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BIFURCATION AND MULTIPLICITY OF SOLUTIONS FOR THE FRACTIONAL LAPLACIAN WITH CRITICAL EXPONENTIAL NONLINEARITY

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Abstract. We study the fractional elliptic equation

$$\begin{gathered}(-\Delta)^{1/2}u = \lambda u + |u|^{p-2}ue^{u^2}, \quad \text{in } (-1,1),\\ u = 0 \quad \text{in } \mathbb{R}\setminus(-1,1),\end{gathered}$$

where λ is a positive real parameter, $p > 2$ and $(-\Delta)^{1/2}$ is the fractional Laplacian operator. We show the multiplicity of solutions for this problem using an abstract critical point theorem of literature in critical point theory. Precisely, we extended the result of Cerami, Fortuno and Struwe [5] for the fractional Laplacian with exponential nonlinearity.

1. Introduction

We study the fractional elliptic equation

$$\begin{gathered}(-\Delta)^{1/2}u = \lambda u + |u|^{p-2}ue^{u^2}, \quad \text{in } (-1,1),\\ u = 0 \quad \text{in } \mathbb{R}\setminus(-1,1),\end{gathered} \tag{1.1}$$

where λ is a positive real parameter, $p > 2$ and $(-\Delta)^{1/2}$ is the fractional Laplacian operator which is defined as follows

$$-(-\Delta)^{1/2}u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}dy \quad \text{for all } x \in \mathbb{R}.$$

The fractional Laplacian operator has been a classical topic in Fourier analysis and nonlinear partial differential equations for a long time. In the recent past the fractional Laplacian operator has been widely studied by many researchers due to its wide range of applications in many fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic lows, multiple scattering, minimal surfaces, materials science, water waves, thin obstacle problem, optimal transport, image reconstruction and many more, see [1, 3, 25, 26] and references therein.

Bifurcation and multiplicity results for the case of Laplace operator with critical polynomial growth was initially studied in [5], where authors showed that the

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problem

$$-\Delta u = \lambda u + u^{2_*-2}u, \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.2}$$

has m pairs of solutions for λ lying in the suitable left neighborhood of any eigenvalue with multiplicity m of the Laplace operator with Dirichlet boundary conditions. Moreover the asymptotic behavior of the solutions is studied when the length of neighborhood goes to zero. Further the bifurcation and multiplicity results for the quasilinear counterpart of the problem in (1.2) have been discussed in [20] using critical point theorem based on a pseudo-index related to the cohomological index. The problems with critical exponential nonlinearity has been discussed in [6], where authors have obtained the local multiplicity results for the problem having singular exponential nonlinearity. Also the asymptotic behavior of the connected unbounded branch of solutions is shown in the radial case.

In the recent past the existence and multiplicity results have been obtained for the problems involving nonlocal fractional operators, see [8, 15, 16, 17, 18, 21] and references therein. In [8], the authors considered the critical exponent problem

$$(-\Delta)^s u = \lambda u + u^{2_*^s-2}u, \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^n$, $s \in (0,1)$ and $2_*^s = \frac{2n}{n-2s}$ is the fractional critical Sobolev exponent. Using the abstract critical point theorem, authors have generalized the results of Cerami, Fortuno and Struwe [5] for the nonlocal setting. For the case of p-fractional operator, Perera Squassina et al. showed the multiplicity results in [21] using the idea of [20].

Recently the fractional nonlocal operators with exponential nonlinearity has been studied in [9, 10, 11, 14]. In [11], authors showed the existence of at least two solutions in the bounded domain of $\mathbb{R}$ by showing the Palais-smale condition in the suitable range. These results are fractional generalizations of results obtained in [7]. In [14], authors obtained the existence result in whole $\mathbb{R}$ by assuming a vanishing potential which tackles the compactness issue. Further in [10], existence and multiplicity results were obtained with superlinear and convex-concave type sign changing critical exponential nonlinearity using idea of Nehari manifold by converting the nonlocal problem into local setting by harmonic extensions introduced in [4]. In [10], authors obtained the existence result for whole $\mathbb{R}$ using the suitable fractional version of Moser-Trudinger inequality.

