Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 216, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

WELL-POSEDNESS AND EXACT CONTROLLABILITY OF A FOURTH ORDER SCHRÖDINGER EQUATION WITH VARIABLE COEFFICIENTS AND NEUMANN BOUNDARY CONTROL AND COLLOCATED OBSERVATION

RUILI WEN, SHUGEN CHAI

Abstract. We consider an open-loop system of a fourth order Schrödinger equation with variable coefficients and Neumann boundary control and collocated observation. Using the multiplier method on Riemannian manifold we show that that the system is well-posed in the sense of Salamon. This implies that the exponential stability of the closed-loop system under the direct proportional output feedback control and the exact controllability of open-loop system are equivalent. So in order to conclude feedback stabilization from well-posedness, we study the exact controllability under a uniqueness assumption by presenting the observability inequality for the dual system. In addition, we show that the system is regular in the sense of Weiss, and that the feedthrough operator is zero.

1. Introduction and statement of main results

There is a wide class of infinite-dimensional linear systems introduced by Salamon and Weiss in the 1980s [20, 21, 24, 27], which cover many control systems described by partial differential equations (PDEs) with the actuators and sensors supported at isolated points, subregions, or on boundaries of the spatial regions. This class of infinite-dimensional systems, although the input and output operators are allowed to be unbounded, may possess many properties that make them similar in many ways to finite-dimensional ones, such as representation, transfer function, internal model based tracking and disturbance rejection, stabilizing controller parametrization, and quadratic optimal control [5]. As of now, many multi-dimensional PDEs have been verified to be well-posed and regular; see [1, 2, 3, 4, 5, 7, 8, 11, 12, 13, 14, 22, 30] and the references therein.

The fourth order Schrödinger equation arises in many scientific fields such as quantum mechanics, plasma physics, nonlinear optics and so on. In quantum mechanics, the solution $\varphi(x,t)$ of system (4.1) denotes the probability amplitude function, and the conservation of the norms validates the Born's statistical interpretation of $\varphi(x,t)$. Further more, $\int_\Omega |\varphi(x,t)|^2 dx$ represents the probability of finding the particle in domain Ω at the time t and the conservation law provides the particle

2010 *Mathematics Subject Classification.* 93C20, 35L35, 35B37.
Key words and phrases. Fourth order Schrödinger equation; variable coefficients; well-posedness; exact controllability; boundary control; boundary observation.
©2016 Texas State University.
Submitted May 25, 2016. Published August 12, 2016.

which will not disappear in Ω. Here we consider the control problem of a fourth order Schrödinger equation with Neumann boundary conditions. On the one hand, we generalize the well-posedness for fourth order Schrödinger equation with Neumann boundary control and collocated observation [25] to the variable coefficients case, and on the other hand, establish the exact controllability of this system. The system that we are concerned with in this paper is described by the PDEs

$$
\begin{gathered}
\mathrm{i}w_t(x,t) + P^2 w(x,t) = 0, \quad x \in \Omega, t > 0, \\
w(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \\
\frac{\partial w(x,t)}{\partial \nu_{\mathcal{A}}} = 0, \quad x \in \Gamma_1,\ t \geqslant 0, \\
\frac{\partial w(x,t)}{\partial \nu_{\mathcal{A}}} = u(x,t), \quad x \in \Gamma_0,\ t \geqslant 0, \\
y(x,t) = -\mathrm{i}\mathcal{A}(A^{-1} w(x,t)), \quad x \in \Gamma_0,\ t \geqslant 0,
\end{gathered} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ is an open bounded region with C^3-boundary $\partial\Omega = \Gamma = \overline{\Gamma_0} \cap \overline{\Gamma_1}$ and assume that Γ_0 ($\mathrm{int}(\Gamma_0) \neq \emptyset$) and Γ_1 are relatively open in $\partial\Omega$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. The operators $\mathcal{A}$ and A are defined in (1.3) and (1.5) later respectively, and P is a second-order partial differential operator

$$
P = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big),
$$

which, for some constants $a, b > 0$, satisfies

$$
\begin{gathered}
a\sum_{i=1}^{n} |\xi_i|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi}_j \leqslant b\sum_{i=1}^{n} |\xi_i|^2, \quad \forall x \in \overline{\Omega},\ \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{C}^n, \\
a_{ij}(x) = a_{ji}(x) \in C^\infty(\mathbb{R}^n), \quad \forall i, j = 1, 2, \dots, n.
\end{gathered} \tag{1.2}
$$

We define the operator $\mathcal{A}$ as

$$
\mathcal{A}f = Pf, \quad D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega), \tag{1.3}
$$

and

$$
\begin{gathered}
\nu_{\mathcal{A}} = \Big(\sum_{k=1}^{n} \nu_k a_{k1}(x), \sum_{k=1}^{n} \nu_k a_{k2}(x), \dots, \sum_{k=1}^{n} \nu_k a_{kn}(x)\Big), \\
\frac{\partial}{\partial \nu_{\mathcal{A}}} = \sum_{i,j=1}^{n} a_{ij}(x)\nu_j \frac{\partial}{\partial x_i},
\end{gathered} \tag{1.4}
$$

where $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ is the unit normal vector of $\partial\Omega$ pointing outward of Ω, u and y are the boundary control and the boundary observation of system (1.1).

Now, let A be the positive self-adjoint operator in $L^2(\Omega)$ defined by

$$
Af = P^2 f, \quad D(A) = H^4(\Omega) \cap H_0^2(\Omega). \tag{1.5}
$$

Just as in [15], one can show that

$$
A^{1/2} = -\mathcal{A}. \tag{1.6}
$$

Let $H = H^{-2}(\Omega)$ and $U = Y = L^2(\Gamma_0)$, where $H^{-2}(\Omega)$ is the dual of $H_0^2(\Omega)$ with respect to the pivot space $L^2(\Omega)$. The following Theorem 1.1 shows that system (1.1) is well-posed with the state space H and the input and output space $U = Y = L^2(\Gamma_0)$ [14].

Theorem 1.1. *System* (1.1) *is well-posed. More precisely, for any $T > 0$, initial value $w_0 \in H$, and the control input $u \in L^2(0,T;U)$, there exists a unique solution $w \in C(0,T;H)$ to* (1.1) *such that*

$$\|w(\cdot,T)\|_H^2 + \|y\|_{L^2(0,T;U)}^2 \leqslant C_T\big[\|w_0\|_H^2 + \|u\|_{L^2(0,T;U)}^2\big], \tag{1.7}$$

where $C_T > 0$ is used to represent the constant that depends only on Ω, Γ_0, and T, although it may have different values in different contexts.

It is proved in [9, Theorem 5.8] (see also [25, Theorem 5.2]) that if the abstract system (2.19) introduced later is well-posed, it must be regular in the sense of Weiss with the zero feedthrough operator. The following result is hence a consequence of Theorem 1.1.

Corollary 1.2. *System* (1.1) *is regular and the feedthrough operator is zero.*

Theorem 1.1 implies that the open-loop system (1.1) is well-posed in the sense of Salamom with the state space H and the same input and output space $U = Y$. From this result and [9, Theorems 6.7 and 6.8] (see also [25, Theorems 5.3 and 5.4]) on the first order abstract system formulation (see also [10] for the second order abstract system), we know that system (1.1) is exactly controllable on some interval $[0,T]$ $(T > 0)$ if and only if its corresponding closed loop systems under the output proportional feedback $u = -ky, k > 0$ is exponentially stable. So, based on this argument, to get the feedback stabilization of system (1.1) from the well-posedness, we need to discuss the exact controllability of the open loop system (1.1). We show that under the assumptions (H1) and (H2) stated below, system (1.1) is exactly controllable on some interval $[0,T]$, $T > 0$.

It should be emphasized that due to the variable coefficients, the classical multipliers method in Euclidean space seems invalid [26] to prove Theorem 1.1 and 1.3, some computations on the Riemannian manifold are needed.

