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THEOREMS ON BOUNDEDNESS OF SOLUTIONS TO STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

YOUSSEF N. RAFFOUL, DAN REN

Abstract. In this report, we provide general theorems about boundedness or bounded in probability of solutions to nonlinear delay stochastic differential systems. Our analysis is based on the successful construction of suitable Lyapunov functionals. We offer several examples as application of our theorems.

1. Introduction

Stochastic differential delay equations (SDDEs) have a wide application in natural or man-made systems. They arise from an approximation to a partial differential equations that describes, e.g., diffusion on some reacting substance or a traveling wave in some medium, see [1]. Moreover, time delay stochastic systems are an important aspect in the modeling of genetic regulation due to slow biochemical reactions such as gene transcription and translation, and protein diffusion between the cytosol and nucleus, for a reference, see [18]. Recently, (SDDEs) have been extensively used in the study of population dynamics and for more on this we refer to [3] and [5]. For more reading on the subject of stochastic systems, we refer to Kushner [6], Mao [7, 8], Hasminskii [2] and the references therein. The report of [9] presents an interesting survey of Lyapunov functions techniques in stochastic differential equation.

Let $B(t) = (B_1(t), B_2(t), \dots, B_m(t))^T$ be an m-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathbf{F}, \mathbf{P})$ and

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n.$$

Let h be a positive constant and we will consider the n-dimensional stochastic differential equation (SDE):

$$\begin{aligned} dx(t) = f\Big(t, x(t), x(t-h), \int_{t-h}^{t} A(t,s)h(s,x(s))ds\Big)dt \\ + g(t, x(t), x(t-h))dB(t), \end{aligned} \tag{1.1}$$

for $t \in \mathbb{R}^+$, with a given deterministic continuous initial condition $\phi : [-h, 0] \to \mathbb{R}^n$, where $A : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $g : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, are all continuous functions on their own domains.

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It should cause no difficulty to start our solution at any time t_0, but for simplicity, we consider the case when $t_0 = 0$ in this paper, i.e., $x(t) = \phi(t)$ for all $t \in [-h, 0]$, and x satisfies the SDE (1.1) starting from time 0.

Let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ denote the set of all non-negative functions that are twice continuously differentiable in the first variable and continuously differentiable in the second variable. Take an arbitrary function $V(t, x)$ in the set, and define the operator $\mathcal{L}V : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{L}V(t,x,y,z) &= V_t(t,x) + \sum_{i=1}^{n} V_{x_i}(t,x) f_i(t,x,y,z) \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} V_{x_i x_j}(t,x) g_{ik}(t,x,y) g_{jk}(t,x,y), \end{aligned} \tag{1.2}$$

where f_i is the ith component of vector f and g_{ij} is the ij-entry of matrix g.

To avoid confusion, we will denote $\mathcal{L}V(t, x, y, z)$, the operator applied to $V(t, x)$, as $\mathcal{L}V(t, x)$. However, readers should keep in mind that $\mathcal{L}V(t, x)$ actually depends on y and z through f and g.

In this article we say Lyapunov functional instead of Lyapunov function to indicate the presence of the variable x in the integrand.

This research is an extension of [20], in which the authors developed a general theory for the analysis of solutions of the nonlinear systems of stochastic differential equation of the form

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), \quad t \geq 0.$$

with initial condition $x(0)$ being a constant.

In our analysis, we obtain inequalities from which we can deduce boundedness in probability of the solutions of (SDDEs) (1.1). In obtaining our inequalities we resort to the suitable construction of Lyapunov functionals. Thus, this paper provides step by step instruction on how these Lyapunov functionals can be easily constructed so that under suitable conditions, the Lyapunov functional is decreasing along the solutions of the (SDDEs) . In addition, our results will be applied to nonlinear stochastic systems with the function f containing terms of the form x^n where n is positive and rational. For a comprehensive review and recent results of stochastic differential and integro-differential equations, we refer the reader to the excellent monograph [13] and to the references therein. For more on (SDDEs), we refer to the survey paper [4].

2. Boundedness of solutions

In this section, we use non-negative definite Lyapunov functionals and establish sufficient conditions to obtain boundedness in probability results on all solutions x of (1.1). The use of an initial function instead of an initial point allows us to observe the past performance of the random variables over a longer period.

Frist, we state some notation and assumptions which will be needed for the rest of the paper.

Let E^ϕ denote the conditional expectation operator associated with the probability measure $\mathbf{P}$ given $x(t) = \phi(t)$ for all $t \in [-h, 0]$, i.e., $E^\phi(\cdot) = E(\cdot \mid x(t) = \phi(t), t \in [-h, 0])$. Let $\| \cdot \|$ denote the Euclidean norm for a vector in $\mathbb{R}^n$.

Definition 2.1. A continuous function $W : \mathbb{R}^+ \to \mathbb{R}^+$ is called a wedge, if $W(0) = 0$, $W(s) > 0$ for $s > 0$, and W is strictly increasing.

In this article a wedge is always denoted as W or W_i where i is a positive integer. We sue the following assumption:

(A1) For any fixed $t \in \mathbb{R}^+$, the following condition holds for all $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$:
$$E^{\phi}\Big[\int_0^t V_{x_i}^2(t, x(s)) g_{ik}^2(s, x(s), x(s-h))\, ds\Big] < \infty. \tag{2.1}$$

A special case of the general condition (2.1) is the following condition.