In this work we obtain the similar results as in [8] for the nonlinearity having critical exponential growth in $\mathbb{R}$. The main difficulty in our problem is the lack of compactness. We use the Moser-Trundinger inequality, see (2.1) below, adapted from [9, 11] to get pass the limit.

Now we define the function space where we look for the solutions of problem (1.1). Also, we discuss the spectrum for the fractional Laplacian and some spectral properties related to our nonlocal operator. In [22], Servedei and Valdinoci introduced such spaces for the problems involving nonlocal operators. We adopt the same idea to define the linear space as follows:

$$X = \Big\{ u : \mathbb{R} \to \mathbb{R} \text{ is measurable, } u|_{(-1,1)} \in L^2((-1,1)) \text{ and } \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy < \infty \Big\},$$

where $Q = \mathbb{R}^2 \setminus (\mathcal{C}(-1,1) \times \mathcal{C}(-1,1))$ and $\mathcal{C}(-1,1) = \mathbb{R} \setminus (-1,1)$. The space X is a normed linear space endowed with the norm

$$\|u\|_X^2 = \|u\|_2^2 + \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy,$$

where $\|\cdot\|_2$ denotes the $L^2((-1,1))$ norm of u. Then we define the space X_0 as

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R} \setminus (-1,1)\}.$$

The space X_0 is a Hilbert space, see [22], with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}\times\mathbb{R}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\,dx\,dy. \tag{1.3}$$

The norm on X_0, induced from the scalar product in (1.3) is defined as

$$\|u\|_{X_0} = \Big(\int_{\mathbb{R}\times\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\Big)^{1/2}.$$

Now we define an equivalent norm on X_0 as follows

$$\|u\| = \Big(\frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\Big)^{1/2}.$$

Then from [19, Proposition 3.6], we have

$$\|u\| = \|(-\Delta)^{1/4}u\|_2^2.$$

Problems of the type (1.1) are motivated from the following Moser-Trudinger inequality, see [12].

Moser-Trudinger inequality: For any $u \in X_0$, $e^{\alpha u^2} \in L^1((-1,1))$ for all $\alpha > 0$. Moreover for $\alpha \le \pi$

$$\sup_{u\in X_0,\ \|u\|\le 1} \int_{-1}^{1} e^{\alpha u^2} dx < \infty. \tag{1.4}$$

The above inequality is sharp in the sense that if $\alpha > \pi$ the integral in (1.4) is not finite. Now we discuss the eigenvalue problem of the type

$$(-\Delta)^{1/2}u = \lambda u \text{ in } (-1,1), \quad u = 0 \text{ in } \mathbb{R} \setminus (-1,1).$$

It is known that there exists a infinite sequence of eigenvalues $\lambda_1 < \lambda_2 \le \dots \le \lambda_n \le \dots$ with $\lambda_n \to +\infty$ as $n \to \infty$. The eigenfunctions $\{\varphi_n\}$ corresponding to each eigenvalue λ_n form an orthonormal basis for $L^2((-1,1))$ and an orthogonal basis for X_0. For more details see the work of Servedei and Valdinoci in [23, 24].

Let $\lambda \in \mathbb{R}$ be given. Denote $\lambda_* = \min\{\lambda_n : \lambda < \lambda_n\}$ then $\lambda_* = \lambda_n$ for some $n \in \mathbb{N}$ and $\lambda_* \ge \lambda_1$. Let λ_* have multiplicity m, then

$$\lambda_* = \lambda_1 < \lambda_2 \quad \text{if } n = 1,$$
$$\lambda_{n-1} < \lambda_* = \lambda_n = \lambda_{n+1} = \dots = \lambda_{n+m-1} < \lambda_{n+m} \quad \text{if } n \ge 2.$$

For more details related to eigenvalue for the square root of the Laplacian in the interval $(-1,1)$ see [13]. Now we state our main result.

Theorem 1.1. *Problem* (1.1) *has m pairs of nontrivial solutions $\{-u_j^\lambda, u_j^\lambda\}$ for λ lying in the sufficiently small left neighborhood of λ_*. Moreover $\|u_j^\lambda\| \to 0$ as $\lambda \to \lambda_*$.*

To prove the Theorem 1.1 we use the following result, see [2, Theorem 2.4].