By the ellipticity condition (1.2), we denote the coefficients matrix and its inverse by $A(x)$ and $G(x)$, respectively, and the determinant of $G(x)$ by $\rho(x)$,

$$\begin{gathered} A(x) = [a_{ij}(x)]_{n\times n}, \quad G(x) = [g_{ij}(x)]_{n\times n} = [a_{ij}(x)]_{n\times n}^{-1} = A(x)^{-1}, \\ \rho(x) = \det[g_{ij}(x)]_{n\times n}, \quad \forall x \in \mathbb{R}^n. \end{gathered} \tag{1.8}$$

Let $\mathbb{R}^n$ be the usual Euclidean space. For each $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, define the inner product and norm over the tangent space $\mathbb{R}^n_x$ of $\mathbb{R}^n$ by

$$\begin{gathered} g(X,Y) := \langle X,Y\rangle_g = \sum_{i,j=1}^n g_{ij}\alpha_i\beta_j, \\ |X|_g := \langle X,X\rangle_g^{1/2}, \quad \forall X = \sum_{i=1}^n \alpha_i\frac{\partial}{\partial x_i},\ Y = \sum_{i=1}^n \beta_i\frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \end{gathered} \tag{1.9}$$

Then $(\mathbb{R}^n, g)$ becomes a Riemannian manifold with Riemannian metric g [28, 29]. Denote by D the Levi-Civita connection with respect to g and let N be a smooth vector field on $(\mathbb{R}^n, g)$. Then for each $x \in \mathbb{R}^n$, the covariant differential DN of N determines a bilinear form on $\mathbb{R}^n_x \times \mathbb{R}^n_x$:

$$DN(X,Y) = \langle D_Y N, X\rangle_g, \quad \forall X, Y \in \mathbb{R}^n_x, \tag{1.10}$$

where $D_Y N$ stands for the covariant derivative of the vector field N with respect to Y.

In this article we use the following assumptions:

(H1) There exists a vector field N on $(\mathbb{R}^n, g)$ such that

$$DN(X,X) = b(x)|X|_g^2, \ \forall\, X \in \mathbb{R}_x^n, \ x \in \Omega, \tag{1.11}$$

where $b(x)$ is a function defined on Ω so that

$$b_0 = \inf_{x\in\Omega} b(x) > 0. \tag{1.12}$$

(H2) (Uniqueness assumption]) The problem

$$\begin{gathered} \mathcal{A}^2 v = \zeta v, \quad x \in \Omega, \\ v = \frac{\partial v}{\partial \nu_{\mathcal{A}}} = 0, \quad x \in \Gamma, \\ \mathcal{A} v = 0, \quad x \in \Gamma_0, \end{gathered} \tag{1.13}$$

possesses a unique zero solution, where ζ is an arbitrary complex number and Γ_0 is relatively open in Γ and satisfies

$$\Gamma_0 = \{x \in \Gamma | N(x) \cdot \nu > 0\}. \tag{1.14}$$

For the variable case, several corollaries were presented to show how to verify Assumption (H1) by means of the Riemannian geometry method in [28]. In fact, when $a_{ij}(x) = \delta_{ij}$, then for some given x_0, the radial field $N = x - x_0$ satisfies Assumption (H1) with $b(x) \equiv 1$. As for Assumption (H2), it is a valid fact ([19, Theorem 4.2] and [12, Theorem 1.3]), but it is not verified, as was indicated in [29], the problem is not a Cauchy problem, and hence many uniqueness theorems cannot be applied. We propose it as an unsolved problem here.

Theorem 1.3. *Under Assumptions* (H1), (H2)*, system* (1.1) *is exactly controllable on some* $[0,T], T > 0$*. That is, given initial data* $w_0 \in H$ *and any time* $T > 0$*, there exists a boundary control* $u \in L^2(0,T; L^2(\Gamma_0))$ *such that the unique solution* $w \in C(0,T;H)$ *of* (1.1) *satisfies* $w(T) = 0$*.*

The following result is a direct consequence of Theorems 1.1 and 1.3.

Corollary 1.4. *Suppose* (1.14) *holds. Then system* (1.1) *is exponentially stable under the proportional output feedback* $u = -ky$ *for any* $k > 0$*.*

This article is organized as follows. In Section 2, we formulate system (1.1) into a collocated abstract first-order system. Some basic knowledge on Riemannian geometry is stated. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

2. Collocated formulation and preliminary results

In this section, we introduce some notations and facts in Riemannian geometry that we need in the following sections. For any $\varphi \in C^2(\mathbb{R}^n)$ and $N =$

$\sum_{i=1}^n h_i(x)\frac{\partial}{\partial x_i}$, denote

$$\begin{gathered}
\operatorname{div}_0(N)=\sum_{i=1}^n\frac{\partial h_i(x)}{\partial x_i},\quad D\varphi=\nabla_g\varphi=\sum_{i,j=1}^n\frac{\partial\varphi}{\partial x_j}a_{ij}(x)\frac{\partial}{\partial x_i},\\
\operatorname{div}_g(N)=\sum_{i=1}^n\frac{1}{\sqrt{\rho(x)}}\frac{\partial}{\partial x_i}(\sqrt{\rho(x)}h_i(x)),\\
\Delta_g\varphi=\sum_{i,j=1}^n\frac{1}{\sqrt{\rho(x)}}\frac{\partial}{\partial x_i}\Big(\sqrt{\rho(x)}a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\Big)=P\varphi-(Dp)\varphi,\\
p=\frac12\ln(\det[a_{ij}(x)]),
\end{gathered}\tag{2.1}$$

where div_0 is the divergence operator in Euclidean space $\mathbb{R}^n$, and $\nabla_g,\mathrm{div_g}$ and Δ_g are the gradient operator, the divergence operator and the Beltrami-Laplace operator in $(\mathbb{R}^n,g)$ respectively.

Let $\mu=\frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g}$ be the unit outward-pointing normal to $\partial\Omega$ in terms of the Riemannian metric g. The following Lemma [23, p. 128,138] provides some useful identities.

Lemma 2.1. *Let $\varphi,\psi\in C^2(\overline{\Omega})$ and let N be a vector field on $(\mathbb{R}^n,g)$. Then we have: (1) Divergence formulae and theorems*

$$\begin{gathered}
\operatorname{div}_0(\varphi N)=\varphi\operatorname{div}_0(N)+N(\varphi),\quad \operatorname{div}_g(\varphi N)=\varphi\operatorname{div}_g(N)+N(\varphi),\\
\int_\Omega\operatorname{div}_0(N)dx=\int_\Gamma N\cdot\nu d\Gamma,\quad \int_\Omega\operatorname{div}_g(N)dx=\int_\Gamma\langle N,\mu\rangle_g d\Gamma.
\end{gathered}$$

(2) Green's formulae

$$\begin{gathered}
\int_\Omega\psi P\varphi dx=\int_\Gamma\psi\frac{\partial\varphi}{\partial\nu_{\mathcal{A}}}d\Gamma-\int_\Omega\langle\nabla_g\varphi,\nabla_g\psi\rangle_g dx,\\
\int_\Omega\psi\Delta_g\varphi dx=\int_\Gamma\psi\frac{\partial\varphi}{\partial\mu}d\Gamma-\int_\Omega\langle\nabla_g\varphi,\nabla_g\psi\rangle_g dx.
\end{gathered}$$

Lemma 2.2. *We denote by $T^2(\mathbb{R}^n_x)$ the set of all covariant tensors of order 2 on $\mathbb{R}^n_x$. Then $T^2(\mathbb{R}^n_x)$ is an inner product space of dimension n^2 with the inner product*

$$\langle F,G\rangle_{T^2(\mathbb{R}^n_x)}=\sum_{i,j=1}^n F(e_i,e_j)G(e_i,e_j),\quad \forall F,G\in T^2(\mathbb{R}^n_x),\tag{2.2}$$

where $\{e_1,e_2,\dots,e_n\}$ is an arbitrarily chosen orthonormal basis for $(\mathbb{R}^n_x,g)$.

