m(l,t)\,dt+\int_0^\tau u_t^m(l,t)(\alpha(t)u^m(l,t))_{tt}\,dt\\
&=K(l)\int_0^\tau\int_0^l fu_t^m\,dx\,dt-\int_0^\tau u_t^m(l,t)\int_0^l Kf\,dx\,dt.
\end{aligned}\tag{3.10}$$

Since $\alpha(0)=\alpha'(0)=0$, $\int_0^l K(x)f(x,0)\,dx=0$, and $c_k(0)=c_k'(0)=0$, it follows that

$$\int_0^\tau\int_0^l u_{tt}^mu_t^m\,dx\,dt=\frac12\int_0^l(u_t^m(x,\tau))^2\,dx,$$

$$\int_0^\tau\int_0^l au_x^mu_{xt}^m\,dx\,dt=\frac12\int_0^l a(u_x^m(x,\tau))^2\,dx-\frac12\int_0^\tau\int_0^l a_t(u_x^m)^2\,dx,$$

$$\begin{aligned}
&\int_0^\tau u_t^m(l,t)\int_0^l Hu^m\,dx\,dt\\
&=-\int_0^\tau u^m(l,t)\int_0^l(Hu^m)_t\,dx\,dt+u^m(l,\tau)\int_0^l Hu^m(x,\tau)\,dx,
\end{aligned}$$

$$\int_0^\tau a(l,t)u^m(l,t)u_t^m(l,t)\,dt=-\frac12\int_0^\tau a_t(l,t)(u^m(l,t))^2\,dt+\frac12 a(l,\tau)(u^m(l,\tau))^2,$$

$$\begin{aligned}
\int_0^\tau u_t^m(l,t)(\alpha(t)u^m(l,t))_{tt}\,dt&=\frac12\alpha(\tau)(u_t^m(l,\tau))^2+\frac32\int_0^\tau\alpha'(t)(u_t^m(l,t))^2\,dt\\
&\quad-\frac12\int_0^\tau\alpha'''(t)(u^m(l,t))^2\,dt+\frac12\alpha''(\tau)(u_t^m(l,\tau))^2,
\end{aligned}$$

$$\int_0^\tau u_t^m(l,t)\int_0^l Kf\,dx\,dt=-\int_0^\tau u^m(l,t)\int_0^l Kf_t\,dx\,dt+u^m(l,\tau)\int_0^l Kf(x,\tau)\,dx\,.$$

From (3.10) we obtain

$$\int_0^l[(u_t^m(x,\tau))^2+a(u_x^m(x,\tau))^2]\,dx+\alpha(\tau)(u_t^m(l,\tau))^2+\frac{3}{K(l)}\int_0^\tau\alpha'(t)(u_t^m(l,t))^2\,dt$$

$$
\begin{aligned}
&= \int_0^\tau \int_0^l [a_t(u_x^m)^2 - 2cu^m u_t^m]\,dx\,dt + \frac{2}{K(l)} \int_0^\tau u^m(l,t) \int_0^l (Hu^m)_t\,dx\,dt \\
&\quad + \frac{1}{K(l)} \int_0^\tau [K'(l)a_t(l,t) - \alpha'''(t)](u^m(l,t))^2\,dt + \frac{2}{K(l)} u^m(l,\tau) \int_0^l Hu^m\,dx \\
&\quad + \frac{1}{K(l)} [K'(l)a(l,\tau) - \alpha''(\tau)](u^m(l,\tau))^2 + 2\int_0^\tau \int_0^l f u_t^m\,dx\,dt \\
&\quad + \frac{2}{K(l)} \int_0^\tau u^m(l,t) \int_0^l K f_t\,dx\,dt - \frac{2}{K(l)} u^m(l,\tau) \int_0^l K f\,dx. \qquad (3.11)
\end{aligned}
$$