Theorem 1.2. *Let H be a Hilbert space with the norm $\|\cdot\|$ and $I : H \to \mathbb{R}$ be a $C^1(H,\mathbb{R})$ functional satisfying the following conditions:*

- (A1) *$I(0) = 0$ and $I(-u) = I(u)$;*
- (A2) *I satisfies the Palais-Smale condition, in short $(PS)_c$, for $c \in (0,\beta)$ for some $\beta > 0$;*
- (A3) *There exists closed subspaces V, W of H and constants ρ, δ, η with $\delta < \eta < \beta$ such that*
  - (i) *$I(u) \le \eta$ for all $u \in W$*
  - (ii) *$I(u) \ge \delta$ for any $u \in V$ with $\|u\| = \rho$*
  - (iii) *$codim(V) < \infty$ and $dim W \ge codim V$.*

Then there exists at least $\dim W - \operatorname{codim} V$ *pairs of critical points of I with critical values belonging to the interval $[\delta, \eta]$.*

2. Bifurcation and multiplicity result

In this section, we define the variational functional related to the problem and we show that the functional satisfies the requirement of critical point Theorem 1.2.

A function $u \in X_0$ is called a weak solution of (1.1) if

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^2}\,dx\,dy \\ &- \lambda\int_{-1}^{1} u(x)\phi(x)dx - \int_{-1}^{1} g(u(x))\phi(x)dx = 0 \end{aligned}$$

for all $\phi \in X_0$.

The variational functional $I_\lambda : X_0 \to \mathbb{R}$ associated with (1.1) is defined as

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{-1}^{1} |u|^2 dx - \int_{-1}^{1} G(u)dx,$$

where $G(t) = \int_0^t g(s)ds$ is the primitive of g. It is clear that the functional $I_\lambda \in C^1(X_0,\mathbb{R})$ and critical points of the functional I_λ are the solutions of (1.1). From the definition, $I_\lambda(0) = 0$ and from g being an odd function, $I_\lambda(-u) = I_\lambda(u)$. Hence the assumption $(A1)$ is satisfied.

Now we show the geometric requirements as in (A3). We assume that

$$W = \operatorname{span}\{\varphi_1, \varphi_2, \dots \varphi_{n+m-1}\}$$

and $V = X_0$ if $n = 1$, otherwise

$$V = \{u \in X_0 : \langle u, \varphi_j\rangle = 0 \ \ \forall\, 1 \le j \le n-1\}.$$

Then both W and V are closed subspaces of X_0 with dim $W = n+m-1$ and $\operatorname{codim} V = n-1$. Now take $u \in W$ then $u(x) = \sum_{j=1}^{n+m-1} \alpha_j\varphi_j(x)$ and

$$\|u\|^2 = \sum_{j=1}^{n+m-1} \alpha_j^2\|\varphi_j\|^2 = \sum_{j=1}^{n+m-1} \lambda_j\alpha_j^2 \le \lambda_n \sum_{j=1}^{n+m-1} \alpha_j^2 = \lambda_n\|u\|_2^2 = \lambda_*\|u\|_2^2.$$

Now using that $G(t) \ge \frac{1}{p}|t|^p$, for $u \in W$, we obtain

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \int_{-1}^{1} G(u(x))dx \le \frac{1}{2}(\lambda_* - \lambda)\|u\|_2^2 - \int_{-1}^{1} G(u(x))dx$$

$$\leq \frac{1}{2}(\lambda_* - \lambda)2^{\frac{p}{p-2}}\|u\|_p^2 - \frac{1}{p}\|u\|_p^p.$$

Define $f(t) = \frac{1}{2}(\lambda_* - \lambda)2^{\frac{p}{p-2}}t^2 - \frac{1}{p}t^p$, then $f(t)$ has a maximum at

$$t_0 = \Big((\lambda_* - \lambda)2^{\frac{p}{p-2}}\Big)^{\frac{1}{p-2}}.$$

Hence

$$I_\lambda(u) \leq \eta = (\frac{1}{2} - \frac{1}{p})\Big((\lambda_* - \lambda)2^{\frac{p}{p-2}}\Big)^{\frac{1}{p-2}}.$$

Note that we can make η to be arbitrary small positive number by choosing λ suitably close to λ_*.

