Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 212, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SUBLINEAR EIGENVALUE PROBLEMS WITH SINGULAR WEIGHTS RELATED TO THE CRITICAL HARDY INEQUALITY

MEGUMI SANO, FUTOSHI TAKAHASHI

Abstract. In this article, we consider a weighted sublinear eigenvalue problem related to an improved critical Hardy inequality. We discuss to what extent the weights can be singular for the existence of weak solutions. Also we study the asymptotic behavior of the first eigenvalues as a parameter involved varies.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $0 \in \Omega$. Here and henceforth, we put $R = \sup_{x\in\Omega} |x|$. In this article, we consider the quasilinear eigenvalue problem with singular weights

$$\begin{gathered} -\Delta_N u - \mu \frac{|u|^{N-2}u}{|x|^N (\log \frac{Re}{|x|})^N} = \lambda f(x)|u|^{q-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian, $1 < q$, $0 \leq \mu < (\frac{N-1}{N})^N$, $\lambda \in \mathbb{R}$ and $f \in L^\infty_{loc}(\Omega \setminus \{0\})$ is a positive weight function which may be unbounded near the origin. We assume that the weight function f satisfies $|\phi|^q f \in L^1(\Omega)$ for any $\phi \in W_0^{1,N}(\Omega)$. This problem is related to the *critical Hardy inequality* due to Adimurthi and Sandeep [2]:

$$\int_\Omega |\nabla u|^N dx \geq \Big(\frac{N-1}{N}\Big)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx, \quad \forall u \in W_0^{1,N}(\Omega). \tag{1.2}$$

In the appendix, we provide a simple proof of (1.2) for the sake of completeness. Thanks to (1.2), the operator

$$L_\mu u = -\Delta_N u - \mu \frac{|u|^{N-2}u}{|x|^N (\log \frac{Re}{|x|})^N}$$

acting on $W_0^{1,N}(\Omega)$ is positive and coercive. We call a function $u \in W_0^{1,N}(\Omega)$ a weak solution of the problem (1.1) if

$$\int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla\phi dx = \mu \int_\Omega \frac{|u|^{N-2}u\phi}{|x|^N (\log \frac{Re}{|x|})^N} dx + \lambda \int_\Omega |u|^{q-2}u\phi f(x) dx$$

2010 *Mathematics Subject Classification.* 35A23, 35J62, 35J20.
Key words and phrases. Critical Hardy inequality; sublinear reaction term; eigenvalue problem.

Submitted July 8, 2016. Published August 10, 2016.

holds whenever $\phi \in W_0^{1,N}(\Omega)$.

When $q-1 = N-1$ case, (1.1) becomes a genuine eigenvalue problem for L_μ, and under suitable integrability assumptions of the indefinite weight function f, the existence of the positive first eigenvalue, its simplicity, and the isolation property are obtained [16]. Also in [19], the authors obtain an unbounded sequence of minimax eigenvalues of L_μ by the use of the cohomological index theory.

When $q-1 > N-1$ ($(N-1)$-superlinear case), since $|u|^{q-2}u$ is subcritical from the view point of Trudinger-Moser inequality, we find several references in which the existence of (multiple) weak solutions is obtained, see for example [18], [19], and the reference therein. See also [9] for the critical growth case and [13], [15] for related results.

In this article, we focus on the $(N-1)$-sublinear case; $0 < q-1 < N-1$. For f in an appropriate class of weight functions, we look for a weak solution $u \in W_0^{1,N}(\Omega)$ of (1.1) by a constrained minimization argument. The solution obtained here corresponds to the first eigenvalue of $\lambda_\mu(f)$ of the operator L_μ:

$$\lambda_\mu(f) = \inf_{u\in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N dx - \mu\int_\Omega \frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N}dx}{\left(\int_\Omega |u|^q f(x)dx\right)^{N/q}}.$$

Furthermore we study the asymptotic behavior of $\lambda_\mu(f)$ as $\mu \nearrow (\frac{N-1}{N})^N$.

To state the main result in this paper, for $0<q<N$, put $\alpha^* = (\frac{N-1}{N})q+1$ and define a class of weight functions

$$\begin{aligned} F_N = \Big\{ &f:\Omega\to\mathbb{R}^+ : f\in L^\infty_{\rm loc}(\Omega\setminus\{0\}) \text{ and } \exists\alpha\in(\alpha^*,N] \text{ such that} \\ &\limsup_{|x|\to 0} f(x)|x|^N\big(\log\frac{Re}{|x|}\big)^\alpha < \infty\Big\}. \end{aligned}$$

Then the main result of the paper reads as follows:

Theorem 1.1. *Let $0<q-1<N-1$. Then for all $f\in F_N$ and $0<\mu<(\frac{N-1}{N})^N$, problem* (1.1) *admits a positive weak solution $u\in W_0^{1,N}(\Omega)$ corresponding to $\lambda = \lambda_\mu(f)>0$. Furthermore, $\lambda_\mu(f)\to\lambda(f)$ as $\mu\nearrow(\frac{N-1}{N})^N$ for a limit $\lambda(f)>0$.*