Under assumptions (H1)–(H5) there exists positive constants c_0, k_i, a_1 such that

$$
\begin{gathered}
\max_{\bar{Q}_T} |c, c_t| \le c_0, \quad \max_{\bar{Q}_T} |a, a_t, a_{xt}| \le a_1, \\
\max_{[0,T]} \int_0^l H^2\,dx \le h_1, \quad \max_{[0,T]} \int_0^l H_t^2\,dx \le h_2, \\
\max_{[0,T]} |K'(l)a(l,\tau) - \alpha'''(\tau)| \le k_1, \quad \max_{[0,T]} |K'(l)a_t(l,t) - \alpha''(t)| \le k_2.
\end{gathered}
$$

Denote

$$
k = \max\{k_1, k_2\}, \quad h = \max\{h_1, h_2\}, \quad \kappa = \int_0^l K^2\,dx.
$$

Let $a(x,t) \ge a_0 > 0$. Using Cauchy, Cauchy-Bunyakovskii inequalities we obtain

$$
\begin{gathered}
2\Big|\int_0^\tau \int_0^l cu^m u_t^m\,dx\,dt\Big| \le c_0 \int_0^\tau \int_0^l [(u^m)^2 + (u_t^m)^2]\,dx\,dt, \\
2\Big|\int_0^\tau u^m(l,t) \int_0^l (Hu^m)_t\,dx\,dt\Big| \le \int_0^\tau (u^m(l,t))^2\,dt + h\int_0^\tau \int_0^l [(u^m)^2 + (u_t^m)^2]\,dx\,dt, \\
2\Big|u^m(l,\tau) \int_0^l Hu^m(x,\tau)\,dx\Big| \le (u^m(l,\tau))^2 + h\int_0^l (u^m(x,\tau))^2\,dx, \\
2\Big|\int_0^\tau \int_0^l f u_t^m\,dx\,dt\Big| \le \int_0^\tau \int_0^l f^2\,dx\,dt + \int_0^\tau \int_0^l (u_t^m)^2\,dx\,dt, \\
2\Big|\int_0^\tau u^m(l,t) \int_0^l K f_t\,dx\,dt\Big| \le \int_0^\tau (u^m(l,t))^2\,dt + \kappa \int_0^\tau \int_0^l f_t^2\,dx\,dt, \\
2\Big|u^m(l,\tau) \int_0^l K f\,dx\,dt\Big| \le (u^m(l,\tau))^2 + \kappa \int_0^l f^2\,dx.
\end{gathered}
$$

Taking into account that $\alpha(t) \ge 0$, $\alpha'(t) \ge 0$ and the inequalities derived above we obtain

$$
\begin{aligned}
&K(l) \int_0^l [(u_t^m(x,\tau))^2 + a(u_x^m(x,\tau))^2]\,dx + \alpha(\tau)(u_t^m(l,\tau))^2 \\
&+ 3\int_0^\tau \alpha'(t)(u_t^m(l,t))^2\,dt \\
&\le C_1 \int_0^\tau \int_0^l [(u^m)^2 + (u_t^m)^2 + (u_x^m)^2]\,dx\,dt + 2\int_0^\tau (u^m(l,t))^2\,dt \\
&\quad + h\int_0^l (u^m(x,\tau))^2\,dx + 2(u^m(l,\tau))^2 + C_2 \int_0^\tau \int_0^l [f^2 + f_t^2]\,dx\,dt
\end{aligned}
$$

$$+\kappa\int_0^l f^2\,dx. \tag{3.12}$$

We estimate the term $\int_0^\tau (u^m(l,t))^2\,dt$, $\int_0^l (u^m(x,\tau))^2\,dx$, and $(u^m(l,\tau))^2$ from right-hand side of (3.12). To do this we apply some inequalities:

$$(u^m(l,\tau))^2\le \tau\int_0^\tau (u_t^m(x,t))^2\,dt \tag{3.13}$$

which follows from representation

$$\begin{gathered}
u^m(l,\tau)=\int_0^\tau u^m(l,t)\,dt,\\
(u^m(l,\tau))^2\le\int_0^l [\varepsilon(u_x^m(x,t))^2+c(\varepsilon)(u^m(x,t))]\,dx
\end{gathered} \tag{3.14}$$

which is a particular case of inequality [21]

$$\int_{\partial\Omega} u^2 ds\le\int_\Omega [\varepsilon(u_x^m(x,t))^2+c(\varepsilon)(u^m(x,t))]\,dx.$$