(A1') There exists a deterministic function $\nu(t)$ such that for all $t \in \mathbb{R}^+, x \in \mathbb{R}^n, y \in \mathbb{R}^n$, and all $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$:
$$\|V_{x_i}(t, x) g_{ik}(t, x, y)\| \le \nu(t) \tag{2.2}$$
and for any fixed $t \in \mathbb{R}^+$,
$$\int_0^t \nu^2(s)\, ds < \infty. \tag{2.3}$$

Theorem 2.2. *Under assumption* (A1)*, suppose that there exists a continuously differentiable Lyapunov functional*
$$V(t, x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$$
such that for all $x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}$ and $t \in \mathbb{R}^+$:
$$W(\|x\|) \le V(t, x), \tag{2.4}$$
$$\mathcal{L}V(t, x) \le -\alpha(t) V^q(t, x) + F(t), \tag{2.5}$$
$$V(t, x) - V^q(t, x) \le \gamma, \tag{2.6}$$
where γ and q are constants with $\gamma \ge 0$, $q \ge 1$, and $\alpha(t)$ and $F(t)$ are positive continuous functions. Then all solutions x of (1.1) *satisfy the following inequality for any given $t \in \mathbb{R}^+$:*
$$E^{\phi}(\|x(t)\|) \le W^{-1}\Big[e^{-\int_0^t \alpha(u)du} V(0, \phi(0)) + \int_0^t e^{-\int_u^t \alpha(s)ds}\big[\gamma\alpha(u) + F(u)\big]du\Big]. \tag{2.7}$$

Proof. Apply Itô's formula to $e^{\int_0^t \alpha(s)ds} V(t, x(t))$:
$$\begin{aligned} &d\left(e^{\int_0^t \alpha(s)ds} V(t, x(t))\right) \\ &= e^{\int_0^t \alpha(s)ds}(\alpha(t)V(t, x(t)) + \mathcal{L}V(t, x(t)))dt + dM(t) \end{aligned}$$
with
$$M(t) = \int_0^t e^{\int_0^u \alpha(s)ds} \sum_{i=1}^n V_{x_i}(u, x(u)) \sum_{k=1}^m g_{ik}(u, x(u), x(u-h)) dB_k(u),$$
and (for short notation)
$$\mathcal{L}V(t, x(t)) = \mathcal{L}V\big(t, x(t), x(t-h), \int_{t-h}^t A(t, s) h(s, x(s)) ds\big).$$
Integrate both sides from 0 to t:
$$e^{\int_0^t \alpha(s)ds} V(t, x(t)) - V(0, x(0))$$

$$
\begin{aligned}
&= \int_0^t e^{\int_0^u \alpha(s)ds}(\alpha(u)V(x(u),u) + \mathcal{L}V(x(u),u))du + M(t) \\
&\le \int_0^t e^{\int_0^u \alpha(s)ds}(\alpha(u)(V(x(u),u) - V^q(x(u),u)) + F(u))du + M(t) \quad (by\ (2.5)) \\
&\le \int_0^t e^{\int_0^u \alpha(s)ds}(\alpha(u)\gamma + F(u))du + M(t) \quad (by\ (2.6))
\end{aligned}
$$

Since $V(0,x(0)) = V(0,\phi(0))$, it follows that

$$
\begin{aligned}
V(t,x(t)) &\le e^{-\int_0^t \alpha(s)ds}V(0,\phi(0)) \\
&\quad + \int_0^t e^{-\int_u^t \alpha(s)\,ds}(\gamma\alpha(u) + F(u))du + e^{-\int_0^t \alpha(s)ds}M(t)
\end{aligned}
$$

Taking expectation E^ϕ on both sides, and noting that $E^\phi\{e^{-\int_0^t \alpha(s)ds}M(t)\} = 0$ under (A1), we obtain

$$
E^\phi[V(t,x(t))] \le e^{-\int_0^t \alpha(s)\,ds}V(0,\phi(0)) + \int_0^t e^{-\int_u^t \alpha(s)\,ds}\Big(\gamma\alpha(u) + F(u)\Big)\,du. \tag{2.8}
$$

Finally, since W is convex, by Jensen's inequality for expectation, we have

$$
W(E^\phi[\|x(t)\|]) \le E^\phi[W(\|x(t)\|)] \le E^\phi[V(t,x(t))].
$$

The proof is completed by noting that W is strictly increasing. □

Considering another situation, we have the following theorem.