Let $\mathcal{X}(\mathbb{R}^n)$ be the set of all vector fields on $\mathbb{R}^n$. We denote by $\Delta:\mathcal{X}(\mathbb{R}^n)\to\mathcal{X}(\mathbb{R}^n)$ the Hodge-Laplace operator. Then [29, (2.2.7),(2.2.14)]*:*

$$\begin{gathered}
\Delta_g(N(\varphi))=(\Delta N)(\varphi)+2\langle DN,D^2\varphi\rangle_{T^2(\mathbb{R}^n_x)}+N(\Delta_g\varphi)+\operatorname{Ric}(N,D\varphi),\\
N(\Delta_g\varphi)=N(\mathcal{A}\varphi)-D^2p(N,D\varphi)-D^2\varphi(N,Dp),\quad \forall\varphi\in C^2(\mathbb{R}^n),
\end{gathered}\tag{2.3}$$

where $\operatorname{Ric}(\cdot,\cdot)$ is the Ricci curvature tensor of the Riemannian metric g, $D^2\varphi$ and D^2p are the Hessian of φ and p, respectively, in terms of the Riemannian metric g.

For a fixed $x\in\mathbb{R}^n$. Let $E_1,E_2,\dots,E_n$ be a frame field normal at x on $(\mathbb{R}^n,g)$, which means that $\langle E_i,E_j\rangle=\delta_{ij}$ in some neighborhood of x and $(D_{E_i}E_j)(x)=0$ for

$1 \leqslant i,j \leqslant n$. Set $N=\sum_{i=1}^n \gamma_i E_i$, then $N(\varphi)=\sum_{i=1}^n \gamma_i E_i(\varphi)$, where $E_i(\varphi)$ is the covariant derivative of φ with respect to E_i under the Riemannian metric g. Then

$$\begin{aligned}\langle Dp, D(N(\varphi))\rangle_g &= E_i(p)E_i(N(\varphi))\\ &= E_i(p)[E_i(\gamma_j)E_j(\varphi)+\gamma_j E_iE_j(\varphi)]\\ &= DN(D\varphi, Dp)+D^2\varphi(N,Dp).\end{aligned}\tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\begin{aligned}\mathcal{A}(N(\varphi)) &= (\Delta_g\varphi + Dp)(N(\varphi))\\ &= \Delta_g(N(\varphi)) + \langle Dp, D(N(\varphi))\rangle_g\\ &= (\Delta N)(\varphi) + 2\langle DN, D^2\varphi\rangle_{T^2(\mathbb{R}^n_x)} + N(\mathcal{A}\varphi) - D^2p(N,D\varphi)\\ &\quad + \operatorname{Ric}(N,D\varphi) + DN(D\varphi, Dp).\end{aligned}\tag{2.5}$$

Lemma 2.3 (see [16, Lemma 4.1]). *Let ψ be a smooth function on $\overline{\Omega}$ and satisfy $\psi|_\Gamma = 0$. Then there exists a continuous function $q(x)$ on Γ which is independent of ψ such that*

$$\Delta_g\psi(x) = \frac{\partial^2\psi}{\partial\mu^2} + q(x)\frac{\partial\psi(x)}{\partial\mu}, \quad \forall x\in\Gamma. \tag{2.6}$$

Moreover, if ψ satisfies $\frac{\partial\psi}{\partial\nu_{\mathcal{A}}}|_\Gamma = 0$, then

$$N(\psi)|_\Gamma = 0 \quad \text{on } \overline{\Omega} \text{ for any vector field } N. \tag{2.7}$$

So,

$$\mathcal{A}(\psi) = \Delta_g\psi + (Df)(\psi) = \Delta_g\psi = \frac{\partial^2\psi}{\partial\mu^2} = \frac{1}{|\nu_{\mathcal{A}}|_g^2}\frac{\partial^2\psi}{\partial\nu_{\mathcal{A}}^2} \quad \text{on } \Gamma, \tag{2.8}$$

and

$$\begin{aligned}\frac{\partial N(\psi)}{\partial\nu_{\mathcal{A}}} &= N\Big(\frac{\partial\psi}{\partial\nu_{\mathcal{A}}}\Big) = \langle N, \frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g}\rangle_g \frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g}\Big(\frac{\partial\varphi}{\partial\nu_{\mathcal{A}}}\Big)\\ &= N\cdot\nu\frac{1}{|\nu_{\mathcal{A}}|_g^2}\frac{\partial^2\psi}{\partial\nu_{\mathcal{A}}^2} = \mathcal{A}\psi N\cdot\nu \quad \text{on } \Sigma.\end{aligned}\tag{2.9}$$

Lemma 2.4. *Let φ be a complex function defined on $\overline{\Omega}$ with suitable regularity. Then there exist some constants C, possibly depending on g, N and Ω, such that:*
(1)

$$\sup_{x\in\overline{\Omega}}|N|_g \leqslant C, \quad \sup_{x\in\overline{\Omega}}|DN|_g \leqslant C, \quad \sup_{x\in\overline{\Omega}}|\operatorname{div}_g(N)| \leqslant C,$$
$$\sup_{x\in\overline{\Omega}}|Dp|_g \leqslant C, \quad \sup_{x\in\overline{\Omega}}|\nabla_g(\operatorname{div}_g N)|_g \leqslant C,$$

(2)

$$|N(\varphi)| \leqslant C|\nabla_g\varphi|_g, \quad |Dp(\varphi)| \leqslant C|\nabla_g\varphi|_g, \quad |DN(\nabla_g\varphi, \nabla_g\overline{\varphi})| \leqslant C|\nabla_g\varphi|_g^2,$$
$$|\langle\nabla_g\varphi, \nabla_g(\operatorname{div}_g N)\rangle_g| \leqslant C|\nabla_g\varphi|_g, \quad |(\Delta N)\varphi|_g \leqslant C|\Delta N|_g|\nabla_g\varphi|_g \leqslant C|\nabla_g\varphi|_g,$$
$$|\langle DN, D^2\varphi\rangle_{T^2(\mathbb{R}^n_x)}| \leqslant C|DN|_g|D^2\varphi|_g \leqslant C|D^2\varphi|_g,$$
$$|D^2p(N,D\varphi)| \leqslant |D^2p|_g|N|_g|D\varphi|_g \leqslant C|D\varphi|_g,$$
$$|D^2\varphi(N,Dp)| \leqslant |D^2\varphi|_g|N|_g|Dp|_g \leqslant C|D^2\varphi|_g,$$
$$|\operatorname{Ric}(N,D\varphi)| \leqslant |\operatorname{Ric}|_g|N|_g|D\varphi|_g \leqslant C|D\varphi|_g,$$

where $p(x) = \frac{1}{2}\ln(\det[a_{ij}(x)])$.

(3)

$$\int_\Omega |\varphi|^2 dx \leqslant C\|\varphi\|^2_{H^2(\Omega)}, \quad \int_\Omega |D\varphi|^2_g dx \leqslant C\|\varphi\|^2_{H^2(\Omega)}, \quad \int_\Omega |D^2\varphi|^2_g dx \leqslant C\|\varphi\|^2_{H^2(\Omega)}.$$

Now we cast the system (1.1) into an abstract first-order system in the state space $H = H^{-2}(\Omega)$ and control and output spaces $U = Y$.

Let A_1 be the positive self -adjoint operator in H induced by the bilinear form $a(\cdot,\cdot)$ defined by

$$\langle A_1 f, g\rangle_{H^{-2}(\Omega)\times H_0^2(\Omega)} = a(f,g) = \int_\Omega Af\overline{Ag}dx, \quad \forall f,g \in H_0^2(\Omega).$$

By the Lax-Milgram theorem, A_1 is a canonical isomorphism from $D(A_1) = H_0^2(\Omega)$ onto H. It is easy to show that $A_1 f = Af$ whenever $f \in H^4(\Omega)\cap H_0^2(\Omega)$ and that $A_1^{-1}g = A^{-1}g$ for any $g \in L^2(\Omega)$. Hence A_1 is an extension of A to the space $H_0^2(\Omega)$.