For the proof of Theorem 1.1, we need an improved version of the critical Hardy inequality (1.2). It is known that the constant $(\frac{N-1}{N})^N$ in (1.2) is optimal and never attained on any bounded domain $\Omega\subset\mathbb{R}^N$ with $0\in\Omega$, see Adimurthi and Sandeep [2]. Therefore there is a possibility to add a nonnegative remainder term to the right-hand side of (1.2). In [2], the authors claim that there exists $C>0$ such that

$$\int_\Omega |\nabla u|^N dx \ge \big(\frac{N-1}{N}\big)^N \int_\Omega \frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N}dx + C\int_\Omega \frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N(\log^{(2)}\frac{R_1}{|x|})^N}dx$$

for any $u\in W_0^{1,N}(\Omega)$, where $R_1\ge(e^e)^{2/N}R$. Here for $k\in\mathbb{N}$, $\log^{(k)}$ is defined inductively by $\log^{(1)}(\cdot)=\log(\cdot)$, $\log^{(k)}(\cdot)=\log(\log^{(k-1)}(\cdot))$ for $k\ge2$. However, the proof of it is omitted in [2]. Barbatis, Filippas and Tertikas [4] proved that, among other things, the improved critical Hardy inequality

$$\int_\Omega |\nabla u|^N dx - \big(\frac{N-1}{N}\big)^N\int_\Omega \frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N}dx$$

$$\geq \frac{1}{2}\Big(\frac{N-1}{N}\Big)^{N-1}\sum_{i=2}^{\infty}\int_{\Omega}\frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N}X_2^2\big(\frac{|x|}{R}\big)\dots X_i^2\big(\frac{|x|}{R}\big)dx$$

where for $t\in(0,1)$ and $i=2,3,\dots$,

$$X_1(t)=(1-\log t)^{-1}=\frac{1}{\log(\frac{e}{t})},\quad X_i(t)=X_1(X_{i-1}(t)).$$

Note that

$$X_2\big(\frac{|x|}{R}\big)=\frac{1}{\log(e\log\frac{eR}{|x|})},\quad X_3\big(\frac{|x|}{R}\big)=\frac{1}{\log(e\log(e\log\frac{eR}{|x|}))},\quad \dots.$$

In [4], the authors use a "vector field approach" as in [3].

In this paper, we obtain another kind of remainder terms for the critical Hardy inequality (1.2) in much simpler way, see Proposition 2.1. We use a classical idea by Brezis and Vázquez [7] combined with a transformation of functions relevant to our study, see (2.3) below.

The organization of this paper is as follows: In §2, an improved critical Hardy inequality is proved. In §3, the optimality of the weight in the improved critical Hardy inequality is discussed. Finally in §4, Theorem 1.1 is proved.

2. Improving the critical Hardy inequality with an idea of Brezis and Vázquez

In this section, we improve the critical Hardy inequality (1.2) by adding a non-negative term to the right hand side. In the proof of Proposition 2.1 below, we utilize the well-known transformation of Brezis and Vázquez [7] combined with an appropriate change of variables.

Proposition 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N\geq 2$, with $0\in\Omega$, and $R=\sup_{x\in\Omega}|x|$. For any $-1<L<N-2$ and $0<q<(\frac{N}{N-1})(N-2-L)$, put*

$$\alpha=\alpha(q,L)=\frac{N-1}{N}q+L+2.$$

Then the inequality

$$\begin{aligned}\int_{\Omega}|\nabla u|^N dx &\geq \Big(\frac{N-1}{N}\Big)^N\int_{\Omega}\frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N}dx\\ &\quad+\omega_N^{1-\frac{N}{q}}C(L,N,q)^{N/q}\Big(\int_{\Omega}\frac{|u|^q}{|x|^N\big(\log\frac{Re}{|x|}\big)^{\alpha}}dx\Big)^{N/q}\end{aligned}\tag{2.1}$$

holds for all $u\in W_0^{1,N}(\Omega)$, where ω_N is the area of the unit sphere in $\mathbb{R}^N$ and

$$C(L,N,q)^{-1}=\int_0^1 s^L\big(\log\frac{1}{s}\big)^{\frac{N-1}{N}q}ds=(L+1)^{-(\frac{N-1}{N}q+1)}\Gamma\big(\frac{N-1}{N}q+1\big),$$

here $\Gamma(\cdot)$ is the Gamma function.

Remark 2.2. Inequality (2.1) does not hold when $L\leq -1$ (see Theorem 3.1). Therefore we see that the weight function in the remainder term of (2.1) is optimal.

First, we recall a simple lemma.

Lemma 2.3 ([10][Lemma 1.1]). *Let $N \geq 2$, and ξ, η be real numbers such that $\xi \geq 0$ and $\xi - \eta \geq 0$. Then*
$$(\xi - \eta)^N + N\xi^{N-1}\eta - \xi^N \geq |\eta|^N. \tag{2.2}$$