Theorem 2.3. *Under assumption* (A1)*, suppose there exists a continuously differentiable Lyapunov functional*

$$
V(t,x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)
$$

such that for all $t \in \mathbb{R}^+$*,*

$$
W_1(\|x(t)\|) \le V(t,x(t)) \le W_2(\|x(t)\|) + \int_{-h}^t \varphi_1(t,s)W_3(\|x(s)\|)ds, \tag{2.9}
$$

$$
\mathcal{L}V(t,x(t)) \le -\alpha_1(t)W_4(\|x(t)\|) - \alpha_2(t)\int_{-h}^t \varphi_2(t,s)W_5(\|x(s)\|)ds + F(t) \tag{2.10}
$$

for positive continuous functions $\alpha_1(t), \alpha_2(t)$*,* $F(t)$*, and* $\varphi_i(t,s)$*,* $i = 1,2$*.*

Moreover, there exists a non-negative constant γ *such that the inequality*

$$
\begin{aligned}
&W_2(\|x(t)\|) - W_4(\|x(t)\|) \\
&+ \int_{-h}^t \Big(\varphi_1(t,s)W_3(\|x(s)\|) - \varphi_2(t,s)W_5(\|x(s)\|)\Big)ds \le \gamma
\end{aligned} \tag{2.11}
$$

holds for all $t \in \mathbb{R}^+$*. Then all solutions of* (1.1) *satisfy*

$$
\begin{aligned}
E^\phi(\|x(t)\|) \le W_1^{-1}\Big\{&e^{-\int_0^t \alpha(s)\,ds}V(0,\phi(0)) \\
&+ \int_0^t e^{-\int_u^t \alpha(s)\,ds}\Big(\gamma\alpha(u) + F(u)\Big)du\Big\}, \quad \forall t \ge 0,
\end{aligned} \tag{2.12}
$$

where $\alpha(t) = \min\{\alpha_1(t), \alpha_2(t)\}$*.*

Proof. Let $x(t)$ be a solution of (1.1) with $x(t)=\phi(t)$, for $-h\le t\le 0$. Then

$$\begin{aligned} &d\Big(e^{\int_0^t\alpha(s)ds}V(t,x(t))\Big)\\ &=e^{\int_0^t\alpha(s)ds}(\alpha(t)V(t,x(t))+\mathcal{L}V(t,x(t)))dt+dM(t) \end{aligned} \tag{2.13}$$

where

$$M(t)=\int_0^t e^{\int_0^u\alpha(s)ds}\sum_{i=1}^n V_{x_i}(u,x(u))\sum_{k=1}^m g_{ik}(u,x(u),x(u-h))dB_k(u),$$

$$\mathcal{L}V(t,x(t))=\mathcal{L}V\big(t,x(t),x(t-h),\int_{t-h}^t A(t,s)h(s,x(s))ds\big).$$

By (2.9), (2.10), (2.11) and the fact that $\alpha(t)=\min\{\alpha_1(t),\alpha_2(t)\}$, we obtain

$$\begin{aligned} &\mathcal{L}V(t,x(t))+\alpha(t)V(t,x(t))\\ &\le-\alpha_1(t)W_4(\|x(t)\|)-\alpha_2(t)\int_{-h}^t\varphi_2(t,s)W_5(|x(s)|)ds\\ &\quad+\ \alpha(t)W_2(\|x(t)\|)+\alpha(t)\int_{-h}^t\varphi_1(t,s)W_3(\|x(s)\|)ds+F(t)\\ &\le\alpha(t)\Big[W_2(\|x(t)\|)-W_4(\|x(t)\|)\\ &\quad+\ \int_{-h}^t\Big(\varphi_1(t,s)W_3(\|x(s)\|)-\varphi_2(t,s)W_5(\|x(s)\|)\Big)ds\Big]+F(t)\\ &\le\gamma\alpha(t)+F(t). \end{aligned} \tag{2.14}$$

Integrating (2.13) from 0 to t, combining with (2.14), we obtain

$$e^{\int_0^t\alpha(s)\,ds}V(t,x(t))\le V(0,\phi(0))+\int_0^t e^{\int_0^u\alpha(s)\,ds}\Big(\gamma\alpha(u)+F(u)\Big)du+M(t).$$

Dividing by $e^{\int_0^t\alpha(s)\,ds}$, taking expectation on both sides, and noting that

$$E^\phi\Big\{e^{-\int_0^t\alpha(s)\,ds}M(t)\Big\}=0$$

in view of (A1), we obtain,

$$E^\phi[V(t,x(t))]\le e^{-\int_0^t\alpha(s)\,ds}V(0,\phi(0))+\int_0^t e^{-\int_u^t\alpha(s)\,ds}\Big(\gamma\alpha(u)+F(u)\Big)du.$$

Finally, since W_1 is convex, by Jensen's inequality for expectation, we have

$$W_1(E^\phi\|x(t)\|)\le E^\phi[W_1(\|x(t)\|)]\le E^\phi[V(t,x(t))].$$

The proof is complete by noting that W_1 is strictly increasing. □

We now consider some special 1-dimensional stochastic processes x, and state relevant results in Theorem 2.4 and 2.5. A sufficient condition for both theorems is stated below: We sue the following assumptin

(A2) Assume that there exist positive functions $\nu,F:\mathbb{R}^+\to\mathbb{R}^+$, such that for all $t\in\mathbb{R}^+,x\in\mathbb{R},y\in\mathbb{R}$:

$$|xg(t,x,y)|\le\nu(t),\quad g^2(t,x,y)\le F(t), \tag{2.15}$$

$$\int_0^t\nu^2(s)ds<\infty,\forall t\ge 0. \tag{2.16}$$

First we consider a particular form of (1.1) given in the (SDDE),

$$dx(t) = \Big(a(t)x(t) + b(t)x(t-h)\Big)dt + g(t, x(t), x(t-h))dB(t), \quad t \ge 0, \tag{2.17}$$

with continuous initial condition ϕ, i.e., any solution of the above SDDE satisfying $x(t) = \phi(t)$, $t \in [-h, 0]$. We have the following theorem.