It is well known that $D(A_1^{1/2}) = L^2(\Omega)$ and $A_1^{1/2}$ is a canonical isomorphism from $L^2(\Omega)$ onto H (see [13]). Define the map $\gamma \in \mathcal{L}(L^2(\Gamma_0), H^{3/2}(\Omega))$ [18, p. 189] so that $\gamma u = \phi$ if and only if

$$\begin{gathered} P^2\phi(x) = 0, \quad x \in \Omega, \\ \phi(x)\big|_\Gamma = 0, \quad \frac{\partial\phi(x)}{\partial\nu_{\mathcal{A}}}\big|_{\Gamma_1} = 0, \quad \frac{\partial\phi(x)}{\partial\nu_{\mathcal{A}}}\big|_{\Gamma_0} = u(x). \end{gathered} \tag{2.10}$$

In terms of the Dirichlet map, we can write (1.1) as

$$\mathrm{i}\dot{w} + A_1(w - \gamma u) = 0. \tag{2.11}$$

It is clear that $D(A_1)$ is dense in H, so is $D(A_1^{1/2})$. We identify H with its dual H'. Then the following Gelfand-triple of continuous and dense inclusions hold:

$$D(A_1^{1/2}) \hookrightarrow H = H' \hookrightarrow D(A_1^{1/2})'. \tag{2.12}$$

Define an extension $\widetilde{A} \in \mathcal{L}(D(A_1^{1/2}), D(A_1^{1/2})')$ of A_1 by

$$\langle \widetilde{A}f, g\rangle_{D(A_1^{1/2})', D(A_1^{1/2})} = \langle A_1^{1/2}f, A_1^{1/2}g\rangle_H, \quad \forall f,g \in D(A_1^{1/2}). \tag{2.13}$$

Hence (2.11) can be written in $D(A_1^{1/2})'$ as

$$\dot{w} = \mathrm{i}\widetilde{A}w + Bu, \tag{2.14}$$

where $B \in \mathcal{L}(U, D(A_1^{1/2})')$ is given by

$$Bu = -\mathrm{i}\widetilde{A}\gamma u, \quad \forall u \in U. \tag{2.15}$$

Define $B^* \in \mathcal{L}(D(A_1^{1/2}), U)$ by

$$\langle B^* f, u\rangle_U = \langle f, Bu\rangle_{D(A_1^{1/2}), D(A_1^{1/2})'}, \quad \forall f \in D(A_1^{1/2}) = H_0^1(\Omega),\ u \in U. \tag{2.16}$$

Then for any $f \in D(A_1^{1/2})$ and $u \in C_0^\infty(\Gamma)$, we have

$$\begin{aligned}\langle Bu, f\rangle_{D(A_1^{1/2})', D(A_1^{1/2})} &= \langle \widetilde{A}^{-1}Bu, \widetilde{A}f\rangle_{D(A_1^{1/2}), D(A_1^{1/2})'} \\ &= \langle A_1^{1/2}\widetilde{A}^{-1}Bu, A_1^{1/2}f\rangle_H \\ &= \langle A_1^{-1}A_1^{1/2}\widetilde{A}^{-1}Bu, A_1^{-1}A_1^{1/2}f\rangle_{H_0^2(\Omega)} \\ &= \langle A_1^{-1/2}(-\mathrm{i}\gamma u), A_1^{-1/2}f\rangle_{H_0^2(\Omega)} \\ &= \langle -\mathrm{i}\gamma u, f\rangle_{L^2(\Omega)} \\ &= \langle -\mathrm{i}\gamma u, AA^{-1}f\rangle_{L^2(\Omega)} = \langle u, -\mathrm{i}\mathcal{A}(A^{-1}f)\rangle_{L^2(\Gamma_0)}.\end{aligned} \tag{2.17}$$

Since $C_0^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, we obtain

$$B^*f = -\mathrm{i}\mathcal{A}(A^{-1}f), \quad \forall f \in D(A_1^{1/2}) = L^2(\Omega). \tag{2.18}$$

Thus, we have formulated the open loop system (1.1) into an abstract first-order form in H:

$$\begin{gathered}\dot{w} = \mathrm{i}\widetilde{A}w + Bu, \\ y = B^*w.\end{gathered} \tag{2.19}$$

where $\widetilde{A}$, B and B^* are defined by (2.13), (2.15) and (2.18), respectively.

3. Proof of Theorem 1.1

We need the following Lemma which comes from [6, Theorem A.1].

Lemma 3.1. *If there exist constants $T > 0$, $C_T > 0$ such that the input and output of system* (1.1) *satisfy*

$$\int_0^T \|y(t)\|_U^2 dt \leqslant C_T \int_0^T \|u(t)\|_U^2 dt, \quad \forall u \in L^2(0,T; L^2(\Gamma_0)), \tag{3.1}$$

with $w(\cdot,0) \equiv 0$, then system (1.1) *is well-posed.*

Proof of Theorem 1.1. Introduce the transformation $z = A_1^{-1}w \in H_0^2(\Omega)$. Then z satisfies

$$\begin{gathered}z_t(x,t) - \mathrm{i}\mathcal{A}^2 z(x,t) = -\mathrm{i}(\gamma u(\cdot,t))(x,t), \quad (x,t) \in \Omega \times (0,T] =: Q, \\ z(x,0) = z_0(x), \quad x \in \Omega, \\ z(x,t) = \frac{\partial z(x,t)}{\partial \nu_{\mathcal{A}}} = 0, \quad (x,t) \in \partial\Omega \times [0,T] =: \Sigma,\end{gathered} \tag{3.2}$$

and from (2.18), the output of system (1.1) is changed into the form

$$y(x,t) = B^*w(x,t) = B^*A_1A_1^{-1}w(x,t) = B^*A_1z(x,t) = -\mathrm{i}\mathcal{A}z(x,t) \quad x \in \Gamma_0,\ t > 0. \tag{3.3}$$

Therefore, by Lemma 3.1, Theorem 1.1 amounts to saying for some (and hence for all) $T > 0$, that the solution to system (3.2) with zero initial data satisfies

$$\int_0^T \int_{\Gamma_0} |\mathcal{A}z(x,t)|^2 d\Gamma dt \leqslant C_T \int_0^T \int_{\Gamma_0} |u(x,t)|^2 d\Gamma dt. \tag{3.4}$$

We proceed with the proof in three steps.

Step 1. (Energy identity) Since $\partial\Omega$ is of class C^3, it follows from [15, Lemma 4.1] that there exists a C^2 vector field N on $\overline{\Omega}$ such that

$$N(x) = \mu(x),\ x \in \Gamma; \quad |N(x)| \leqslant 1,\ x \in \Omega. \tag{3.5}$$

Now, multiply both sides of the first equation in (3.2) by $N(\overline{z})$ and integrate over Q to obtain

$$\int_Q z_t N(\overline{z})dQ - \mathrm{i}\int_Q \mathcal{A}^2 z N(\overline{z})dQ = -\mathrm{i}\int_Q \gamma u N(\overline{z})dQ. \tag{3.6}$$

Compute the first term on the left-hand side of (3.6) to yield

$$\begin{aligned}
&\int_Q z_t N(\overline{z})dQ \\
&= \int_\Omega zN(\overline{z})dx\Big|_0^T - \int_Q zN(\overline{z}_t)dQ \\
&= \Big(\int_\Omega \mathrm{div}_g(|z|^2 N)dx - \int_\Omega \overline{z}N(z)dx - \int_\Omega |z|^2\,\mathrm{div}_g(N)dx\Big)\Big|_0^T \\
&\quad - \Big(\int_Q \mathrm{div}_g(z\overline{z}_t N)dQ - \int_Q \overline{z}_t N(z)dQ - \int_Q z\overline{z}_t\,\mathrm{div}_g(N)dQ\Big),
\end{aligned} \tag{3.7}$$

and hence

$$\begin{aligned}
2\mathrm{i}\,\mathrm{Im}\int_Q z_t N(\overline{z})dQ &= \int_Q z\overline{z}_t\,\mathrm{div}_g(N)dQ - \Big(\int_\Omega \overline{z}N(z)dx + \int_\Omega |z|^2\,\mathrm{div}_g(N)dx\Big)\Big|_0^T \\
&= \int_Q \mathrm{i}\overline{\gamma u}z\,\mathrm{div}_g(N)dQ - \int_Q \mathrm{i}\mathcal{A}^2\overline{z}z\,\mathrm{div}_g(N)dQ \\
&\quad - \Big(\int_\Omega \overline{z}N(z)dx + \int_\Omega |z|^2\,\mathrm{div}_g(N)dx\Big)\Big|_0^T.
\end{aligned} \tag{3.8}$$