Proof of Proposition 2.1.
Step 1: First we prove the inequality (2.1) when Ω is a ball $B_R(0) \subset \mathbb{R}^N$ and for smooth nonnegative radially non-increasing functions $u \in C_0^\infty(B_R(0))$. We write $u(x) = u(r)$ with $r = |x|$ for radially symmetric functions u. We define the transformation
$$\begin{gathered} v(s) = (\log \frac{Re}{r})^{-\frac{N-1}{N}} u(r), \quad \text{where } r = |x|,\ s = s(r) = \big(\log \frac{Re}{r}\big)^{-1} \in [0,1], \\ s'(r) = \frac{s(r)}{r \log \frac{Re}{r}} \geq 0. \end{gathered} \tag{2.3}$$
Note that $v(0) = v(1) = 0$ since $u(0)$ is finite and $u(R) = 0$, and
$$u'(r) = -\big(\frac{N-1}{N}\big)\big(\log \frac{Re}{r}\big)^{-1/N} \frac{v(s(r))}{r} + \big(\log \frac{Re}{r}\big)^{\frac{N-1}{N}} v'(s(r))s'(r) \leq 0. \tag{2.4}$$
Now we observe that
$$\begin{aligned} I &= \int_{B_R(0)} |\nabla u|^N dx - \big(\frac{N-1}{N}\big)^N \int_{B_R(0)} \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx \\ &= \omega_N \int_0^R |u'(r)|^N r^{N-1} dr - \big(\frac{N-1}{N}\big)^N \omega_N \int_0^R \frac{|u(r)|^N}{r(\log \frac{Re}{r})^N} dr \\ &= \omega_N \int_0^R \Big(\frac{N-1}{N}(\log \frac{Re}{r})^{-1/N} \frac{v(s(r))}{r} - \big(\log \frac{Re}{r}\big)^{\frac{N-1}{N}} v'(s(r))s'(r)\Big)^N r^{N-1} dr \\ &\quad - \big(\frac{N-1}{N}\big)^N \omega_N \int_0^R \frac{|v(s(r))|^N}{r \log \frac{Re}{r}} dr. \end{aligned}$$
Here, we can apply Lemma 2.3 with the choice
$$\xi = \frac{N-1}{N}\big(\log \frac{Re}{r}\big)^{-1/N} \frac{v(s(r))}{r} \quad \text{and} \quad \eta = \big(\log \frac{Re}{r}\big)^{\frac{N-1}{N}} v'(s(r))s'(r).$$
By noticing the cancellation of the term ξ^N in (2.2) and using the boundary conditions $v(0) = v(1) = 0$, we obtain
$$\begin{aligned} I &\geq -\omega_N N\big(\frac{N-1}{N}\big)^{N-1} \int_0^R v(s(r))^{N-1} v'(s(r))s'(r) dr \\ &\quad + \omega_N \int_0^R |v'(s(r))|^N (s'(r))^N (r \log \frac{Re}{r})^{N-1} dr \\ &= -\omega_N N\big(\frac{N-1}{N}\big)^{N-1} \int_0^1 v(s)^{N-1} v'(s) ds + \omega_N \int_0^1 |v'(s)|^N s^{N-1} ds \\ &= \omega_N \int_0^1 |v'(s)|^N s^{N-1} ds. \end{aligned} \tag{2.5}$$
When $N = 2$, actually this inequality becomes the equality. On the other hand, by using the estimate
$$|v(s)| = \Big|\int_s^1 v'(t) dt\Big| = \Big|\int_s^1 v'(t) t^{\frac{N-1}{N} - \frac{N-1}{N}} dt\Big|$$

$$\le \Big(\int_0^1 |v'(t)|^N t^{N-1}\,dt\Big)^{1/N}\Big(\log\frac{1}{s}\Big)^{\frac{N-1}{N}},$$

we obtain

$$\int_0^1 |v(s)|^q s^L ds \le \Big(\int_0^1 |v'(s)|^N s^{N-1} ds\Big)^{q/N}\int_0^1 s^L\Big(\log\frac{1}{s}\Big)^{\frac{N-1}{N}q} ds.$$

Note that the last integral is finite when $L > -1$ and $q > 0$. Therefore, we have

$$\int_0^1 |v'(s)|^N s^{N-1} ds \ge C(L,N,q)^{N/q}\Big(\int_0^1 |v(s)|^q s^L\,ds\Big)^{N/q}. \tag{2.6}$$

Consequently, by (2.5) and (2.6), we obtain

$$\begin{aligned}
I &\ge \omega_N C(L,N,q)^{N/q}\Big(\int_0^1 |v(s)|^q s^L\,ds\Big)^{N/q}\\
&= \omega_N C(L,N,q)^{N/q}\Big(\int_0^R \frac{|u(r)|^q}{r(\log\frac{Re}{r})^\alpha} dr\Big)^{N/q}\\
&= \omega_N^{1-\frac{N}{q}} C(L,N,q)^{N/q}\Big(\int_{B_R(0)} \frac{|u|^q}{|x|^N(\log\frac{Re}{|x|})^\alpha} dx\Big)^{N/q}.
\end{aligned}$$

where $\alpha = \alpha(q,L) = \frac{N-1}{N}q + L + 2$.

Step 2: Let $u^\#$ denote the symmetric decreasing rearrangement (the Schwarz symmetrization) of $u \in C_0^\infty(\Omega)$:

$$u^\#(x) = u^\#(|x|) = \inf\{\lambda > 0 : \big|\{x \in \Omega : |u(x)| > \lambda\}\big| \le |B_{|x|}(0)|\},$$

where $|A|$ denotes the measure of the set $A \subset \mathbb{R}^N$. Assume $|\Omega| = |B_{\tilde R}(0)|$ for some $\tilde R > 0$. Note that the function $r \mapsto \frac{1}{r^N(\log\frac{Re}{r})^\alpha}$ is monotonically decreasing on $[0,R]$ since $\alpha \le N$. Thus by using the symmetrization argument, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^N dx &\ge \int_{B_{\tilde R}(0)} |\nabla u^\#|^N dx\\
&\ge \Big(\frac{N-1}{N}\Big)^N \int_{B_{\tilde R}(0)} \frac{|u^\#|^N}{|x|^N(\log\frac{\tilde R e}{|x|})^N} dx\\
&\quad + \omega_N^{1-\frac{N}{q}} C(L,N,q)^{N/q}\Big(\int_{B_{\tilde R}(0)} \frac{|u^\#|^q}{|x|^N(\log\frac{\tilde R e}{|x|})^\alpha} dx\Big)^{N/q}\\
&\ge \Big(\frac{N-1}{N}\Big)^N \int_{B_{\tilde R}(0)} \frac{|u^\#|^N}{|x|^N(\log\frac{Re}{|x|})^N} dx\\
&\quad + \omega_N^{1-\frac{N}{q}} C(L,N,q)^{N/q}\Big(\int_{B_{\tilde R}(0)} \frac{|u^\#|^q}{|x|^N(\log\frac{Re}{|x|})^\alpha} dx\Big)^{N/q}\\
&\ge \Big(\frac{N-1}{N}\Big)^N \int_\Omega \frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N} dx\\
&\quad + \omega_N^{1-\frac{N}{q}} C(L,N,q)^{N/q}\Big(\int_\Omega \frac{|u|^q}{|x|^N(\log\frac{Re}{|x|})^\alpha} dx\Big)^{N/q}
\end{aligned}$$

