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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR TWO-POINT FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this note we present an existence and uniqueness of a continuous solution for a fractional boundary-value problem which depends on the Riemmann-Liouville operator. We conclude this article by presenting an illustrative example.

1. Introduction

In the book by Kelley and Peterson [4] the following result is established:

Theorem 1.1 ([4, Theorem 7.7]). *Assume $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to the second variable on $[a, b]\times\mathbb{R}$ with Lipschitz constant K; that is,*

$$|f(t, x) - f(t, y)| \le K|x - y|,$$

for all $(t, x), (t, y) \in [a, b] \times \mathbb{R}$. If

$$b - a < \frac{2\sqrt{2}}{\sqrt{K}},$$

then the boundary value problem

$$y''(t) = -f(t, y(t)), \quad a < t < b,$$
$$y(a) = A, \; y(b) = B, \quad A, B \in \mathbb{R},$$

has a unique continuous solution.

In this work we want to extend the above result by considering a fractional Riemmann-Liouville derivative (we refer the reader to [5] for the definitions and basic results on fractional calculus) instead of the classical operator y'', i.e., we prove the existence and uniqueness of solutions for the fractional differential boundary value problem

$$_aD^\alpha y(t) = -f(t, y(t)), \quad a < t < b, \tag{1.1}$$
$$y(a) = 0, \quad y(b) = B, \tag{1.2}$$

where $1 < \alpha \le 2$. Existence and uniqueness results for fractional IVPs and BVPs have been obtained before in the literature (cf. [1, 3] and the references cited

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therein). Nevertheless we believe that our results are new and provide useful tools in the study of fractional boundary value problems.

2. Main Results

We start by writing the boundary value problem (1.1)–(1.2) in its integral form.

Lemma 2.1. *Suppose that f is a continuous function. A function $y \in C[a,b]$ is a solution of* (1.1)–(1.2) *if and only if y satisfies the integral equation*

$$y(t) = B\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} + \int_a^b G(t,s)f(s,y(s))ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \le s \le t \le b,\\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}, & a \le t \le s \le b. \end{cases}$$

Proof. The proof is somewhat standard. Nevertheless, for completeness, we provide it here.

It is well known that solving (1.1)–(1.2) is equivalent to solving the integral equation

$$y(t) = c\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + d\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,y(s))ds,$$

where c and d are some real constants. Now, $d = 0$ by the first boundary condition. On the other hand, $y(b) = B$ implies

$$B = c\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,y(s))ds,$$

which after some manipulations yields

$$c = \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\Big(B + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,y(s))ds\Big).$$

Hence,

$$\begin{aligned} y(t) &= \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\Big(B + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,y(s))ds\Big)\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\\ &\quad - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,y(s))ds,\\ &= B\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s,y(s))ds\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\\ &\quad - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,y(s))ds, \end{aligned}$$

and the proof is complete. □

The next result is essential for proving our main result.

Proposition 2.2. *Let G be the Green function given in Lemma 2.1. Then*

$$\int_a^b |G(t,s)|ds \le \frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha}.$$

Proof. It is known [2, Lemma 2.2] that $G(t,s) \geq 0$ for all $a \leq t, s \leq b$. Therefore,

$$\begin{aligned}
\int_a^b |G(t,s)|ds &= \frac{1}{\Gamma(\alpha)}\Big(\int_a^t \big(\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}\big)ds \\
&\quad + \int_t^b \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}ds\Big) \\
&= \frac{1}{\Gamma(\alpha)}\Big(-\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-t)^{\alpha}}{\alpha} + \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-a)^{\alpha}}{\alpha} \\
&\quad - \frac{(t-a)^{\alpha}}{\alpha} + \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-t)^{\alpha}}{\alpha}\Big) \\
&= \frac{1}{\Gamma(\alpha)}\Big((t-a)^{\alpha-1}\frac{b-a}{\alpha} - \frac{(t-a)^{\alpha}}{\alpha}\Big) \\
&= \frac{1}{\Gamma(\alpha)}\frac{(t-a)^{\alpha-1}(b-t)}{\alpha}.
\end{aligned}$$

Define $g : [a,b] \to \mathbb{R}$ by

$$g(t) = \frac{(t-a)^{\alpha-1}(b-t)}{\alpha}.$$

Differentiating the function g we immediately find that its maximum is achieved at the point

$$t^* = \frac{(\alpha-1)b+a}{\alpha}.$$

Moreover,

$$g(t^*) = \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha},$$

which completes the proof. □

Theorem 2.3. *Assume $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to the second variable on $[a,b] \times \mathbb{R}$ with Lipschitz constant K; that is,*

$$|f(t,x) - f(t,y)| \leq K|x-y|,$$

for all $(t,x),(t,y) \in [a,b] \times \mathbb{R}$. If

$$b - a < \Gamma^{1/\alpha}(\alpha)\frac{\alpha^{(\alpha+1)/\alpha}}{K^{1/\alpha}(\alpha-1)^{(\alpha-1)/\alpha}}, \tag{2.1}$$

then the boundary-value problem

$$_aD^{\alpha}y(t) = -f(t,y(t)), \quad a < t < b, \tag{2.2}$$

$$y(a) = 0,\ y(b) = B, \quad B \in \mathbb{R}, \tag{2.3}$$

has a unique continuous solution.

Proof. Let $\mathcal{B}$ be the Banach space of continuous functions defined on $[a,b]$ with the norm

$$\|x\| = \max_{t\in[a,b]} |x(t)|.$$

By Lemma 2.1, $y \in C[a,b]$ is a solution of (2.2)–(2.3) if and only if it is a solution of the integral equation

$$y(t) = B\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} + \int_a^b G(t,s)f(s,y(s))ds.$$

Define the operator $T:\mathcal{B}\to\mathcal{B}$ by

$$Ty(t)=B\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}+\int_a^b G(t,s)f(s,y(s))ds,$$

for $t\in[a,b]$. We will show that the operator T has a unique fixed point.

Let $x,y\in\mathcal{B}$. Then

$$\begin{aligned}|Tx(t)-Ty(t)|&\le\int_a^b|G(t,s)||f(s,x(s))-f(s,y(s))|ds\\&\le\int_a^b|G(t,s)|K|x(s)-y(s)|ds\\&\le K\int_a^b G(t,s)ds\|x-y\|\\&\le K\frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha}\|x-y\|,\end{aligned}$$

where we have used Proposition 2.2. By (2.1) we conclude that T is a contracting mapping on $\mathcal{B}$, and by the Banach contraction mapping theorem we get the desired result. □

Remark 2.4. We note that when $\alpha=2$ in Theorem 2.3, one immediately obtains Theorem 1.1 (apart from the restriction $A=0$ ($y(a)=0$), which we have to assume in order to consider continuous solutions on $[a,b]$ to (2.2)).

As an example we consider the initial-value problem

$$ {}_0D^{3/2}y(t)=-1-\sin(y(t)),\quad 0<t<1, \tag{2.4}$$

$$ y(0)=0,\quad y(1)=0. \tag{2.5}$$

Here $f(t,y)=-1-\sin(y)$ and, therefore,

$$|f_y(t,y)|=|\cos(y)|\le 1=K.$$

Since $\alpha=3/2$, we have

$$\Gamma^{1/\alpha}(\alpha)\frac{\alpha^{(\alpha+1)/\alpha}}{(\alpha-1)^{(\alpha-1)/\alpha}}=\frac{3}{4}\pi^{1/3}3^{2/3},$$

and therefore (2.1) is satisfied. Now an application of Theorem 2.3 proves that (2.4)–(2.5) has a unique solution.

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