Theorem 2.4. *Suppose that* (A2) *holds for some functions* ν *and* F*. We define the function*

$$\xi(t) = \frac{e^{\int_0^t 2a(s)ds}}{1 + 2h\int_t^{t+\tau} e^{\int_0^u 2a(s)ds}du}, \tag{2.18}$$

for some constant $\tau \ge 0$*, such that for all* $t \in \mathbb{R}^+$*:*

$$|b(t)| \le k\xi(t), \tag{2.19}$$

for some positive constant k*, and*

$$2a(t) + 2k\xi(t) \le -\alpha(t), \tag{2.20}$$

for some positive continuous function $\alpha(t)$*. Then all solutions of* (2.17) *satisfy*

$$E^{\phi}(|x(t)|) \le \Big\{e^{-\int_0^t \alpha(s)ds}V(0) + \int_0^t e^{-\int_u^t \alpha(s)ds}F(u)du\Big\}^{1/2}, \tag{2.21}$$

with

$$V(0) = \phi^2(0) + k\xi(0)\int_{-h}^0 \phi^2(s)ds.$$

Proof. For $t \ge 0$, define a process V via:

$$V(t) = x^2(t) + k\xi(t)\int_{t-h}^t x^2(s)ds.$$

Apply Itô's formula to $e^{\int_0^t \alpha(s)ds}V(t)$:

$$d\Big(e^{\int_0^t \alpha(s)ds}V(t)\Big) = e^{\int_0^t \alpha(s)ds}(\alpha(t)V(t) + \mathcal{L}V(t))dt + dM(t) \tag{2.22}$$

where

$$\begin{aligned} \mathcal{L}V(t) &= (2a(t) + k\xi(t))x^2(t) + 2b(t)x(t)x(t-h) - k\xi(t)x^2(t-h) \\ &\quad + k\xi'(t)\int_{t-h}^t x^2(s)\,ds + g^2(t, x(t), x(t-h)), \end{aligned} \tag{2.23}$$

$$M(t) = 2\int_0^t e^{\int_0^u \alpha(s)ds}x(u)g(u, x(u), x(u-h))dB(u). \tag{2.24}$$

Moreover, by (A2), (2.18), (2.19), and the fact that

$$\begin{aligned} \xi'(t) &= 2a(t)\xi(t) + 2k\xi^2(t) - 2k\xi^2(t)e^{\int_t^{t+\tau} 2a(s)\ ds} \\ &\le 2a(t)\xi(t) + 2k\xi^2(t)2b(t)x(t)x(t-h) \\ &\le |b(t)|(x^2(t) + x^2(t-h)), \end{aligned}$$

equation (2.23) reduces to

$$
\begin{aligned}
\mathcal{L}V(t) &\le \Big(2a(t)+|b(t)|+k\xi(t)\Big)x^2(t)+\Big(|b(t)|-k\xi(t)\Big)x^2(t-h)\\
&\quad +\ k\Big(2a(t)\xi(t)+2k\xi^2(t)\Big)\int_{t-h}^{t}x^2(s)ds+F(t)\\
&\le \Big(2a(t)+2k\xi(t)\Big)x^2(t)+k\xi(t)\Big(2a(t)+2k\xi(t)\Big)\int_{t-h}^{t}x^2(s)ds+F(t)\\
&= \Big(2a(t)+2k\xi(t)\Big)\Big[x^2(t)+k\xi(t)\int_{t-h}^{t}x^2(s)ds\Big]+F(t)\\
&\le -\alpha(t)V(t)+F(t).
\end{aligned}
\tag{2.25}
$$

We integrate from 0 to t, divide by $e^{\int_0^t\alpha(s)ds}$, and then take expectation on both sides of (2.22). It follows that

$$
\begin{aligned}
&E^{\phi}\big(V(t)\big)-e^{-\int_0^t\alpha(s)ds}V(0)\\
&\le E^{\phi}\Big[\int_0^t e^{-\int_u^t\alpha(s)ds}F(u)du\Big]+E^{\phi}\Big[e^{-\int_0^t\alpha(s)ds}M(t)\Big]
\end{aligned}
\tag{2.26}
$$

Since under Assumption (A2), $e^{-\int_0^t\alpha(s)ds}M(t)$ is a martingale with expectation 0, we have

$$
E^{\phi}\big(v(t)\big)\le e^{-\int_0^t\alpha(s)ds}V(0)+E^{\phi}\Big[\int_0^t e^{-\int_u^t\alpha(s)ds}F(u)du\Big] \tag{2.27}
$$

The proof is complete by observing that

$$
E^{\phi}(|x(t)|)\le (E^{\phi}(x^2(t)))^{1/2}\le (E^{\phi}(V(t)))^{1/2}.
$$

$\square$

Next, we turn our attention to the (SDDE)

$$
\begin{aligned}
dx(t)&=\Big(a(t)x(t)+b(t)x(t-h)+\int_{t-h}^{t}A(t,s)f(s,x(s))ds\Big)dt\\
&\quad+g(t,x(t),x(t-h))dB(t),\quad t\ge 0,
\end{aligned}
\tag{2.28}
$$

with given continuous initial condition ϕ, such that $x(t)=\phi(t)$, $t\in[-h,0]$.