where the first inequality comes from the Pólya-Szegö inequality, the second one comes from Step 1, the third one comes from the fact that $R \ge \tilde R$, and the last one comes from the Hardy-Littlewood inequality: $\int_{B_{\tilde{R}}(0)} f^{\#} g^{\#} \geq \int_{\Omega} fg$ for nonnegative measurable functions f and g. Finally, a density argument assures (2.1) holds true for all $u \in W_0^{1,N}(\Omega)$. The proof is complete. $\square$

From Proposition 2.1, we easily have the following result.

Corollary 2.4 (Adimurthi-Sandeep [2, Theorem 1.3]). *Let $N \geq 2$. The best constant $\left(\frac{N-1}{N}\right)^N$ in the inequality (1.2) is never attained in $W_0^{1,N}(\Omega)$.*

3. Optimality of weights

In this section, we discuss the optimality of the weight function in the improved critical Hardy inequality (2.1).

Theorem 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $0 \in \Omega$, with $R = \sup_{x \in \Omega} |x|$. For $0 < q < N$, put*

$$\alpha^* = \Big(\frac{N-1}{N}\Big) q + 1$$

and define

$$F_N = \Big\{ f : \Omega \to \mathbb{R}^+ : f \in L^\infty_{\rm loc}(\Omega \setminus \{0\}) \text{ and } \exists \alpha \in (\alpha^*, N] \text{ s.t. } \limsup_{|x| \to 0} f(x)|x|^N \big(\log \frac{Re}{|x|}\big)^\alpha < \infty \Big\},$$

and

$$G_N = \Big\{ f : \Omega \to \mathbb{R}^+ : f \in L^\infty_{\rm loc}(\Omega \setminus \{0\}) \text{ and } \liminf_{|x| \to 0} f(x)|x|^N (\log \frac{Re}{|x|})^{\alpha^*} > 0 \Big\}.$$

If $f \in F_N$, then there exists $\lambda(f) > 0$ such that the inequality

$$\int_\Omega |\nabla u|^N dx \geq \Big(\frac{N-1}{N}\Big)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx + \lambda(f) \Big(\int_\Omega f(x) |u|^q \, dx \Big)^{N/q} \tag{3.1}$$

holds for all $u \in W_0^{1,N}(\Omega)$. If $f \in G_N$, then no inequality of type (3.1) can hold.

Especially, we cannot replace α in the remainder term of (2.1) by α^*. Also by Theorem 3.1, we see $\int_\Omega f(x)|u|^q \, dx < \infty$ for any $u \in W_0^{1,N}(\Omega)$ if $f \in F_N$.

Remark 3.2. There exist functions f with $f \notin F_N$ and $f \notin G_N$. For example, $f_\gamma(x) = |x|^{-N} (\log \frac{Re}{|x|})^{-\alpha^*} \big(\log |\log \frac{Re}{|x|}| \big)^{-\gamma}$ for $\gamma > 0$ are such functions.

To prove Theorem 3.1, we follow the argument of the proof in Adimurthi-Chaudhuri-Ramaswamy [1, Corollary 1.2].

Proof of Theorem 3.1. If $f \in F_N$, then there exists $\alpha \in (\alpha^*, N]$ such that

$$\lim_{\varepsilon \to 0} \sup_{x \in B_\varepsilon} f(x) |x|^N \Big(\log \frac{Re}{|x|} \Big)^\alpha < \infty.$$

Hence for sufficiently small $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$f(x) < \frac{C}{|x|^N (\log \frac{Re}{|x|})^\alpha} \quad \text{in } B_\varepsilon(0).$$

Outside of B_ε, f is a bounded function and hence C can be chosen so that this inequality holds in the whole of Ω. Then, it is easy to check that (3.1) follows from the improved critical Hardy inequality (2.1).

For the proof of the latter half part of Theorem, let $f \in G_N$. Then we can find $C > 0$, $b > 0$ such that $f(x) \geq \frac{C}{|x|^N (\log(Re/|x|))^{\alpha^*}}$ in $0 \leq |x| \leq \frac{bRe}{2}$. We may assume that $B_{bRe}(0) \subset \Omega$ ($\subset B_R(0)$). Let $s < \frac{N-1}{N}$ be a positive parameter and we define

$$u_s(x) = \begin{cases} (\log \frac{Re}{|x|})^s & \text{if } 0 \leq |x| \leq \frac{bRe}{2} \\ \text{smooth} & \text{if } \frac{bRe}{2} \leq |x| \leq bRe \\ 0 & \text{if } bRe \leq |x|. \end{cases} \tag{3.2}$$