Straightforward computations yield

$$\begin{aligned}
&\int_Q \mathcal{A}^2\overline{z}z\,\mathrm{div}_g(N)dQ \\
&= \int_Q |\mathcal{A}z|^2\,\mathrm{div}_g(N)dQ + \int_Q z\mathcal{A}(\mathrm{div}_g(N))\mathcal{A}\overline{z}dQ + 2\int_Q \mathcal{A}\overline{z}\langle\nabla_g z, \nabla_g\,\mathrm{div}_g(N)\rangle_g dQ,
\end{aligned} \tag{3.9}$$

where we used the fact $\mathcal{A}(\varphi\psi) = \psi\mathcal{A}\varphi + \varphi\mathcal{A}\psi + 2\langle\nabla_g\varphi, \nabla_g\psi\rangle_g$. Substituting (3.9) in (3.8) yileds

$$\begin{aligned}
\mathrm{Im}\int_Q z_t N(\overline{z})dQ &= \frac{1}{2}\int_Q \overline{\gamma u}z\,\mathrm{div}_g(N)dQ - \frac{1}{2}\int_Q |\mathcal{A}z|^2\,\mathrm{div}_g(N)dQ \\
&\quad - \frac{1}{2}\int_Q z\mathcal{A}(\mathrm{div}_g(N))\mathcal{A}\overline{z}dQ - \int_Q \mathcal{A}\overline{z}\langle\nabla_g z, \nabla_g\,\mathrm{div}_g(N)\rangle_g dQ \\
&\quad + \frac{\mathrm{i}}{2}\Big(\int_\Omega \overline{z}N(z)dx + \int_\Omega |z|^2\,\mathrm{div}_g(N)dx\Big)\Big|_0^T.
\end{aligned} \tag{3.10}$$

Next we compute the second term on the left-hand side of (3.6) to yield

$$
\begin{aligned}
&\operatorname{Im} \mathrm{i} \int_Q \mathcal{A}^2 z N(\overline{z}) dQ \\
&= \operatorname{Re} \int_Q \mathcal{A}^2 z N(\overline{z}) dQ = \operatorname{Re} \int_Q \Delta_g(\mathcal{A}z) N(\overline{z}) dQ + \operatorname{Re} \int_Q (Dp)(\mathcal{A}z) N(\overline{z}) dQ \\
&= -\operatorname{Re} \int_\Sigma \frac{\partial (N(\overline{z}))}{\partial \mu} \mathcal{A}z d\Sigma + \operatorname{Re} \int_Q \Delta_g(N(\overline{z})) \mathcal{A}z dQ \\
&\quad + \operatorname{Re} \int_Q (Dp)(\mathcal{A}z) N(\overline{z}) dQ \\
&= -\operatorname{Re} \int_\Sigma |\mathcal{A}z|^2 d\Sigma + \operatorname{Re} \int_\Sigma \mathcal{A}z (Dp)(\overline{z}) d\Sigma + \operatorname{Re} \int_Q (\Delta N)(\overline{z}) \mathcal{A}z dQ \\
&+ 2 \operatorname{Re} \int_Q \mathcal{A}z \langle DN, D^2 \overline{z} \rangle_{T^2(\mathbb{R}^n_x)} dQ + \operatorname{Re} \int_Q N(\mathcal{A}\overline{z}) \mathcal{A}z dQ \\
&\quad - \operatorname{Re} \int_Q D^2 p(N, D\overline{z}) \mathcal{A}z dQ - \operatorname{Re} \int_Q D^2 \overline{z}(N, Dp) \mathcal{A}z dQ \\
&\quad + \operatorname{Re} \int_Q \operatorname{Ric}(N, D\overline{z}) \mathcal{A}z dQ + \operatorname{Re} \int_Q (Dp)(\mathcal{A}z) N(\overline{z}) dQ,
\end{aligned}
\tag{3.11}
$$

where we have used (2.3) and the fact that

$$
\overline{z}\big|_\Gamma = \frac{\partial \overline{z}}{\partial \mu}\Big|_\Gamma = 0 \Rightarrow \frac{\partial^2 \overline{z}}{\partial \mu^2}\Big|_\Gamma = \Delta_g \overline{z}\big|_\Gamma,
$$

while

$$
\operatorname{Re} \int_Q N(\mathcal{A}\overline{z}) \mathcal{A}z dQ = \frac{1}{2} \int_\Sigma |\mathcal{A}z|^2 d\Sigma - \frac{1}{2} \int_Q |\mathcal{A}z|^2 \operatorname{div}_g(N) dQ, \tag{3.12}
$$

$$
\begin{aligned}
&\operatorname{Re} \int_Q Dp(\mathcal{A}z) N(\overline{z}) dQ \\
&= -\operatorname{Re} \int_Q \mathcal{A}z Dp(N(\overline{z})) dQ - \operatorname{Re} \int_Q N(\overline{z}) \mathcal{A}z \operatorname{div}_g(Dp) dQ.
\end{aligned}
\tag{3.13}
$$

Combining (3.6), (3.10), (3.11), (3.12) and (3.13) to obtain

$$
\begin{aligned}
&\frac{1}{2} \int_\Sigma |\mathcal{A}z|^2 d\Sigma \\
&= \frac{1}{2} \int_Q z \mathcal{A}(\operatorname{div}_g(N)) \mathcal{A}\overline{z} dQ + \int_Q \mathcal{A}\overline{z} \langle \nabla_g z, \nabla_g(\operatorname{div}_g(N)) \rangle_g dQ \\
&\quad + \operatorname{Re} \int_Q (\Delta N)(\overline{z}) \mathcal{A}z dQ + 2 \operatorname{Re} \int_Q \mathcal{A}z \langle DN, D^2 \overline{z} \rangle_{T^2(\mathbb{R}^n_x)} dQ \\
&\quad - \operatorname{Re} \int_Q D^2 p(N, D\overline{z}) \mathcal{A}z dQ - \operatorname{Re} \int_Q D^2 \overline{z}(N, Dp) \mathcal{A}z dQ \\
&\quad + R_1 + R_2 + b_{0,T},
\end{aligned}
\tag{3.14}
$$

where

$$\begin{aligned} R_1 &= \mathrm{Re}\int_Q \mathrm{Ric}(N, D\overline{z})\mathcal{A}z dQ - \mathrm{Re}\int_Q (\mathcal{A}z)Dp(N(\overline{z}))dQ \\ &\quad - \mathrm{Re}\int_Q N(\overline{z})\mathcal{A}z\,\mathrm{div}_g(Dp)dQ, \\ R_2 &= -\frac{1}{2}\int_Q \overline{\gamma u}z\,\mathrm{div}_g(N)dQ - \mathrm{Re}\int_Q \gamma u N(\overline{z})dQ, \\ b_{0,T} &= -\frac{\mathrm{i}}{2}\Big(\int_\Omega \overline{z}N(z)dx + \int_\Omega |z|^2\,\mathrm{div}_g(N)dx\Big)\Big|_0^T \end{aligned}$$

Step 2. (Estimation of R_1). Let $\gamma u = 0$ in the first identity of (3.2) and note that $z = A_1^{-1}w \in H_0^2(\Omega)$. We know that the solution to (3.2) is associated with a C_0-group on the space $H_0^2(\Omega)$. That is to say, for any $z_0 \in H_0^2(\Omega)$, there exists a unique solution $z \in H_0^2(\Omega)$ to (3.2), which depends continuously on z_0. This fact together with (3.14) implies that

$$\frac{1}{2}\int_\Sigma |\mathcal{A}z|^2 d\Sigma \leqslant C_T\|z_0\|_{H_0^2(\Omega)}. \tag{3.15}$$

This shows that the operator B^* is admissible, and so is B [4]. In other words,

$$u \mapsto w \text{ is continuous from } L^2(\Sigma) \text{ to } C(0,T;H^{-2}(\Omega)). \tag{3.16}$$

Moreover, by (3.16),

$$z = A_1^{-1}w \in H_0^2(\Omega) \text{ depends continuously on } u \in L^2(0,T;L^2(\Gamma_0)). \tag{3.17}$$

Therefore,

$$R_1 \leqslant C_T\|u\|^2_{L^2(0,T;L^2(\Gamma_0))}, \quad \forall u \in L^2(0,T;L^2(\Gamma_0)). \tag{3.18}$$

where we have used Lemma 2.4.