Then

$$\begin{gathered}
\int_0^\tau (u^m(l,t))^2\,dt\le C_3\int_0^\tau\int_0^l [(u_x^m)^2+(u^m)^2]\,dx\,dt,\\
\int_0^l (u^m(x,\tau))^2\,dx\le\tau\int_0^\tau\int_0^l (u_t^m)^2\,dx\,dt,\\
(u^m(l,\tau))^2\le\varepsilon\int_0^l (u_x^m(x,\tau))^2\,dx+\tau c(\varepsilon)\int_0^\tau\int_0^l (u_t^m)^2\,dx\,dt.
\end{gathered}$$

We choose $\varepsilon=a_0K(l)/8$ to provide $K(l)a_0-2\varepsilon>0$, add (3.13) to (3.12) and obtain

$$\begin{aligned}
&\int_0^l [(u^m(x,\tau))^2+(u_t^m(x,\tau))^2+(u_x^m(x,\tau))^2]\,dx\\
&+\alpha(\tau)(u_t^m(l,\tau))^2+\int_0^\tau \alpha'(t)(u_t^m(l,t))^2\,dt\\
&\le C_5\int_0^\tau\int_0^l [(u^m)^2+(u_t^m)^2+(u_x^m)^2]\,dx\,dt+C_6\int_0^\tau\int_0^l (f^2+f_t^2)\,dx\,dt,
\end{aligned} \tag{3.15}$$

where C_5, C_6 do not depend on m. Applying Gronwall's lemma to (3.15) and integrating over $(0,T)$ we obtain the *first a priori estimate*

$$\|u^m\|_{W_2^1(Q_T)}\le R_1. \tag{3.16}$$

It follows that for $\tau>0$,

$$\|u^m\|_{L_2(\Gamma_l)}\le r_1. \tag{3.17}$$

To derive the second a priori estimate we differentiate (3.7) with respect to t, multiply by $c_j''(t)$, sum over $j = 1, \dots, m$ and integrate over $(0, \tau)$. So, we obtain

$$
\begin{aligned}
&K(l)\int_0^\tau\int_0^l (u^m_{ttt}u^m_{tt} + au^m_{xt}u^m_{xtt} + cu^m_t u^m_{tt} + a_t u^m_x u^m_{xtt} + c_t u^m u^m_{tt})\,dx\,dt\\
&+\int_0^\tau u^m_{tt}(l,t)\int_0^l Hu^m_t\,dx\,dt + \int_0^\tau u^m_{tt}(l,t)\int_0^l H_t u^m\,dx\,dt\\
&\quad - K'(l)\int_0^\tau a(l,t)u^m_t(l,t)u^m_{tt}(l,t)\,dt - K'(l)\int_0^\tau a_t(l,t)u^m(l,t)u^m_{tt}(l,t)\,dt\\
&\quad + \int_0^\tau u^m_{tt}(l,t)(\alpha(t)u^m(l,t))_{ttt}\,dt\\
&= K(l)\int_0^\tau\int_0^l f_t u^m_{tt}\,dx\,dt - \int_0^\tau u^m_{tt}(l,t)\int_0^l Kf_t\,dx\,dt.
\end{aligned}
\tag{3.18}
$$

Using integrating by parts in (3.18) and taking into account initial data $c_k(0) = c_k'(0) = 0$ we obtain