For the second part, we make use of the fact that for $u \in V$, we have $\|u\|^2 \geq \lambda_*\|u\|_2^2$ and $G(t) \leq \frac{1}{p}e^{t^2}|t|^p$. Therefore

$$I_\lambda(u) \geq \big(\frac{\lambda_* - \lambda}{2\lambda_*}\big)\|u\|^2 - \frac{1}{p}\int_{-1}^{1}|u|^p e^{u^2}dx \geq \big(\frac{\lambda_* - \lambda}{2\lambda_*}\big)\|u\|^2 - \frac{C}{p}\|u\|^p = \delta > 0$$

for $\|u\| = \rho$ with sufficiently small value of $\rho > 0$. Note that the second integral in the above expression is well defined by Moser-Trudinger inequality (1.4) for the choice of $\rho > 0$ taken. Now we prove few Lemmas which are useful for showing compactness of Palais-Smale sequence.

Lemma 2.1. *Every Palais-Smale sequence of I_λ is bounded in X_0.*

Proof. Let $\{u_k\}$ be a Palais-Smale sequence that is

$$|I_\lambda(u_k)| \leq C \quad \text{and} \quad \langle I'_\lambda(u_k), u_k\rangle \leq C\|u_k\|, \tag{2.1}$$

for any $k \in \mathbb{N}$. Also using the fact that $G(t) \leq \frac{1}{p}g(t)t$, we obtain

$$\begin{aligned} I_\lambda(u_k) - \frac{1}{2}I'_\lambda(u_k)(u_k) &\geq \big(\frac{1}{2} - \frac{1}{p}\big)\int_{-1}^{1} g(u_k)u_k dx \\ &\geq \big(\frac{1}{2} - \frac{1}{p}\big)\int_{-1}^{1}|u_k|^p dx \geq C_0\|u_k\|^p. \end{aligned} \tag{2.2}$$

On the other hand, from (2.1), we obtain

$$I_\lambda(u_k) - \frac{1}{2}\langle I'_\lambda(u_k)(u_k)\rangle \leq C(1 + \|u_k\|). \tag{2.3}$$

Now, from (2.2) and (2.3), we obtain $\|u_k\|^p \leq C(1 + \|u_k\|)$, which implies that sequence $\{u_k\}$ is bounded in X_0. □

Lemma 2.2. *Let $\{u_k\}$ be a Palais-Smale sequence for I_λ then $g(u_k) \to g(u)$ in $L^1((-1,1))$ for some $u \in X_0$.*

Proof. From Lemma 2.1, we obtain that $\{u_k\}$ is bounded, therefore

$$\|u\| \leq C, \quad \int_{-1}^{1} g(u_k)u_k dx \leq C, \quad \int_{-1}^{1} G(u_k)dx \leq C. \tag{2.4}$$

So there exists $u \in X_0$ such that $u_k \rightharpoonup u$ in X_0, $u_k \to u$ in $L^\gamma((-1,1))$ for all $\gamma > 1$ and $u_k(x) \to u(x)$ a.e. in $(-1,1)$. Now the proof follows from Lebesgue dominated convergence theorem, see [7, Lemma 2.1]. □

Now we state a version of higher integrability Lemma. Proof is adapted from [10, Lemma 4.1].

Lemma 2.3. *Let $\{u_k\}$ be a sequence in X_0 with $\|u_k\| = 1$ and $u_k \rightharpoonup u$ weakly in X_0. Then for any r such that $1 < r < \frac{1}{1-\|u\|^2}$, we have*

$$\sup_k \int_{-1}^{1} e^{\alpha r (u_k)^2} dx < \infty, \quad \text{for all } 0 < \alpha < \pi.$$

Proof. First we note that from Young's inequality, for $\frac{1}{\mu} + \frac{1}{\nu} = 1$, we have

$$e^{s+t} \le \frac{1}{\mu} e^{\mu s} + \frac{1}{\nu} e^{\nu t}. \tag{2.5}$$

Now using the inequality $u_k^2 \le (1+\epsilon)(u_k - w_0)^2 + C(\epsilon)u^2$ and (2.5), we obtain

$$\begin{aligned} e^{\alpha r u_k^2} &\le e^{\alpha r \left((1+\epsilon)(u_k-u)^2 + c(\epsilon)u^2\right)} \\ &\le \frac{1}{\mu} e^{\alpha r \mu \left((1+\epsilon)(u_k-u)^2\right)} + \frac{1}{\nu} e^{\alpha r \nu \left(c(\epsilon)u^2\right)}. \end{aligned}$$