Step 3. (Estimation of R_2 and $b_{0,T}$). This can be easily obtained from the representations of R_2 and $b_{0,T}$ in (3.14) and (3.17) that

$$R_2 + b_{0,T} \leqslant C_T\|u\|^2_{L^2(0,T;L^2(\Gamma_0))},\ \forall u \in L^2(0,T;L^2(\Gamma_0)). \tag{3.19}$$

Finally, it follows from (3.14), (3.18), and (3.19) that (3.4) holds. The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.3

In this section, we show the exact controllability by means of the Hilbert Uniqueness Method (HUM) [17]. Since by Theorem 1.1, system (1.1) is well-posed which is cast into the abstract first-order formulation (2.19) and $(\mathrm{i}\widetilde{A})^* = -\mathrm{i}\widetilde{A}$ in $H^{-2}(\Omega)$, it follows that $\dot{w} = \mathrm{i}\widetilde{A}w + Bu$ is exactly controllable if and only if $\dot{w} = \mathrm{i}\widetilde{A}w, y = B^*w$ is exactly observable. More precisely, the exact controllability of system (1.1) is equivalent to the exact observability of the dual system of system (1.1) as follows:

$$\begin{gathered} \mathrm{i}\varphi_t + P^2\varphi = 0 \quad \text{in } \Omega\times(0,T] =: Q, \\ \varphi = 0, \quad \frac{\partial\varphi}{\partial\nu_{\mathcal{A}}} = 0 \quad \text{on } \partial\Omega\times[0,T] =: \Sigma, \\ \varphi(x,0) = \varphi^0(x) \quad \text{in } \Omega, \end{gathered} \tag{4.1}$$

with the output $y = -\mathrm{i}\mathcal{A}\varphi$. That is to say, the "observability inequality" holds for system (4.1) in the sense of (cf. (3.2) and (3.4)):

$$\int_0^T \int_{\Gamma_0} \big|\mathcal{A}\varphi\big|^2 d\Gamma dt \geqslant C_T \|\varphi^0\|^2_{H_0^2(\Omega)}, \quad \forall \varphi^0 \in H_0^2(\Omega), \tag{4.2}$$

for some (and hence for all) positive $T > 0$.

To prove (4.2), we let A be defined by (1.5) and let φ be a solution to (4.1). Then $\mathrm{i}A$ generates a strongly continuous unitary group on the space $H_0^2(\Omega)$ and hence

$$\begin{aligned} \|\varphi(t)\|_{H_0^2(\Omega)} &= \|A^{1/2}\varphi(t)\|_{L^2(\Omega)} = \|e^{\mathrm{i}At}\varphi^0\|_{H_0^2(\Omega)} \\ &= \|\varphi^0\|_{H_0^2(\Omega)} = \|A^{1/2}\varphi^0\|_{L^2(\Omega)}. \end{aligned} \tag{4.3}$$

In this Section, let N be an arbitrary vector field on $(\mathbb{R}^n, g)$. Assume that φ solves problem (4.1). Multiply the both sides of the first equation in (4.1) by $N(\overline{\varphi})$ and integrate on Q to obtain

$$\int_Q \varphi_t N(\overline{\varphi}) dQ - \mathrm{i} \int_Q \mathcal{A}^2 \varphi N(\overline{\varphi}) dQ = 0. \tag{4.4}$$

Now use the same process as the computation from (3.7) to (3.10) to obtain

$$\begin{aligned} \operatorname{Im} \int_Q \varphi_t N(\overline{\varphi}) dQ &= -\frac{1}{2} \int_Q |\mathcal{A}\varphi|^2 \operatorname{div}_0(N) dQ - \frac{1}{2} \int_Q \varphi \mathcal{A}(\operatorname{div}_0(N)) \mathcal{A}\overline{\varphi} dQ \\ &\quad - \int_Q \mathcal{A}\overline{\varphi} \langle \nabla_g \varphi, \nabla_g \operatorname{div}_0(N) \rangle_g dQ \\ &\quad + \frac{\mathrm{i}}{2} \Big(\int_\Omega \overline{\varphi} N(\varphi) dx + \int_\Omega |\varphi|^2 \operatorname{div}_0(N) dx \Big) \Big|_0^T. \end{aligned} \tag{4.5}$$

Next we compute the second term on the left hand side of (4.4)

$$\begin{aligned} &\operatorname{Im} \mathrm{i} \int_Q \mathcal{A}^2 \varphi N(\overline{\varphi}) dQ \\ &= \operatorname{Re} \int_Q \mathcal{A}^2 \varphi N(\overline{\varphi}) dQ \\ &= -\operatorname{Re} \int_\Sigma \frac{\partial (N(\overline{\varphi}))}{\partial \nu_{\mathcal{A}}} \mathcal{A}(\overline{\varphi}) d\Sigma + \operatorname{Re} \int_Q \mathcal{A}(N(\overline{\varphi})) \mathcal{A}\varphi dQ \\ &= -\frac{1}{2} \int_\Sigma |\mathcal{A}\varphi|^2 N \cdot \nu d\Sigma + \operatorname{Re} \int_Q \mathcal{A}\varphi [(\Delta N)(\overline{\varphi}) + 2 \langle DN, D^2 \overline{\varphi} \rangle_{T^2(\mathbb{R}^n_x)} \\ &\quad + \operatorname{Ric}(N, D\overline{\varphi}) - D^2 p(N, D\overline{\varphi}) + DN(D\overline{\varphi}, Dp)] dQ \\ &\quad - \frac{1}{2} \operatorname{Re} \int_Q |\mathcal{A}\varphi|^2 \operatorname{div}_0(N) dQ, \end{aligned} \tag{4.6}$$

where we have used (2.5), (2.7) and (2.9).

To obtain the observability inequality, we define $T \in T^2(\mathbb{R}^n_x)$ for any $x \in \overline{\Omega}$ as follows:

$$T(X, Y) = DN(X, Y) + DN(Y, X), \quad \forall X, Y \in \mathbb{R}^n_x. \tag{4.7}$$

It is clear that $T(\cdot, \cdot)$ is symmetric, and from (1.11), we have

$$DN(X, Y) + DN(Y, X) = 2b(x) \langle X, Y \rangle_g, \quad \forall X, Y \in \mathbb{R}^n_x,\ x \in \overline{\Omega}. \tag{4.8}$$

Fix $x \in \overline{\Omega}$, and let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $(\mathbb{R}_x^n, g)$. By (4.8), we have

$$\begin{aligned} \langle DN, D^2\varphi\rangle_{T^2(\mathbb{R}_x^n)} &= \sum_{i,j=1}^{n} DN(e_i, e_j) D^2\varphi(e_i, e_j) \\ &= b(x)\Delta_g\varphi = b(x)(\mathcal{A}\varphi - Dp(\varphi)). \end{aligned} \tag{4.9}$$