Theorem 2.5. *Assume that* (A2) *holds. Let $f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be a nonlinear continuous function, and a,b and A are assumed to be continuous on their respective domains. Suppose that there exists a positive constant λ such that for all $t\in\mathbb{R}^+, x\in\mathbb{R}$,*

$$
|f(t,x)|\le\lambda|x|\,. \tag{2.29}
$$

Let k be a positive constant and assume that there are two positive constants α_1 and α_2 so that for all $t\in\mathbb{R}^+$:

$$
2a(t)+1+b^2(t)+k\int_t^{\infty}|A(u,t)|du\le-\alpha_1, \tag{2.30}
$$

$$
1-k\int_t^{\infty}|A(u,t-h)|du\le 0, \tag{2.31}
$$

$$
\lambda^2\int_{t-h}^{t}|A(t,s)|ds-k\le-\alpha_2 k. \tag{2.32}
$$

Finally, we assume that there exists some positive constant β, such that for all $s, t(s \le t)$,

$$|A(t,s)| \ge \beta \int_t^\infty |A(u,s)| du. \tag{2.33}$$

Then all solutions of (2.28) satisfy

$$E^\phi(|x(t)|) \le \Big\{ V(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-u)} F(u) du \Big\}^{1/2}, \tag{2.34}$$

with

$$\alpha = \min\{\alpha_1,\ \alpha_2\beta\},$$

$$V(0) = \phi(0)^2 + k\int_{-h}^0 \int_0^\infty |A(u,s)| du \phi^2(s) ds.$$

Proof. To see this we define a process

$$V(t) = x^2(t) + k\int_{t-h}^t \int_t^\infty |A(u,s)| du x^2(s) ds.$$

Apply Itô's formula to $e^{\alpha t}V(t)$:

$$d\left(e^{\alpha t}V(t)\right) = e^{\alpha t}(\alpha V(t) + \mathcal{L}V(t))dt + dM(t) \tag{2.35}$$

where

$$M(t) = 2\int_0^t e^{\alpha s} x(s) g(s, x(s), x(s-h)) dB(s)$$

$$\begin{aligned}
\mathcal{L}V(t) &= 2x(t)\Big(a(t)x(t) + b(t)x(t-h) + \int_{t-h}^t A(t,s) f(s,x(s)) ds\Big) \\
&\quad + kx^2(t)\int_t^\infty |A(u,t)| du - kx^2(t-h)\int_t^\infty |A(u,t-h)| du \\
&\quad - k\int_{t-h}^t |A(t,s)| x^2(s) ds + g^2(t, x(t), x(t-h)) \\
&\le 2a(t)x^2(t) + 2b(t)x(t)x(t-h) + 2x(t)\int_{t-h}^t A(t,s) f(s,x(s)) ds \\
&\quad + kx^2(t)\int_t^\infty |A(u,t)| du - kx^2(t-h)\int_t^\infty |A(u,t-h)| du \\
&\quad - k\int_{t-h}^t |A(t,s)| x^2(s) ds + F(t)
\end{aligned}$$

Using the fact that $2ab \le a^2 + b^2$, and Schwartz inequality we simplify the following terms:

$$2b(t)x(t)x(t-h) \le b^2(t)x^2(t) + x^2(t-h),$$

$$\begin{aligned}
2x(t)\int_{t-h}^t A(t,s) f(s,x(s)) ds &\le x^2(t) + \Big(\int_{t-h}^t A(t,s) f(s,x(s)) ds\Big)^2 \\
&\le x^2(t) + \lambda^2\Big(\int_{t-h}^t |A(t,s)|^{\frac12} |A(t,s)|^{\frac12} |x(s)| ds\Big)^2 \\
&\le x^2(t) + \lambda^2 \int_{t-h}^t |A(t,s)| ds \int_{t-h}^t |A(t,s)| x^2(s) ds.
\end{aligned}$$

Using the above two inequalities and conditions (2.30)-(2.33) we arrive at

$$\begin{aligned}\mathcal{L}V(t,x) &\le \Big(2a(t)+1+b^2(t)+k\int_t^\infty |A(u,t)|du\Big)x^2(t)\\ &\quad+\Big(1-k\int_t^\infty |A(u,t-h)|\Big)x^2(t-h)\\ &\quad+\Big(\lambda^2\int_{t-h}^t |A(t,s)|ds-k\Big)\int_{t-h}^t |A(t,s)|x^2(s)ds+F(t)\\ &\le -\alpha_1x^2(t)-\alpha_2\beta k\int_{t-h}^t\int_t^\infty |A(u,s)|dux^2(s)ds+F(t)\\ &\le -\alpha V(t,x)+F(t).\end{aligned}$$

where $\alpha=\min(\alpha_1,\alpha_2\beta)$.