Direct calculations show that

$$\Big(\int_\Omega \frac{|u_s|^q}{|x|^N (\log \frac{Re}{|x|})^{\alpha^*}} dx \Big)^{N/q} = \Big(\omega_N \frac{1}{(\frac{N-1}{N} - s)q} \big(\log \frac{2}{b} \big)^{(s - \frac{N-1}{N})q} \Big)^{N/q} + O(1), \tag{3.3}$$

$$\int_\Omega |\nabla u_s|^N dx = \omega_N \frac{-s^N}{(s-1)N+1} \big(\log \frac{2}{b} \big)^{(s-1)N+1} + O(1), \tag{3.4}$$

$$\int_\Omega \frac{|u_s|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx = \omega_N \frac{-1}{(s-1)N+1} \big(\log \frac{2}{b} \big)^{(s-1)N+1} + O(1) \tag{3.5}$$

as $s \to \frac{N-1}{N}$. By (3.3), (3.4), (3.5) and $N/q > 1$, we have

$$\begin{aligned} &\frac{\int_\Omega |\nabla u_s|^N dx - \big(\frac{N-1}{N}\big)^N \int_\Omega \frac{|u_s|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx}{(\int_\Omega f(x)|u_s|^q dx)^{N/q}} \\ &\leq \frac{\int_\Omega |\nabla u_s|^N dx - \big(\frac{N-1}{N}\big)^N \int_\Omega \frac{|u_s|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx}{C(\int_\Omega \frac{|u_s|^q}{|x|^N (\log \frac{Re}{|x|})^{\alpha^*}} dx)^{N/q}} \\ &= C\big(\frac{N-1}{N} - s\big)^{\frac{N}{q}-1} \to 0 \end{aligned}$$

as $s \to \frac{N-1}{N}$. Thus the inequality (2.1) does not hold for f as above. $\square$

4. Proof of Theorem 1.1

To prove the Theorem 1.1, we need the following lemmas.

Lemma 4.1 (Boccardo-Murat [5, Thm. 2.1])**.** *Let $\{u_m\}_{m=1}^\infty \subset W_0^{1,p}(\Omega)$ be such that, as $m \to \infty$, $u_m \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and satisfies*

$$-\Delta_p u_m = f_m + g_m \quad \text{in } \mathcal{D}'(\Omega),$$

where $f_m \to 0$ in $W_0^{-1,p'}(\Omega)$ and g_m is bounded in $\mathcal{M}(\Omega)$, the space of Radon measures on Ω, i.e.

$$|\langle g_m, \phi \rangle| \leq C_K \|\phi\|_\infty$$

for all $\phi \in \mathcal{D}(\Omega)$ with $\operatorname{supp} \phi \subset K$. Then there exists a subsequence u_{m_k} such that

$$u_{m_k} \to u \quad \text{in } W_0^{1,\gamma}(\Omega) \quad \forall \gamma < p.$$

Lemma 4.2 (Brezis-Lieb [6])**.** *For $p \in (0, +\infty)$, let $\{g_m\}_{m=1}^\infty \subset L^p(\Omega, \mu)$ be a sequence of functions on a measurable space (Ω, μ) such that*

(i) $\|g_m\|_{L^p(\Omega,\mu)} \le \ ^\exists C < \infty$ *for all* $m \in \mathbb{N}$*, and*
(ii) $g_m(x) \to g(x)$ μ *a.e.* $x \in \Omega$ *as* $m \to \infty$*.*

Then

$$\lim_{m\to\infty}(\|g_m\|^p_{L^p(\Omega,\mu)} - \|g_m - g\|^p_{L^p(\Omega,\mu)}) = \|g\|^p_{L^p(\Omega,\mu)}.$$

We may apply Lemma 4.2 to $\mu(dx) = f(x)dx$, where f is any nonnegative $L^1(\Omega)$ function. Next we have a compactness theorem for the embedding $W_0^{1,N}(\Omega)$ into a weighted Lebesgue space $L^q(\Omega, f) = \{u \in L^1_{loc}(\Omega) : \int_\Omega |u|^q f(x)dx < \infty\}$.

Lemma 4.3. *For any* $0 < q < N$ *and any* $\alpha > \alpha^* = \frac{N-1}{N}q + 1$*, there exists* $C > 0$ *such that the inequality*

$$\int_\Omega |\nabla u|^N dx \ge C\Big(\int_\Omega \frac{|u|^q}{|x|^N(\log\frac{Re}{|x|})^\alpha}dx\Big)^{N/q} \tag{4.1}$$

holds for all $u \in W_0^{1,N}(\Omega)$*. Moreover, for*

$$f_\alpha(x) = \frac{1}{|x|^N(\log(Re/|x|))^\alpha},$$

the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega, f_\alpha)$ *is compact for* $1 \le q < N$*.*

Recently, inequality (4.1) was proved by Machihara-Ozawa-Wadade [12]. In the following, we provide a simpler proof of (4.1) than the one in [12].

Proof. By Hölder inequality and the critical Hardy inequality (1.2), we have

$$\begin{aligned}
&\int_\Omega \frac{|u|^q}{|x|^N(\log\frac{Re}{|x|})^\alpha}dx \\
&\le \Big(\int_\Omega \frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N}dx\Big)^{q/N}\Big(\int_\Omega \frac{1}{|x|^N(\log\frac{Re}{|x|})^{\frac{N}{N-q}(\alpha-q)}}dx\Big)^{1-\frac{q}{N}} \\
&\le \Big((\frac{N-1}{N})^{-N}\int_\Omega |\nabla u|^N dx\Big)^{q/N}\Big(\int_\Omega \frac{1}{|x|^N(\log\frac{Re}{|x|})^{\frac{N}{N-q}(\alpha-q)}}dx\Big)^{1-\frac{q}{N}}.
\end{aligned}$$

Since $\alpha > \alpha^* = \frac{N-1}{N}q + 1$, the exponent $\frac{N}{N-q}(\beta - q) > 1$, so the last integral is finite. Thus we have (4.1).