Combining (4.4), (4.5), (4.6) and (4.9) we obtain

$$\frac{1}{2}\int_\Sigma |\mathcal{A}\varphi|^2 N\cdot\nu d\Sigma = M_1 + M_2 + M_3 + M_4 \tag{4.10}$$

where

$$M_1 = 2\int_Q b(x)|\mathcal{A}\varphi|^2 dQ,$$

$$\begin{aligned} M_2 = \Big[&\frac{1}{2}\int_Q \varphi\mathcal{A}(\operatorname{div}_0(N))\mathcal{A}\overline{\varphi} dQ + \int_Q \mathcal{A}\overline{\varphi}\langle\nabla_g\varphi, \nabla_g \operatorname{div}_0(N)\rangle_g dQ \\ &+ \operatorname{Re}\int_Q \mathcal{A}\varphi[(\Delta N)(\overline{\varphi}) + \operatorname{Ric}(N, D\overline{\varphi}) - D^2p(N, D\overline{\varphi}) + DN(D\overline{\varphi}, Dp)]dQ \\ &- 2\operatorname{Re}\int_Q b(x)\mathcal{A}\varphi Dp(\overline{\varphi})dQ\Big], \end{aligned}$$

$$M_3 = -\frac{\mathrm{i}}{2}\Big(\int_\Omega \overline{\varphi}N(\varphi)dx\Big|_0^T,$$

$$M_4 = -\frac{\mathrm{i}}{2}\Big(\int_\Omega |\varphi|^2 \operatorname{div}_0(N)dx\Big)\Big|_0^T$$

Now define the energy function for (4.1) as

$$E(t) = E(\varphi, t) = \frac{1}{2}\int_\Omega |\mathcal{A}\varphi|^2 dx. \tag{4.11}$$

Then $E(t) = E(0)$ for all $t > 0$. Set

$$L(t) = \int_\Omega (|\varphi|^2 + |\nabla_g\varphi|_g^2)dx \tag{4.12}$$

be the lower order terms in composition of $E(t)$.

Lemma 4.1. *Suppose that* (H2) *holds. Let φ is the solution of* (4.1) *with $\mathcal{A}\varphi = 0$ on Σ_0. Then $\varphi \equiv 0$ in Q.*

Proof. Let

$$J = \{\varphi \in X = C(0,T; H_0^2(\Omega)); \varphi \text{ is the solution of } (4.1) \text{ with } \mathcal{A}\varphi|_{\Sigma_0} = 0\}.$$

We shall prove $J = 0$. First, note that for any given initial data $\varphi^0 \in H_0^2(\Omega)$, Equation (4.1) admits a unique weak solution

$$\varphi(t) \in C(0,T; H_0^2(\Omega)). \tag{4.13}$$

From this and (4.24) below, we have

$$E(0) \leqslant C(\|\mathcal{A}\varphi\|^2_{L^2(\Sigma_0)} + \|\varphi\|^2_{L^\infty(0,T;H_0^1(\Omega))}), \quad \forall\varphi \in X\text{solves } (4.1). \tag{4.14}$$

Now, we show that there exists a constant $C > 0$ such that for any $\varphi \in X$ satisfying

$$\|\varphi\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leqslant C(\|\mathcal{A}\varphi\|^2_{L^2(\Sigma_0)} + \|\varphi\|^2_{L^\infty(0,T;L^2(\Omega))}). \tag{4.15}$$

Actually, if (4.15) does not hold, then there exists a solution sequence $\{\varphi_n\} \in X$ to (4.1) satisfying

$$\|\mathcal{A}\varphi_n\|^2_{L^2(\Sigma_0)} + \|\varphi_n\|^2_{L^\infty(0,T;L^2(\Omega))} \to 0, \quad \text{as } n \to \infty, \tag{4.16}$$

$$\|\varphi_n\|^2_{L^\infty(0,T;H_0^1(\Omega))} = 1. \tag{4.17}$$

It is easy to learn from (4.13) and (4.14) that $\{\varphi_n\}$ is bounded in X and hence is relatively compact in $L^\infty(0,T;H_0^1(\Omega))$. Without loss of generality, we extract a subsequence $\{\varphi_n\}$ and assume it converges strongly to $\varphi \in L^\infty(0,T;H_0^1(\Omega))$, by (4.17), and satisfies

$$\|\varphi\|^2_{L^\infty(0,T;H_0^1(\Omega))} = 1. \tag{4.18}$$

However, (4.16) implies $\varphi \equiv 0$ in Q, which contradicts (4.18). So (4.15) holds.

From (4.14) and (4.15), we have

$$E(0) \leqslant C(\|\mathcal{A}\varphi\|^2_{L^2(\Sigma_0)} + \|\varphi\|^2_{L^2(0,T;L^2(\Omega))}), \quad \forall \varphi \in X \text{that solves (4.1)}. \tag{4.19}$$

Then (4.19) still holds for $\varphi \in L^\infty(0,T;L^2(\Omega))$ satisfying (4.1) by a denseness argument. Thus, we have proved that $\varphi \in J$ implies that $\psi = \dot{\varphi}$ satisfies (4.1) with $\mathcal{A}\psi|_{\Sigma_0} = 0$ and $\psi \in L^\infty(0,T;L^2(\Omega))$. This together with (4.19) gets

$$\psi(0) \in H_0^2(\Omega). \tag{4.20}$$

At last, because of (4.13), $\psi \in X$, it follows from (4.19) that the map $\frac{\partial}{\partial t} : \varphi \to \dot{\varphi}$ is continuous from J to J and the injection of $\{\varphi \in J; \dot{\varphi} \in J\}$ is compact. Therefore, J is a finite dimensional space. There must be an $\eta \in \mathbb{C}$ and $\varphi \in J\backslash\{0\}$ such that $\dot{\varphi} = \eta\varphi$, which implies

$$\varphi(x,t) = e^{\eta t}\varphi(x,0). \tag{4.21}$$

Substitute (4.21) into (4.1) to obtain (1.13) with $v(x) = \varphi(x,0)$ and $\zeta = -\mathrm{i}\eta$. By (H2), we obtain $\varphi(x,t) \equiv 0$, hence $J = \{0\}$. □

Next, we evaluate the terms on the right-hand side of (4.10).

$$\begin{aligned}
M_1 &= 2\,\mathrm{Re}\int_Q b(x)|\mathcal{A}\varphi|^2 dQ \geqslant 4b_0 T E(0),\\
|M_2| &\leqslant C_1\varepsilon T E(0) + \frac{C_2}{\varepsilon}\int_0^T L(t)dt,\\
|M_3| &= \Big| -\frac{\mathrm{i}}{2}\int_\Omega \overline{\varphi} N(\varphi)dx\Big| \leqslant \varepsilon E(0) + \frac{1}{16\varepsilon}L(t),\\
|M_4| &= \Big| -\frac{\mathrm{i}}{2}\int_\Omega |\varphi|^2 \operatorname{div}_g(N)dx\Big|\\
&= \Big| -\frac{\mathrm{i}}{2}\int_\Omega \varphi\overline{\varphi}\operatorname{div}_g(N)dx\Big|\\
&\leqslant C_3\varepsilon E(0) + \frac{C_4}{\varepsilon}L(t),
\end{aligned} \tag{4.22}$$

where we have used Lemma 2.4 in the evaluation of M_2, and (4.3) in the evaluations of M_3 and M_4, respectively. So

$$\begin{aligned} &\frac{1}{2}\int_{\Sigma_0} |\mathcal{A}\varphi|^2 N\cdot\nu d\Sigma \\ &\geqslant \frac{1}{2}\int_{\Sigma} |\mathcal{A}\varphi|^2 N\cdot\nu d\Sigma \\ &\geqslant 4b_0\big[T-\frac{C_1T+2+2C_3}{4b_0}\varepsilon\big]E(0) \\ &\quad -\frac{C_2}{\varepsilon}\int_0^T L(t)dt-\frac{1+16C_4}{16\varepsilon}L(T)-\frac{1+16C_4}{16\varepsilon}L(0). \end{aligned} \tag{4.23}$$

Setting $\varepsilon>0$ small enough, we obtain

$$E(0)\leqslant C_T\int_{\Sigma_0}|\mathcal{A}\varphi|^2 d\Sigma + C\Big(\int_0^T L(t)dt+L(T)+L(0)\Big). \tag{4.24}$$