Now using that $\|u_k - u\|^2 = 1 - \|u\|^2 + o_k(1)$, we obtain

$$\alpha r \mu \left((1+\epsilon)(u_k-u)^2\right) = \left(\alpha r \mu (1+\epsilon)(1 - \|u\|^2 + o_k(1))\right) \Big(\frac{(u_k - u)}{\|u_k - u\|}\Big)^2.$$

Hence for any $1 < r < \frac{1}{1-\|u\|^2}$, and $\epsilon > 0$ small enough and $\mu > 1$ close to 1, we have

$$\alpha r \mu (1+\epsilon)(1 - \|u\|^2) < \pi$$

and the proof follows from (1.4). □

Now we show a compactness result. The proof follows closely [7, Proposition 2.1].

Proposition 2.4. *Let $\{u_k\}$ be a Palais-Smale sequence, that is*

$$I_\lambda(u_k) = c + o_k(1) \quad \text{and} \quad I'_\lambda(u_k) = o_k(1) \tag{2.6}$$

then it has a strongly convergent subsequence for $c \in (0, \frac{\pi}{2})$.

Proof. From Lemma 2.1, there exists $u \in X_0$ such that $u_k \rightharpoonup u$ in X_0, $u_k \to u$ in $L^\gamma((-1,1))$ for all $\gamma > 1$ and $u_k(x) \to u(x)$ a.e. in $(-1,1)$. Following the Lebesgue dominated convergence theorem and the relation (2.4), we have $G(u_k) \to G(u)$ in $L^1((-1,1))$ and

$$\begin{aligned} \lim_{k\to\infty} \|u_k\|^2 &= 2c + \lambda \|u\|_2^2 + 2\int_{-1}^{1} G(u)dx, \\ \lim_{k\to\infty} \int_{-1}^{1} g(u_k)u_k dx &= 2c + 2\int_{-1}^{1} G(u)dx. \end{aligned} \tag{2.7}$$

Now from the weak convergence of u_k in X_0, strong convergence in $L^2((-1,1))$ and the Lemma 2.2, u solves the problem (1.1). Moreover,

$$\|u\|^2 - \lambda \|u\|_2^2 = \int_{-1}^{1} g(u)u \, dx \ge 2\int_{-1}^{1} G(u)\, dx.$$

Therefore $I_\lambda(u) \ge 0$. Now we divide the proof into three cases.

Case 1: $c = 0$. In this case $\liminf I_\lambda(u_k) = 0$. Therefore $\|u_k\| \to \|u\|$, and the proof is complete.

Case 2: $c \neq 0, u = 0$**.** As $u = 0$, so from (2.7) we obtain $\lim_{k\to\infty} \|u_k\| = 2c$. Hence for every $\epsilon > 0$ and for large k, $\|u_k\| \leq 2c + \epsilon$. Now for some $q > 1$ and close to 1, we have the estimate

$$\begin{aligned} \int_{-1}^{1} |g(u_k)|^q dx &= \int_{-1}^{1} |u_k|^{qp} e^{qu_k^2} dx \\ &\leq \|u_k\|^{qp} \Big(\int_{-1}^{1} e^{qlu_k^2} dx \Big)^{1/l} \\ &\leq C \Big(\sup_{\|v\|\leq 1} \int_{-1}^{1} e^{lq\|u_k\|^2 v^2} dx \Big)^{1/l} < \infty \end{aligned} \tag{2.8}$$

for $c < \pi/2$, $l > 1$ close to 1 and ϵ sufficiently small. Now from (2.6), we obtain

$$\big| \|u_k\|^2 - \lambda \|u_k\|_2^2 - \int_{-1}^{1} g(u_k) u_k dx \big| \leq \epsilon_k \|u_k\|. \tag{2.9}$$

Now from the estimate in (2.8) inequality (2.9) and $u_k \to 0$ in $L^\gamma((-1,1))$ we obtain $\|u_k\| \to 0$, which is the contradiction from (2.7) as $c \neq 0$.