We integrate from 0 to t, divide by $e^{\alpha t}$, and then take expectation on both sides of (2.35). It follows that

$$E^\phi\big(v(t)\big)-e^{-\alpha t}V(0)\le\int_0^t e^{-\alpha(t-u)}F(u)du+E^\phi[e^{-\alpha t}M(t)] \tag{2.36}$$

Again, by (A2), $e^{-\alpha t}M(t)$ is a martingale with expectation 0. Therefore,

$$E^\phi\big(v(t)\big)\le e^{-\alpha t}V(0)+\int_0^t e^{-\alpha(t-u)}F(u)du \tag{2.37}$$

The proof is complete by observing that

$$E^\phi(|x(t)|)\le(E^\phi(x^2(t)))^{1/2}\le(E^\phi(V(t)))^{1/2}.$$

□

3. Examples

In this section, we provide several examples as the application of the results we obtained in the previous section. To illustrate the application of Theorem 2.2, we consider the following two dimensional stochastic system of nonlinear Volterra integro-differential equations.

Example 3.1. Given continuous scalar functions $A_i:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$, $f_i:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ $(i=1,2)$, such that $|A_1(\cdot,\cdot)|\ge|A_2(\cdot,\cdot)|$, $f_1(\cdot,\cdot)\ge 0$ and $|f_2(\cdot,\cdot)|\le f_1(\cdot,\cdot)$. Suppose the process $y=(y_1,y_2)$ satisfies the SDDE

$$\begin{aligned}dy_1(t)&=\Big(y_2(t)-y_1(t)|y_1(t)|-y_1(t)\,y_2^2(t)\int_{t-h}^t|A_1(t,s)|f_1(y_1(s),y_2(s))ds\Big)dt\\ &\quad+g_{11}(t,y(t),y(t-h))dB_1(t)+g_{12}(t,y(t),y(t-h))dB_2(t),\\ dy_2(t)&=\Big(-y_1(t)-y_2(t)|y_2(t)|+y_1^2(t)\,y_2(t)\int_{t-h}^t A_2(t,s)f_2(y_1(s),y_2(s))ds\Big)dt\\ &\quad+g_{21}(t,y(t),y(t-h))dB_1(t)+g_{22}(t,y(t),y(t-h))dB_2(t),\end{aligned}$$

and $(y_1(t),y_2(t))=(\varphi_1(t),\varphi_2(t))$, $(-h\le t\le 0)$, for some given initial continuous functions $\varphi_1(t),\varphi_2(t)$.

We assume that the functions $g_{ij} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(i, j = 1, 2)$ are given nonlinear continuous functions and for some $M(t)$ and $N(t)$:

$$K^2(t, y(t), y(t-h)) := \sum_{i,j=1}^{2} g_{ij}^2(t, y(t), y(t-h)) \leq M(t)$$

$$\sum_{i,j=1}^{2} y_i(t) g_{ij}^2(t, y(t), y(t-h)) \leq N(t)$$

where $\int_0^t N(s)ds < \infty$, for all $t \in \mathbb{R}^+$.

Take $V(t, y(t)) := \frac{1}{2}(y_1^2(t) + y_2^2(t))$, $W(y) := \frac{1}{2}y^2$. Then $W(|y|) \leq V(y, t)$, and

$$\begin{aligned}
&\mathcal{L}V(t, y(t)) \\
&= -y_1^2|y_1| - y_2^2|y_2| + \frac{1}{2}K^2(t, y(t), y(t-h)) \\
&\quad - y_1^2 y_2^2 \Big(\int_{t-h}^{t} |A_1(t,s)| f_1(y_1(s), y_2(s)) ds - \int_{t-h}^{t} A_2(t,s) f_2(y_1(s), y_2(s)) ds \Big) \\
&\leq -\big(|y_1|^3 + |y_2|^3\big) + y_1^2 y_2^2 \int_{t-h}^{t} \Big(|A_2(t,s)| - |A_1(t,s)| \Big) f_1(y_1(s), y_2(s)) ds + \frac{M(t)}{2} \\
&\leq -2\Big[\frac{|y_1|^3}{2} + \frac{|y_2|^3}{2}\Big] + \frac{M(t)}{2} \\
&= -2\Big[\frac{(|y_1|^2)^{3/2}}{2} + \frac{(|y_2|^2)^{3/2}}{2}\Big] + \frac{M(t)}{2} \\
&\leq -2\big(|y_1|^2 + |y_2|^2\big)^{3/2} 2^{-3/2} + \frac{M(t)}{2} \\
&= -2V^{\frac{3}{2}}(y(t), t) + \frac{M(t)}{2}.
\end{aligned}$$

where we have used the inequality $(\frac{a+b}{2})^l \leq \frac{a^l}{2} + \frac{b^l}{2}$, $a, b > 0$, $l > 1$. Here y_i $(i = 1, 2)$ are the short notations for $y_i(t)$. Then $\alpha(t) = 2$, $q = \frac{3}{2}$, $F(t) = \frac{M(t)}{2}$.