$$
\begin{aligned}
&K(l)\int_0^l [(u^m_{tt}(x,\tau))^2 + a(u^m_{xt}(x,\tau))^2]\,dx + \alpha(\tau)(u^m_{tt}(l,\tau))^2\\
&+ 5\int_0^\tau \alpha'(t)(u^m_{tt}(l,t))^2\,dt\\
&= K(l)\int_0^l (u^m_{tt}(x,0))^2\,dx + K(l)\int_0^\tau\int_0^l a_t(u^m_{xt})^2\,dx\,dt\\
&\quad + 2K(l)\int_0^\tau\int_0^l a_{tt}u^m_x u^m_{xt}\,dx\,dt - 2K(l)\int_0^l a_t(x,\tau)u^m_x(x,\tau)u^m_{xt}(x,\tau)\,dx\\
&\quad - 2K(l)\int_0^\tau\int_0^l (cu^m_t u^m_{tt} + c_t u^m u^m_{tt})\,dx\,dt\\
&\quad + 2\int_0^\tau u^m_t(l,t)\int_0^l (Hu^m)_{tt}\,dx\,dt + \int_0^\tau [5\alpha'''(t) - 2a_t(l,t)](u^m_t(l,t))^2\,dt\\
&\quad + 2\int_0^\tau [\alpha^{IV} - a_{tt}(l,t)]u^m_t u^m\,dt + 2a_t(l,\tau)u^m(l,\tau)u^m(l,\tau)\\
&\quad - 3\alpha''(\tau)(u^m_t(l,\tau))^2 + 2K(l)\int_0^\tau\int_0^l f_t u^m_{tt}\,dx\,dt\\
&\quad + 2\int_0^\tau u^m_t(l,t)\int_0^l Kf_{tt}\,dx\,dt - 2u^m_t(l,\tau)\int_0^l Kf_t(x,\tau)\,dx.
\end{aligned}
\tag{3.19}
$$

Let us begin to derive the second estimate. This process is complicated by the presence of $u^m_{tt}(x,0)$ but this difficulty can be overcome as follows. Multiplying (3.7) by $c_j''(0)$ and summing over $j = 1, \dots, m$ we obtain

$$
\int_0^l (u^m_{tt}(x,0))^2\,dx = \int_0^l f(x,0)u^m_{tt}(x,0)\,dx.
$$

Hence

$$
\|u^m_{tt}(x,0)\|_{L_2(0,l)} \le \|f(x,0)\|_{L_2(0,l)}. \tag{3.20}
$$

Now using the same technique to derive the first estimate, inequalities (3.16), (3.17), (3.20) and Gronwall's lemma we obtain the second a priori estimate

$$\|u_{tt}^m\|_{L_2(Q_T)} + \|u_{xt}^m\|_{L_2(Q_T)} \le R_2, \quad \|u_{tt}^m(l,t)\|_{L_2(0,T)} \le r_2. \tag{3.21}$$

3: Passage to the limit. Multiplying (3.7) by $d \in C^1(0,T)$ with $d(T) = 0$ and integrating with respect to t over $(0,T)$ we obtain

$$\begin{aligned} &K(l)\int_0^T d(t)\int_0^l (u_{tt}^m w_j + au_x^m w_j' + cu^m w_j)\,dx\,dt \\ &+ \int_0^T d(t)w_j(l)\int_0^l Hu^m\,dx - K'(l)\int_0^T d(t)u^m(l,t)w_j(l)a(l,t)\,dt \\ &- \int_0^T d'(t)(\alpha(t)u^m(l,t))_t w_j(l)\,dt \\ &= K(l)\int_0^T d(t)\int_0^l fw_j\,dx\,dt - \int_0^T d(t)w_j(l)\int_0^l Kf\,dx\,dt. \end{aligned} \tag{3.22}$$

By using (3.16), (3.17) and (3.21) we can extract a subsequence $\{u^\mu\}$ from $\{u^m\}$ such that as $\mu \to \infty$,

$$\begin{gathered} u^\mu \to u \quad \text{weakly in } W(Q_T), \\ u_{tt}^\mu \to u_{tt} \quad \text{weakly in } L_2(Q_T \cup \Gamma_l), \\ u_{tt}^\mu \to u_{tt} \quad \text{weakly in } L_2(Q_T \cup \Gamma_l), \\ u^\mu, u_t^\mu \to u, u_t \quad \text{a.e. on } \Gamma_l, u^\mu(x,0), u_t^\mu(x,0) \to u, u_t \quad \text{a.e. on } (0,l). \end{gathered}$$