Case 3: $c \neq 0$**,** $u \neq 0$**.** In this case we claim that $I_\lambda(u) = c$. If the claim is proved then from (2.7), $\lim_{k\to\infty} \|u_k\| = \|u\|$ and hence the proof of the Lemma. Suppose not, then $I_\lambda(u) < c$ and consequently

$$\|u\|^2 < 2c + \lambda \|u\|_2^2 + \int_{-1}^{1} G(u) dx. \tag{2.10}$$

Now choose $w_k = u_k / \|u_k\|$ then $\|w_k\| = 1, w_k \rightharpoonup w$ for

$$w = \frac{u}{2c + \lambda \|u\|_2^2 + \int_{-1}^{1} G(u) dx}$$

and $\|w\| < 1$. Therefore, using Lemma 2.3, we have

$$\int_{-1}^{1} e^{r\alpha w_k^2} dx < \infty \quad \text{for all } \alpha < \pi \text{ and } r < (1 - \|w\|^2)^{-1}.$$

Using the above inequality we have the estimate

$$\begin{aligned} \int_{-1}^{1} |g(u_k)|^q dx &= \int_{-1}^{1} |u_k|^{qp} e^{qu_k^2} dx \leq \|u_k\|^{qp} \Big(\int_{-1}^{1} e^{lqu_k^2} dx \Big)^{1/l} \\ &\leq C \Big(\int_{-1}^{1} e^{lq\|u_k\|^2 w_k^2} dx \Big)^{1/l} < \infty \end{aligned} \tag{2.11}$$

for

$$lq \|u_k\|^2 < \frac{\pi}{1 - \|w\|^2} = \pi \frac{c + \frac{\lambda}{2} \|u\|_2^2 + \int_{-1}^{1} G(u) dx}{c - I_\lambda(u)}$$

which holds when l, q are greater than 1 and sufficiently close to 1 and $c < \frac{\pi}{2}$.

Now using the above estimate and (2.6) with $(u_k - u)$ as test function, and using weak convergence of u_k in X_0 we obtain that $\|u_k\|^2 \to \|u\|^2$ as $k \to \infty$. However, from (2.7) and (2.10) we obtain a contradiction. $\square$

Proof of Theorem 1.1. By taking λ in the sufficiently small neighborhood of λ_*, we have $\delta < \eta < \beta$. Now invoking Theorem 1.2, we obtain at least m pairs of critical points of I_λ, hence the solutions of the problem (1.1), with critical values in the range $[\delta, \eta]$.

Now we estimate the X_0 bound of the solutions using the relation $I_\lambda(u_j^\lambda) < \eta$ for all $1 \le j \le m$ as follows

$$\begin{aligned}\big(\frac{1}{2}-\frac{1}{p}\big)\Big((\lambda_*-\lambda)2^{\frac{p}{p-2}}\Big)^{\frac{1}{p-2}} &\ge I_\lambda(u_j^\lambda) = I_\lambda(u_j^\lambda) - \frac{1}{2}\langle I'_\lambda(u_j^\lambda), u_j^\lambda\rangle \\ &= \frac{1}{2}\int_{-1}^{1} g(u_j^\lambda(x))u_j^\lambda(x)dx - \int_{-1}^{1} G(u_j^\lambda(x))dx \\ &\ge \big(\frac{1}{2}-\frac{1}{p}\big)\int_{-1}^{1} g(u_j^\lambda(x))u_j^\lambda(x)dx \\ &\ge \big(\frac{1}{2}-\frac{1}{p}\big)\|u_j^\lambda\|_p^p \ge C\|u_j^\lambda\|^p.\end{aligned}$$

Now letting $\lambda \to \lambda_*$ we obtain $\|u_j^\lambda\| \to 0$. □

References

[1] D. Applebaum; *Lévy processes and stochastic calculus*, Second edition. Cambridge Studies in Advanced Mathematics, 116, Cambridge University Press, Cambridge, 2009.

[2] P. Bartolo, V. Benci; *Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity*, Nonlinear analysis: Theory, methods & applications, 7(9), (1983), 981-1012.