Next, by straightforward calculation,

$$\begin{aligned}
V(t, y) - V^q(t, y) &= V(t, y) - V^{\frac{3}{2}}(t, y) \\
&= y_1^2 + y_2^2 - (y_1^2 + y_2^2)^{\frac{3}{2}} 2^{-3/2} \leq \frac{4}{27}.
\end{aligned}$$

Hence, we have $\gamma = \frac{4}{27}$. By Theorem 2.2, all solutions of the above two dimensional stochastic system satisfy

$$\begin{aligned}
&E^{\varphi_1, \varphi_2}\Big[\sqrt{y_1^2(t) + y_2^2(t)}\Big] \\
&\leq \Big(2\Big\{ \frac{1}{2}(\varphi_1^2(0) + \varphi_2^2(0)) e^{-\int_0^t 2ds} + \int_0^t \big[\frac{4}{27}(2) + \frac{M(u)}{2}\big] e^{-\int_u^t 2ds} du \Big\} \Big)^{1/2} \\
&= \Big\{ e^{-2t}\Big[\varphi_1^2(0) + \varphi_2^2(0) + \int_0^t e^{2u} M(u)\, du\Big] + \frac{8}{27}(1 - e^{-2t}) \Big\}^{1/2}.
\end{aligned}$$

Next we present an example of Theorem 2.3.

Example 3.2. Let $\phi(t)$ be a given bounded continuous initial function and consider the scalar stochastic Volterra integro-differential equation

$$
\begin{gathered}
dx(t) = \Big(a(t)x(t) + \int_{t-h}^{t} A(t,s)(x(s))^{2/3}ds\Big)dt + g(t,x(t),x(t-h))dB(t), \quad t \geq 0 \\
x(t) = \phi(t) \quad \text{for } -h \leq t \leq 0,
\end{gathered} \tag{3.1}
$$

where $g : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given nonlinear continuous function, which satisfies (A2) for some functions M and F. Also, the functions $a : \mathbb{R}^+ \to \mathbb{R}$ and $A : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, are assumed to be continuous on their respective domains. If

$$
-2a(t) - \int_{t-h}^{t} |A(t,s)|ds - \int_{t}^{\infty} |A(u,t)|du > 0,
$$

and

$$
\frac{|A(t,s)|}{3} \geq \int_{t}^{\infty} |A(u,s)|du
$$

then all solutions of (3.1) satisfy inequality (2.12) with $\gamma = 0$, where

$$
\alpha(t) = \min\{-2a(t) - \int_{t-h}^{t} |A(t,s)|ds - \int_{t}^{\infty} |A(u,t)|du,\ 1\},
$$

and

$$
V(0) = \phi^2(0) + \int_{-h}^{0}\int_{0}^{\infty} |A(u,s)|du\phi^2(s)ds.
$$

To see this we define the process

$$
V(t) = x^2(t) + \int_{-h}^{t}\int_{t}^{\infty} |A(u,s)|du x^2(s)ds.
$$

which satisfy (2.9) with $W_1(x) = W_2(x) = W_3(x) = x^2$, $\phi_1(t,s) = \int_t^\infty |A(u,s)|du$; Then along solutions of (3.1) we have

$$
\begin{aligned}
\mathcal{L}V(t) &= 2x(t)\Big(a(t)x(t) + \int_{t-h}^{t} A(t,s)(x(s))^{2/3}ds\Big)dt + g^2(t,x(t),x(t-h)) \\
&\quad + \int_{t}^{\infty} |A(u,t)|x^2(t)du - \int_{-h}^{t} |A(t,s)|x^2(s)ds \\
&\leq 2a(t)x^2(t) + 2\int_{t-h}^{t} |A(t,s)|\,|x(t)|(x(s))^{2/3}ds + F(t) \\
&\quad + \int_{t}^{\infty} |A(u,t)|x^2(t)du - \int_{-h}^{t} |A(t,s)|x^2(s)ds.
\end{aligned}
$$

Using the fact that $ab \leq a^2/2 + b^2/2$, the above inequality simplifies to

$$
\begin{aligned}
\mathcal{L}V(t) &\leq 2a(t)x^2(t) + \int_{t-h}^{t} |A(t,s)|(x^2(t) + x^{4/3}(s))ds \\
&\quad + \int_{t}^{\infty} |A(u,t)|x^2(t)du - \int_{-h}^{t} |A(t,s)|x^2(s)ds + F(t)
\end{aligned} \tag{3.2}
$$

To further simplify (3.2) we use Young's inequality, which says for any two nonnegative real numbers w and z, we have

$$
wz \leq \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } \frac{1}{e} + \frac{1}{f} = 1.
$$

Thus, for $e = 3$ and $f = 3/2$, we obtain

$$\begin{aligned}\int_{t-h}^{t} |A(t,s)| x^{4/3}(s)ds &= \int_{t-h}^{t} |A(t,s)|^{1/3} |A(t,s)|^{2/3} x^{4/3}(s)ds \\ &\le \int_{t-h}^{t} \Big(\frac{|A(t,s)|}{3} + \frac{2}{3}|A(t,s)|x^2(s)\Big)ds. \\ &\le \int_{t-h}^{t} \frac{|A(t,s)|}{3} ds + \int_{-h}^{t} \frac{2}{3}|A(t,s)|x^2(s)ds.\end{aligned}$$

Then by substituting the above inequality into (3.2), we arrive at

$$\begin{aligned}\mathcal{L}V(t) &\le \Big(2a(t) + \int_{t-h}^{t} |A(t,s)|ds + \int_{t}^{\infty} |A(u,t)|du\Big)x^2(t) \\ &\quad - \int_{-h}^{t} \frac{|A(t,s)|}{3} x^2(s)ds + \frac{1}{3}\int_{t-h}^{t} |A(t,s)|ds + F(t)\end{aligned}$$

Then (2.10) holds for

$$W_4(x) = W_5(x) = x^2, \quad \phi_2(t,x) = \frac{|A(t,s)|}{3}$$

$$\alpha_1(t) = -2a(t) - \int_{t-h}^{t} |A(t,s)|ds - \int_{t}^{\infty} |A(u,t)|du > 0, \quad \alpha_2(t) = 1$$

and $F(t)$ replaced by $L(t) = \frac{1}{3}\int_{t-h}^{t} |A(t,s)|ds + F(t)$.