Thus, we are able to pass to the limit in (3.22) to obtain

$$\begin{aligned} &K(l)\int_0^T d(t)\int_0^l (u_{tt}w_j + au_x w_j' + cuw_j)\,dx\,dt + \int_0^T d(t)w_j(l)\int_0^l Hu\,dx \\ &- K'(l)\int_0^T d(t)u(l,t)w_j(l)a(l,t)\,dt - \int_0^T d'(t)(\alpha(t)u(l,t))_t w_j(l)\,dt \\ &= K(l)\int_0^T d(t)\int_0^l fw_j\,dx\,dt - \int_0^T d(t)w_j(l)\int_0^l Kf\,dx\,dt. \end{aligned} \tag{3.23}$$

All integrals in (3.23) are defined for any function $d \in C^1(0,T)$, $d(T) = 0$. Taking into account that $\{w_j(x)\}$ is dense in $W_2^1(0,l)$ we conclude that (3.5) holds.

4: Uniqueness. Suppose that u_1 and u_2 are two solutions to (3.1)–(3.4). Then for fixed t and for every function $\omega \in W_2^1(0,l)$ $u = u_1 - u_2$ satisfies $u(x,0) = 0$, $u_t(x,0) = 0$ and the identity

$$\begin{aligned} &K(l)\int_0^l (u_{tt}\omega + au_x\omega_x + cu\omega)\,dx + \omega(l)\int_0^l Hu\,dx \\ &- K'(l)a(l,t)u(l,t)\omega(l) + \omega(l)(\alpha(t)u(l,t))_{tt} = 0. \end{aligned} \tag{3.24}$$

For fixed $t \in [0,T]$ let $\omega(x) = u_t(x,t)$. Then from (3.24),

$$\begin{aligned} &K(l)\frac{\partial}{\partial t}\int_0^l (u_t^2 + u_x^2)\,dx + 2K(l)\int_0^l cuu_t\,dx + 2u_t(l,t)\int_0^l Hu\,dx \\ &- 2K'(l)a(l,t)u(l,t)u_t(l,t) + 2u_t(\alpha(t)u(l,t))_{tt} = 0 \end{aligned}$$

and integrating over $(0,\tau),\ \tau\in[0,T]$, we obtain

$$
\begin{aligned}
&K(l)\int_0^l (u_t^2+u_x^2)|_{t=\tau}\,dx + 2K(l)\int_0^\tau\int_0^l cuu_t\,dx\,dt\\
&+2\int_0^\tau u_t(l,t)\int_0^l Hu\,dx\,dt - 2K'(l)\int_0^\tau a(l,t)u(l,t)u_t(l,t)\,dt\\
&+2\int_0^\tau u_t(\alpha u)_{tt}\,dt = 0.
\end{aligned}
\tag{3.25}
$$

Integrating some terms of (3.25), we obtain

$$
\begin{aligned}
&\int_0^\tau u_t(l,t)\int_0^l Hu\,dx\,dt\\
&= -\int_0^\tau u(l,t)\int_0^l Hu_t\,dx\,dt - \int_0^\tau u(l,t)\int_0^l H_t u\,dx\,dt + u(l,\tau)\int_0^l Hu\,dx,\\
&2\int_0^\tau u_t(l,t)(\alpha(t)u(l,t))_{tt}\,dt\\
&= 3\int_0^\tau \alpha'(t)u_t^2(l,t)\,dt - \int_0^\tau \alpha^{'''}(t)u^2(l,t)\,dt + \alpha(\tau)u_t^2(l,\tau) + \alpha''(\tau)u^2(l,\tau),\\
&-2K'(l)\int_0^\tau a(l,t)u(l,t)u_t(l,t)\,dt = K'(l)\int_0^\tau a_t(l,t)u^2(l,t)\,dt - K'(l)a(l,\tau)u^2(l,\tau).
\end{aligned}
$$