Next, use the standard compact uniqueness argument to absorb the lower-order terms in (4.24). That is to say, we want to show that there exists a constant $C>0$ such that

$$\|\varphi\|^2_{L^\infty(0,T;H_0^1(\Omega))}\leqslant C\int_{\Sigma_0}|\mathcal{A}\varphi|^2 d\Sigma, \tag{4.25}$$

for the solution φ of (4.1). We will assume (4.25) is not true to obtain a contradiction. To this purpose, let $\{\varphi_n\}$ be the solution sequence of (4.1) such that

$$\int_{\Sigma_0}|\mathcal{A}\varphi_n|^2 d\Sigma\to 0,\quad n\to\infty, \tag{4.26}$$

$$\|\varphi_n\|^2_{L^\infty(0,T;H_0^1(\Omega))}=1. \tag{4.27}$$

Then it follows from (4.24) that $\{\varphi_n\}$ is a bounded sequence in $C(0,T;H_0^2(\Omega))$, and so relatively compact in $L^\infty(0,T;H_0^1(\Omega))$ because of the injection

$$C(0,T;H_0^2(\Omega))\to L^\infty(0,T;H_0^1(\Omega))$$

is compact. Without loss of generality, we extract a subsequence $\{\varphi_n\}$ and assume that $\{\varphi_n\}$ converges strongly to $\varphi\in L^\infty(0,T;H_0^1(\Omega))$. From (4.27),

$$\|\varphi\|^2_{L^\infty(0,T;H_0^1(\Omega))}=1. \tag{4.28}$$

Furthermore, $\{\varphi_n\}$ converges to φ in $L^\infty(0,T;H_0^2(\Omega))$ in weak star topology. Therefore, φ is a solution to (4.1) with

$$\varphi\in C(0,T;H_0^2(\Omega)). \tag{4.29}$$

By (3.15), we know that

$$\frac{1}{2}\int_{\Sigma_0}|\mathcal{A}\varphi|^2 d\Sigma\leqslant C_T\|\varphi_0\|_{H_0^2(\Omega)}.$$

From this fact and (4.26) to have

$$\mathcal{A}\varphi=0\quad\text{on }\Sigma_0. \tag{4.30}$$

Finally, by Lemma 4.1, we have

$$\varphi=0\quad\text{in }Q, \tag{4.31}$$

contradicting (4.28). So the proof of Theorem 1.3 is complete.

Acknowledgments. This work was supported by the National Natural Science Foundation of China for the Youth (No.61503230), and by the National Natural Science Foundation of China for the Youth (No. 61403239).

References

[1] K. Ammari; Dirichlet boundary stabilization of the wave equation, *Asymptotic Anal.*, 30 (2002), 117-130.

[2] K. Ammari, M. Tucsnak; Stabilization of second-order evolution equations by a class of unbounded feedbacks, *ESAIM Control Optim. Calc.Var.*, 6 (2001), 361-386.

[3] C. I. Byrnes, D. S. Gilliam, V. I. Shubov, G. Weiss; Regular linear systems governed by a boundary controlled heat equation, *J. Dyn. Control Syst.*, 8 (2002), 341-370.

[4] R. F. Curtain; The Salamon-Weiss class of well-posed infinite dimensional linear systems: A survey, *IMA J. Math. Control Inform.*, 14 (1997), 207-223.

[5] R. F. Curtain; Linear operator inequalities for strongly stable weakly regular linear systems, *Math. Control Signals Syst.*, 14 (2001), 299-337.

[6] S. G. Chai, B. Z. Guo; Well-posedness and regularity of Naghdi's shell equation under boundary control, *J. Differential Equations*, 249 (2010), 3174-3214.

[7] S. G. Chai, B. Z. Guo; Well-posedness and regularity of weakly coupled wave-plate equation with boundary control and observation, *J. Dyn. Control Syst.*, 15 (2003), 331-358.

[8] S. G. Chai, B. Z. Guo; Feedthrough operator for linear elasticity system with boundary control and observation, *SIAM J. Control Optim.*, 48 (2010), 3708-3734.

[9] B. Z. Guo, S. G. Chai; *Infinite-Dimensional Linear Dystem Control Theory*, Science Press, Beijing, 2012 (in Chinese).

[10] B. Z. Guo, Y. H. Luo; Controllability and Stability of a seconder order hyperbolic system with collocated sensor/actuator, *Systems Control Lett.*, 46 (2002), 45-65.

[11] B. Z. Guo, Z. C. Shao; Regularity of a Schrödinger equation with Dirichlet control and collocated observation, *Systems Control Lett.* 54 (2005), 1135-1142.

[12] B. Z. Guo, Z. C. Shao; On well-posedness, regularity and exact controllability for problems of transmission of plate equation with variable coefficients, *Quart. Appl. Math.*, 65 (2007), 705-736.

[13] B. Z. Guo, Z. C. Shao; Regularity of an Euler-Bernoulli equation with Neumann control and collocated observation, *J. Dyn. Control Syst.*, 12(2006), 405-418.

[14] B. Z. Guo, X. Zhang; The Regularity of wave equation with partial Dirichlet control and observation, *SIAM J. Control Optim.*, 44 (2005), 1598-1613.

[15] B. Z. Guo, Z. X. Zhang; On the well-posedness and regularity of wave equations with variable coefficients and partial boundary Dirichlet control and collocated observation, *ESAIM Control Optim. Calc. Var.*, 13 (2007), 776-792.

[16] B. Z. Guo, Z. X. Zhang; Well-posedness and regularity for an Euler-Bernoulli plate with variable coefficients and boundary control and observation, *Math. Control Signals Syst.*, 19 (2007), 337-360.

[17] J. L. Lions; Exact contrllability, Stabilization and perturbations for distributed systems, *SIAM Rev.*, 30 (1988), 1-68.

[18] J. L. Lions, E. Magenes; *Non-Homogeneous Boundary Value Problems and Application*, Vol. I, II, Springer-Verlag, Birlin, 1972.

[19] W. Liu, G. H. Williams; Exact controllability for problems of transmission of the plate equation with lower order terms, *Quart. Appl. Math.*, 58 (2000), 37-68.

[20] D. Salamon; Infinite dimensional systems with unbounded control and observation: a functional analytic approach, *Trans. Amer. Math. Soc.*, 300 (1987), 383-481.

[21] D. Salamon; Realization theory in Hilbert space, *Math. Systems Theory,* 21 (1989), 147-164.

[22] O. J. Staffans, G. Weiss; A physically motivated class of scattering passive linear systems, *SIAM J. Control Optim.*, 50 (2012), 3083-3112.

[23] M. E. Taylor; *Partial Diffirential Equations I: Basic Theory,* Springer-Verlag, New York, 1996.

[24] G. Weiss; Admissible observation operators for linear semigroups, *Israel J. Math.*, 65 (1989), no. 1, 17-43.

[25] R. L. Wen, S. G. Chai, B. Z. Guo; Well-posedness and exact controllability of fourth order Schrödinger equation with boundary control snd collocated onservation, *SIAM J. Control Optim.*, 52 (2014), 365-396.
[26] R. L. Wen, S. G. Chai, B. Z. Guo; Well-posedness and regularity of Euler-Bernoulli equation with variable coefficient and Dirichlet boundary control and collocated observation, *Math. Meth. Appl. Sci.*, 37 (2014), 2889-2905.
[27] G. Weiss; Admissibility of unbounded control operators, *SIAM J. Control Optim.*, 27 (1989), 527-545.
[28] P. F. Yao; On the observatility inequalities for exact controllability of wave equations with variable coefficients, *SIAM Journal on Control and Optimization* 37 (1999), 1568-1599.
[29] P. F. Yao; Observatility inequalities for the Euler-Bernoulli plate with variable coefficients, *Contemporary. Mathematics*, Amer. Math. Soc., Providence, RI 268 (2000), 383-406.
[30] H. Zwart; Transfer functions for infinite-dimensional systems, *Systems Control Lett.*, 52 (2004), 247-255.

Ruili Wen
School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China
E-mail address: wenruili@sxu.edu.cn

Shugen Chai (corresponding author)
School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China
E-mail address: sgchai@sxu.edu.cn