For the proof of the latter half part, we follow the argument by Chaudhuri-Ramaswamy [8, Proposition 2.1]. The continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega, f_\alpha)$ comes from the inequality (4.1). To prove that this embedding is compact, let $\{u_m\}$ be a bounded sequence in $W_0^{1,N}(\Omega)$. Then we have a subsequence $\{u_{m_k}\}$ such that

$$\begin{gathered}
u_{m_k} \rightharpoonup u \quad \text{weakly in } W_0^{1,N}(\Omega) \text{ as } k \to \infty, \\
u_{m_k} \to u \quad \text{strongly in } L^\gamma(\Omega) \text{ as } k \to \infty\ \forall 1 \le \gamma < \infty.
\end{gathered}$$

Take β such that $\alpha > \beta > \alpha^*$ and note that $\lim_{|x|\to 0} |x|^N(\log\frac{Re}{|x|})^\beta f_\alpha(x) = 0$. Then for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\sup_{B_\delta(0)} |x|^N\Big(\log\frac{Re}{|x|}\Big)^\beta f_\alpha(x) \le \varepsilon \quad \text{and} \quad \|f_\alpha\|_{L^\infty(\Omega\setminus B_\delta(0))} < \infty.$$

Thus

$$\|u_{m_k} - u\|^q_{L^q(\Omega,f_\alpha)} = \int_{\Omega\setminus B_\delta(0)} |u_{m_k} - u|^q f_\alpha(x)dx + \int_{B_\delta(0)} |u_{m_k} - u|^q f_\alpha(x)dx$$

$$
\begin{aligned}
&\le \|f_\alpha\|_{L^\infty(\Omega\setminus B_\delta(0))}\|u_{m_k}-u\|^q_{L^q(\Omega)} + \varepsilon\int_\Omega \frac{|u_{m_k}-u|^q}{|x|^N(\log\frac{Re}{|x|})^\beta}dx\\
&\le \|f_\alpha\|_{L^\infty(\Omega\setminus B_\delta(0))}\|u_{m_k}-u\|^q_{L^q(\Omega)} + \varepsilon C\|\nabla(u_{m_k}-u)\|^q_{L^N(\Omega)}\\
&= o(1)+\varepsilon O(1) \quad \text{as } k\to\infty,
\end{aligned}
$$

here the second inequality comes from (4.1). Finally, letting $\varepsilon \to 0$, we obtain $\|u_{m_k}-u\|^q_{L^q(\Omega,f_\alpha)} \to 0$ and the proof is complete. $\square$

Proof of Theorem 1.1. We use the methods similar to the proof in [1, Theorem 1.2]. We look for a minimizer of the functional

$$
J_\mu(u)=\int_\Omega |\nabla u|^N\,dx - \mu\int_\Omega \frac{|u|^N}{(|x|\log\frac{Re}{|x|})^N}\,dx \quad \forall u\in W_0^{1,N}(\Omega)
$$

over the manifold $M=\{u\in W_0^{1,N}(\Omega): \int_\Omega |u|^q f(x)dx=1\}$. Since $f\in F_N$, M is well-defined and non empty by Theorem 3.1. Note that J_μ is continuous, Găteaux differentiable and coercive on $W_0^{1,N}(\Omega)$ for any $\mu\in[0,\big(\frac{N-1}{N}\big)^N)$ thanks to the Hardy inequality (1.2). Thus it is clear that $\lambda_\mu(f)=\inf_{u\in M}J_\mu(u)$ is positive. Let $\{u_m\}_{m=1}^\infty\subset M$ be a minimizing sequence of $\lambda_\mu(f)$. By Ekeland's Variational Principle, we may assume $J'_\mu(u_m)\to 0$ in $W_0^{-1,N'}(\Omega)$ as $m\to\infty$ without loss of generality. The coercivity of J_μ implies that $\{u_m\}_{m=1}^\infty$ is a bounded sequence in $W_0^{1,N}(\Omega)$, hence we have a subsequence $\{u_{m_k}\}_{k=1}^\infty$ and $u\in W_0^{1,N}(\Omega)$ such that

$$
u_{m_k}\rightharpoonup u \quad \text{weakly in } W_0^{1,N}(\Omega) \text{ as } k\to\infty, \tag{4.2}
$$

$$
u_{m_k}\rightharpoonup u \quad \text{weakly in } L^N\Big(\Omega,(|x|\log\frac{Re}{|x|})^{-N}\Big) \text{ as } k\to\infty, \tag{4.3}
$$

$$
u_{m_k}\to u \quad \text{strongly in } L^\gamma(\Omega) \text{ as } k\to\infty\ \forall 1\le\gamma<\infty), \tag{4.4}
$$

$$
u_{m_k}\to u \quad \text{a.e. in } \Omega \text{ as } k\to\infty \tag{4.5}
$$

for some $u\in W_0^{1,N}(\Omega)$. Note that the second convergence (4.3) comes from the fact that

$$
\Big(L^N(\Omega,(|x|\log\frac{Re}{|x|})^{-N})\Big)^*\subset W^{-1,N'}(\Omega)=(W_0^{1,N}(\Omega))^*,
$$

which is a consequence of the Hardy inequality (1.2), and (4.2). Recall that for $f\in F_N$, there exist $C>0$ and $\alpha\in(\alpha^*,N]$ such that

$$
f(x)\le \frac{C}{|x|^N\big(\log\frac{Re}{|x|}\big)^\alpha} \quad \text{in } \Omega.
$$

Thus $W_0^{1,N}(\Omega)$ is compactly embedded in $L^q(\Omega,f)$ by Lemma 4.3. Hence M is weakly closed in $W_0^{1,N}(\Omega)$ and $u\in M$.