So

$$
\begin{aligned}
&K(l)\int_0^l (u_t^2+u_x^2)|_{t=\tau}\,dx + \alpha(\tau)u_t^2(l,\tau) + 3\int_0^\tau \alpha'(t)u_t^2(l,t)\,dt\\
&= 2\int_0^\tau u(l,t)\int_0^l Hu_t\,dx\,dt + 2\int_0^\tau u(l,t)\int_0^l H_t u\,dx\,dt\\
&\quad - 2u(l,\tau)\int_0^l Hu\,dx + \int_0^\tau \alpha'''(t)u^2(l,t)\,dt - 2K(l)\int_0^\tau\int_0^l cuu_t\,dx\,dt\\
&\quad - K'(l)\int_0^\tau a_t(l,t)u^2(l,t)\,dt + K'(l)a(l,\tau)u^2(l,\tau) - \alpha''(\tau)u^2(l,\tau).
\end{aligned}
\tag{3.26}
$$

To estimate the right-hand side of (3.26) we use the same technique as above in the subsection *a priori estimate*, the inequalities Cauchy, Cauhy-Bunyakovskii as well (3.13) and (3.14). As a result, we obtain

$$
\begin{aligned}
&\int_0^l [u^2(x,\tau)+u_t^2(x,\tau)+u_x^2(x,\tau)]\,dx + \alpha(\tau)u_t^2(l,\tau) + 3\int_0^\tau \alpha'(t)u_t^2(l,t)\,dt\\
&\le A\int_0^\tau\int_0^l [u^2+u_t^2+u_x^2]\,dx\,dt.
\end{aligned}
\tag{3.27}
$$

By using Gronwall's lemma for all $t\in(0,T)$ we obtain

$$
\int_0^l [u^2(x,\tau)+u_t^2(x,\tau)+u_x^2(x,\tau)]\,dx \le 0.
$$

This implies that $u=0$ in Q_T The proof of Theorem 3.2 is complete. $\square$

Remark 3.3. We use homogeneous initial conditions for technical reasons only. Nonhomogeneous initial data also can be considered with little restrictions. In fact, suppose that initial conditions are imposed as follows

$$u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x)$$

where $\varphi,\psi\in W_2^2(0,l)$, $\varphi'(0)=\psi'(0)=0$. Using the transformation $v(x,t)=u(x,t)-\varphi(x)-t\psi(x)$, we obtain

$$\begin{gathered} v_{tt}-(av_x)_x+cv=F(x,t),\\ v(x,0)=0, \quad v_t(x,0)=0, \quad v_x(0,t)=0,\\ \alpha(t)v(l,t)+\int_0^l K(x)v(x,t)\,dx+g(t)=0. \end{gathered}$$

Here $F(x,t)=f(x,t)+(a(x,t)\Phi_x(x,t))_x-c(x,t)\Phi(x,t)$, $\Phi(x,t)=\varphi(x)+t\psi(x)$, $g(t)=\alpha(t)\Phi(l,t)+\int_0^l K(x)\Phi(x,t)\,dx$. If $\int_0^l K(x)\varphi(x)\,dx=0$, $\int_0^l K(x)\psi(x)\,dx=0$ the compatibility conditions

$$\int_0^l K(x)v(x,0)\,dx+g(0)=0, \quad \int_0^l K(x)v_t(x,0)\,dx+g'(0)=0$$

(as $\alpha(0)=\alpha'(0)=0$) holds. The nonhomogeneous nonlocal condition can be reduced to the following dynamical nonlocal condition

$$\begin{aligned} &K(l)a(l,t)v_x(l,t)-K'(l)a(l,t)v(l,t)+(\alpha(t)v(l,t))_{tt}+\int_0^l H(x,t)v(x,t)\,dx\\ &+\int_0^l K(x)F(x,t)\,dx+\alpha''(t)\Phi(l,t)+2\alpha'(t)\Phi'(l,t)=0. \end{aligned}$$

If $\varphi,\psi\in W_2^2(0,l)$, then $F,F_t\in L_2(Q_T)$ and we are able to obtain necessary a priori estimates and pass to limit by method of Section 3. Of course, nonhomogeneous initial data complicates calculations, but does not affect the final result.

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