It is obvious that (2.11) holds for $\gamma = 0$. Hence by (2.34), all solutions of (3.1) satisfy:

$$E^{\phi}(\|x(t)\|) \le \Big[e^{-\int_0^t \alpha(u)du} V(0,\phi(0)) + \int_0^t e^{-\int_u^t \alpha(s)ds} L(u)du\Big]^{1/2}.$$

where $\alpha(t) = \min(\alpha_1(t),$

$$\alpha_2(t)) = \min(-2a(t) - \int_{t-h}^{t} |A(t,s)|ds - \int_{t}^{\infty} |A(u,t)|du, 1).$$

Now we present an example of Theorem 2.4.

Example 3.3. Given some positive constant c, let

$$dx(t) = [-cx(t) - ce^{-2ct}x(t-h)]dt + e^{-ct}\frac{\min(1,|x(t-h)|)}{|x(t)| + |x(t-h)|}\, dB(t)$$

i.e., $a(t) = -c$, $b(t) = -ce^{-2ct}$,

$$g(t, x(t), x(t-h)) = e^{-ct}\frac{\min(1,|x(t-h)|)}{|x(t)| + |x(t-h)|}$$

in (2.17). Then (A2) holds for $M(t) = e^{-ct}, F(t) = e^{-2ct}$.

Let $\tau = 0$, then $\xi(t) = e^{-2ct}$. Then (2.19) and (2.20) hold for $k = c$, $\alpha(t) = 2c(1 - e^{-2ct})$. By applying Theorem 2.4,

$$E^{\phi}(|x(t)|) \le e^{-ct - \frac{1}{2}e^{-2ct}}\Big(e \cdot V(0) + \int_0^t e^{e^{-2cu}}du\Big)^{1/2},$$

with $V(0) = \phi(0)^2 + c\int_{-h}^{0} \phi^2(s)ds$.

We conclude this paper with an application of Theorem 2.5.

Example 3.4. Given some positive constant β, let

$$\begin{aligned} dx(t) = &\Big(-2\big(e^{\frac{t+h}{\beta}}+1\big)x(t) + x(t-h) + \int_{t-h}^{t} \frac{2}{\beta} e^{-t/beta}|x(s)|ds\Big)dt \\ &+ e^{-\alpha t/2}\frac{\min(1,|x(t-h)|)}{|x(t)|+|x(t-h)|}dB(t) \end{aligned} \tag{3.3}$$

where α is defined later as in (3.4). Obviously, $a(t) = -2\big(e^{\frac{t+h}{\beta}}+1\big)$, $b(t) = 1$, $A(t,s) = e^{-\frac{1}{\beta}(t-s)}$, $f(t,x) = \frac{2}{\beta}e^{-t/beta}|x|$.

Then (2.29), (2.31) and (2.33) hold for $\lambda = \frac{2}{\beta}$ and $k = \frac{4}{\beta}e^{h/\beta}$. Now consider for (2.30):

$$2a(t) + 1 + b^2(t) + k\int_t^\infty |A(u,t)|du = -4\left(e^{\frac{t+h}{\beta}}+1\right) + 2 + 4e^{\frac{h}{\beta}} \le -2$$

which implies that (2.30) holds for $\alpha_1 = 2$

Then we consider, for (2.32),

$$\lambda^2\int_{t-h}^{t}|A(t,s)|ds - k = \frac{4}{\beta}e^{\frac{h}{\beta}}\big(-e^{-\frac{2h}{\beta}} + e^{-\frac{h}{\beta}} - 1\big) \le \frac{4}{\beta}e^{\frac{h}{\beta}}\big(-\frac{3}{4}\big)$$

implying that (2.32) holds for $\alpha_2 = 3/4$. Therefore,

$$\alpha = \min\{\alpha_1, \alpha_2\beta\} = \min\{2, \frac{3\beta}{4}\} \tag{3.4}$$

By Theorem 2.5, straightforward calculation suggests that for any $t \in \mathbb{R}^+$,

$$E^\phi(|x(t)|) \le e^{-\alpha t/2}(V(0)+t)^{1/2}$$

with $V(0) = \phi^2(0) + 4\int_{-h}^{0} e^{\frac{s+h}{\beta}}\phi^2(s)ds$.

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Youssef N. Raffoul
Department of Mathematics, University of Dayton, Dayton, OH 45469-2316, USA
E-mail address: `yraffoul1@udayton.edu`

Dan Ren
Department of Mathematics, University of Dayton, Dayton, OH 45469-2316, USA
E-mail address: `dren01@udayton.edu`