Furthermore since $\|J'_\mu(u_m)\|_{W^{-1,N'}(\Omega)}\to 0$, u_m satisfies

$$
-\Delta_N u_m=\mu\frac{|u_m|^{N-2}u_m}{\big(|x|\log\frac{Re}{|x|}\big)^N}+\lambda_m|u_m|^{q-2}u_m f+f_m
$$

in $\mathcal{D}'(\Omega)$, where $f_m\to 0$ in $W^{-1,N'}(\Omega)$ and $\lambda_m\to\lambda$ as $m\to\infty$. Putting

$$
g_m=\mu\frac{|u_m|^{N-2}u_m}{(|x|\log\frac{Re}{|x|})^N}+\lambda_m|u_m|^{q-2}u_m f,
$$

one can check that g_m is bounded in $\mathcal{M}(\Omega)$. Thus we have

$$\nabla u_{m_k} \to \nabla u \quad \text{a.e. in } \Omega \tag{4.6}$$

from Lemma 4.1. By using Lemma 4.2, (4.2), (4.3), (4.5), (4.6), and the Hardy inequality (1.2), we obtain

$$\begin{aligned} \lambda_\mu(f) &= \|\nabla u_{m_k}\|_N^N - \mu \|u_{m_k}\|^N_{L^N(\Omega,(|x|\log\frac{Re}{|x|})^{-N})} + o(1) \\ &= \|\nabla (u_{m_k} - u)\|_N^N - \mu \|u_{m_k} - u\|^N_{L^N(\Omega,(|x|\log\frac{Re}{|x|})^{-N})} \\ &\quad + \|\nabla u\|_N^N - \mu \|u\|^N_{L^N(\Omega,(|x|\log\frac{Re}{|x|})^{-N})} + o(1) \\ &\geq \Big(\big(\frac{N-1}{N}\big)^N - \mu \Big) \|u_{m_k} - u\|^N_{L^N(\Omega,(|x|\log\frac{Re}{|x|})^{-N})} + \lambda_\mu(f) + o(1) \end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$. As $\mu < \big(\frac{N-1}{N}\big)^N$, we conclude that

$$\begin{gathered} \|u_{m_k} - u\|^N_{L^N(\Omega,(|x|\log\frac{Re}{|x|})^{-N})} \to 0 \quad \text{as } k \to \infty, \\ \|\nabla (u_{m_k} - u)\|_N^N \to 0 \quad \text{as } k \to \infty. \end{gathered} \tag{4.7}$$

Hence we have the strong convergence of $\{u_{m_k}\}$ which implies $J_\mu(u) = \lambda_\mu(f)$ and $\lambda = \lambda_\mu(f)$. Since $J_\mu(|u|) = J_\mu(u)$ and the strong maximum principle of Δ_N, we can take $u > 0$ in Ω. Then using Lemma 4.1 and (4.7), we assure that u is a distributional solution of (1.1) corresponding to $\lambda = \lambda_\mu(f)$. Moreover u is a weak solution of (1.1) from density argument.

Finally, Theorem 3.1 implies

$$\lambda(f) = \inf_{u \in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N dx - \big(\frac{N-1}{N}\big)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx}{\big(\int_\Omega |u|^q f(x) dx \big)^{N/q}} > 0$$

if $f \in F_N$. Since it is trivial that $\lambda_\mu(f) \to \lambda(f)$ as $\mu \nearrow \big(\frac{N-1}{N}\big)^N$, this completes the proof. $\square$

Remark 4.4. By using the test function u_s defined by (3.2), we check that

$$\inf_{u \in W_0^{1,N}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^N dx}{\big(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{Re}{|x|})^{\alpha^*}} dx \big)^{N/q}} = 0.$$

Thus we cannot replace α in the inequality (4.1) by α^*. By this reason, if we define the class of weight functions

$$\mathcal{F}_N = \big\{ f : \Omega \to \mathbb{R}^+ : f \in L^\infty_{\rm loc}(\Omega\setminus\{0\}) \text{ and } \limsup_{|x|\to 0} f(x)|x|^N \big(\log \frac{Re}{|x|}\big)^{\alpha^*} < \infty \big\},$$

then we do not know the solvability of (1.1) for $f \in \mathcal{F}_N$.

5. Appendix

In this appendix, we prove the following result.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain. Then the inequality*

$$\int_\Omega \big| \frac{x}{|x|} \cdot \nabla u \big|^N dx \geq \big(\frac{N-1}{N} \big)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx \tag{5.1}$$

holds for all $u \in W_0^{1,N}(\Omega)$.

Proof. We argue as in [17]. It is sufficient to prove (5.1) for $u \in C_0^\infty(\Omega)$. By the identity

$$\operatorname{div}\Big(\frac{x}{|x|^N(\log\frac{Re}{|x|})^{N-1}}\Big) = \frac{N-1}{|x|^N(\log\frac{Re}{|x|})^N},$$

integration by parts and Hölder's inequality yield

$$\begin{aligned}
&\int_\Omega \frac{|u(x)|^N}{|x|^N(\log\frac{Re}{|x|})^N}dx \\
&= \Big|\frac{1}{N-1}\int_\Omega |u|^N \operatorname{div}\Big(\frac{x}{|x|^N(\log\frac{Re}{|x|})^{N-1}}\Big)dx\Big| \\
&= \Big| -\frac{1}{N-1}\int_\Omega \nabla(|u|^N)\cdot\frac{x}{|x|^N(\log\frac{Re}{|x|})^{N-1}}dx\Big| \\
&= \Big| -\frac{N}{N-1}\int_\Omega |u|^{N-2}u\nabla u\cdot\frac{x/|x|}{|x|^{N-1}(\log\frac{Re}{|x|})^{N-1}}dx\Big| \\
&\le |\frac{N}{N-1}|\Big(\int_\Omega \frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N}dx\Big)^{(N-1)/N}\Big(\int_\Omega |\frac{x}{|x|}\cdot\nabla u|^N dx\Big)^{1/N}.
\end{aligned}$$