[3] L. Caffarelli; Non local diffusions, drifts and games, Nonlinear Partial Differential Equations: The Abel Symposium, 7, (2012), 37-52.

[4] L. Caffarelli and L. Silvestre; *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, 32, (2007), 1245–1260.

[5] G. Cerami, D. Fortuno, M. Struwe; *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. Henri poincare, 1(5), (1984), 341-350.

[6] R. Dhanya, J. Giacomoni, S. Prashanth, K. Saoudi; *Global bifurcation and local multiplicity results for elliptic equations with singular nonlinearity of super exponential growth in* $\mathbb{R}^2$, Adv. Differential Equations, 17(3-4), (2012), 369–400.

[7] D. G. de Figueiredo, O. H. Miyagaki, B. Ruf; *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var., 3, (1995), 139–153.

[8] A. Fiscellaa , G. M. Bisci, R. Servadei; *Bifurcation and multiplicity results for critical nonlocal fractional Laplacian problems*, Bulletin des Sciences Mathmatiques, 140(1), (2016), 14–35.

[9] J. Giacomoni, P. K. Mishra, K. Sreenadh; *Fractional elliptic equations with critical exponential nonlinearity*, Advances in Nonlinear Analysis, 5, (2016), 57-74.

[10] J. Giacomoni, P. K. Mishra, K. Sreenadh; *Critical growth problems for* $\frac{1}{2}$ *- Laplacian in* $\mathbb{R}$, Differential equations and applications, 8, (2016), 295-317.

[11] A. Iannizzotto, M. Squassina; *1/2-Laplacian problems with exponential nonlinearity*, J. Math. Anal. Appl., 414, (2014), 372–385.

[12] S. Iula, A. Maalaoui, L. Martinazzi; *A fractional Moser-Trudinger type inequality in one dimension and its critical points*, Differential Integral Equations, 29(5-6), (2016), 455-492.

[13] M. Kwaśnicki; *Eigenvalues of the fractional Laplace operator in the interval*, Journal of Functional Analysis, 262, (2012), 2379–2402.

[14] J. Marcus Do Ó, O. H. Miyagaki, M. Squassina; *Nonautonomous fractional problems with exponential growth*, Nonlinear Differential Equations and Applications NoDEA, 22(5), (2015), 1395–1410.

[15] G. Molica Bisci, R. Servadei; *A bifurcation result for non-local fractional equations*, Anal. Appl., 13(4), (2015), 371–394.

[16] G. Molica Bisci, R. Servadei; *Lower semicontinuity of functionals of fractional type and applications to nonlocal equations with critical Sobolev exponent*, Adv. Differential Equations, 20, (2015), 635–660.

[17] G. Molica Bisci, V. Radulescu; *Ground state solutions of scalar field fractional Schroedinger equations*, Calc. Var. Partial Differential Equations, 54, (2015), 2985-3008.

[18] G. Molica Bisci, V. Radulescu, R. Servadei; *Variational methods for nonlocal fractional problems. With a foreword by Jean Mawhin*, Encyclopedia of Mathematics and its Applications, 162, Cambridge University Press, Cambridge, 2016, ISBN: 978-1-107-11194-3.

[19] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchhikers guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136, (2012), 521–573.

[20] K. Perera, M. Squassina, Y. Yang; *Bifurcation and multiplicity results for critical p-Laplacian problems*, Topological Methods in Nonlinear Analysis, 47(1), (2016), 187-194.

[21] K. Perera, M. Squassina, Y. Yang; *Bifurcation and multiplicity results for critical fractional p-Laplacian problems*, Math. Nachr., 289(2-3), (2016), 332-342.

[22] R. Servadei, E. Valdinoci; *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl., 389, (2012), 887-898.

[23] R. Servadei, E. Valdinoci; *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal., 12(6), (2013), 2445–2464.

[24] R. Servadei, E. Valdinoci; *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst., 33(5), (2013), 2105–2137.

[25] E. Valdinoci; *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. SeMA, 49, (2009), 33–44.

[26] J. L. Vázquez; *Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators*, Discrete Contin. Dyn. Syst.- Series S, 7(4), (2014), 857–885.

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