After some manipulations, we obtain (5.1). □

Remark 5.2. The same proof as above yields *the critical Hardy inequality in a sharp form*:

$$\int_\Omega \Big|\frac{x}{|x|}\cdot\nabla u\Big|^N dx \ge \big(\frac{N-1}{N}\big)^N\int_\Omega \frac{|u|^N}{|x|^N(\log\frac{R}{|x|})^N}dx, \quad \forall u\in W_0^{1,N}(\Omega), \qquad (5.2)$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N\ge 2$). Note that the weight function $\frac{1}{|x|^N(\log\frac{R}{|x|})^N}$ is singular both on the origin and on the boundary. When $\Omega = B_R(0)$ case, Ioku and Ishiwata [11] showed that the constant $(\frac{N-1}{N})^N$ in the inequality (5.2) is optimal and never attained in $W_0^{1,N}(B_R(0))$. Furthermore in [14], the current authors provide a remainder term for the inequality (5.2) when $\Omega = B_R(0)$.

Acknowledgments. This work was supported in part by JSPS Grant-in-Aid for Fellows (DC2), No. 16J07472 (MS), by JSPS Grant-in-Aid for Scientific Research (B), No. 23340038, and by JSPS Grant-in-Aid for Challenging Exploratory Research, No. 26610030 (FT).

References

[1] Adimurthi, N. Chaudhuri and M. Ramaswamy; *An improved Hardy-Sobolev inequality and its application*, Proc. Amer. Math. Soc. **130** (2002), No. 2, 489-505 (electronic).

[2] Adimurthi, and K. Sandeep; *Existence and non-existence of the first eigenvalue of the perturbed Hardy-Sobolev operator*, Proc. Roy. Soc. Edinburgh Sect. A. **132** (2002), No. 5, 1021-1043.

[3] G. Barbatis, S. Filippas, A. Tertikas; *A unified approach to improved L^p Hardy inequalities with best constants*, Trans. Amer. Math. Soc. **356** (2004), no. 6, 2169-2196.

[4] G. Barbatis, S. Filippas, A. Tertikas; *Series expansion for L^p Hardy inequalities*, Indiana Univ. Math. J. **52** (2003), no. 1, 171-190.

[5] L. Boccardo, F. Murat; *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. TMA. **19** (1992), 581-597.

[6] H. Brezis, E. Lieb; emphA relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. **88** (1983), 486-490.

[7] H. Brezis, J. L. Vázquez; *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), No. 2, 443-469.
[8] N. Chaudhuri and M. Ramaswamy; *Existence of positive solutions of some semilinear elliptic equations with singular coefficients*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), No. 6, 1275-1295.
[9] L. F. O. Faria, O. H. Miyagaki, F. R. Pereira; *A nonhomogeneous quasilinear elliptic problem involving critical growth and Hardy potentials*, Differential Integral Equations, **27**, (2014), no. 11-12, 1171-1186.
[10] F. Gazzola, H.-C. Grunau, E. Mitidieri; *Hardy inequalities with optimal constants and remainder terms*, Trans. Amer. Math. Soc. **356** (2003), No.6, 2149-2168.
[11] N. Ioku, M. Ishiwata; *A scale invariant form of a critical Hardy inequality*, IMRN, International Mathematics Research Notices. (2015), No.18, 8830-8846.
[12] S. Machihara, T. Ozawa, H. Wadade; *Generalizations of the logarithmic Hardy inequality in critical Sobolev-Lorentz spaces*, J. Inequal. Appl. 2013, 2013:381, 14 pp.
[13] V. Raghavendra, K. Sreenadh; *Nontrivial solutions for perturbations of a Hardy-Sobolev operator on unbounded domains,* J. Math. Anal. Appl. **288** (2003), no. 1, 314-325.
[14] M. Sano and F. Takahashi; *Scale invariance structures of the critical and the subcritical Hardy inequalities and their improvements*, submitted.
[15] K. Sandeep, K. Sreenadh; *Asymptotic behaviour of the first eigenfunction of a Hardy-Sobolev operator,* Nonlinear Anal. **54** (2003), no. 3, 545-563.
[16] K. Sreenadh; *On the eigenvalue problem for the Hardy-Sobolev operator with indefinite weights*, Electron. J. Differential Equations, **2002**,(2002), No. 33, 12 pp. (electronic).
[17] F. Takahashi; *A simple proof of Hardy's inequality in a limiting case*, Archiv der Math. **104**, (2015), no.1, 77-82.
[18] J. Tyagi; *Multiple solutions for singular N-Laplace equations with a sign changing nonlinearity*, Commun. Pure Appl. Anal. **12** (2013), no. 6, 2381-2391.
[19] G. Zhang, J-Y. Shao, S. Liu; *Linking solutions for N -Laplace elliptic equations with Hardy-Sobolev operator and indefinite weights*, Commun. Pure Appl. Anal. **10** (2011), no. 2, 571–581.

Megumi Sano
Department of Mathematics, Graduate School of Science, Osaka City University, Sumiyoshi-ku, Osaka, 558-8585, Japan
E-mail address: `megumisano0609@st.osaka-cu.ac.jp`

Futoshi Takahashi
Department of Mathematics, Osaka City University & OCAMI, Sumiyoshi-ku, Osaka, 558-8585, Japan
E-mail address: `futoshi@sci.osaka-cu.ac.jp`