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MULTIPLICITY OF SOLUTIONS FOR EQUATIONS INVOLVING A NONLOCAL TERM AND THE BIHARMONIC OPERATOR

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Abstract. In this work we study the existence and multiplicity result of solutions to the equation

$$\Delta^2 u - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = \lambda |u|^{q-2}u + |u|^{2^{**}}u \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 5$, $1 < q < 2$ or $2 < q < 2^{**}$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. Since there is a competition between the function M and the critical exponent, we need to make a truncation on the function M. This truncation allows to define an auxiliary problem. We show that, for λ large, exists one solution and for λ small there are infinitely many solutions for the auxiliary problem. Here we use arguments due to Brezis-Nirenberg [12] to show the existence result and genus theory due to Krasnolselskii [29] to show the multiplicity result. Using the size of λ, we show that each solution of the auxiliary problem is a solution of the original problem.

1. Introduction

In this work we deal with questions of existence and multiplicity of solutions to an equation involving a nonlocal term and biharmonic operator. More precisely we study the equation

$$\begin{gathered} \Delta^2 u - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = \lambda |u|^{q-2}u + |u|^{2^{**}-2}u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < q < 2$ or $2 < q < 2^{**}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that satisfies conditions which will be stated later. Here $2^{**} = \frac{2N}{N-4}$ with $N \geq 5$ and Δ^2 is the biharmonic operator; that is,

$$\Delta^2 u = \sum_{i=1}^{N} \frac{\partial^4}{\partial x_i^4} u + \sum_{i \neq j}^{N} \frac{\partial^4}{\partial x_i^2 x_j^2} u.$$

Our study was strongly motivated by extensible beam equation type or of a stationary Berger plate equation, as can be seen below.

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In 1950, Woinowsky-Krieger [42] studied the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4} - \Big(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L |\frac{\partial u}{\partial x}|^2\, dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where L is the length of the beam in the rest position, E is the Young modulus of the material, I is the cross-sectional moment of inertia, ρ is the mass density, H is the tension in the rest position and A is the cross-sectional area. This model was proposed to modify the theory of the dynamic Euler-Bernoulli beam, assuming a nonlinear dependence of the axial strain on the deformation of the gradient. Owing to its importance in engineering, physics and material mechanics, since such model was proposed, this class of problems has been studied. These studies are focused on the properties of its solutions, as can be seen in [5, 6, 18, 33] and references therein. More recent references with important details about the physical motivation of (1.2) can be seen in [3, 28, 30, 35].

In 1955, Berger [8] studied the equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + \Big(Q + \int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(u, u_t, x), \tag{1.3}$$

which is called the Berger plate model [16], as a simplification of the von Karman plate equation which describes large deflection of plate, where the parameter Q describes in-plane forces applied to the plate and the function f represents transverse loads which may depend on the displacement u and the velocity u_t.

Problem (1.1) is a generalization of the stationary problem associated with problem (1.2) in dimension one or problem (1.3) in dimension two. Before stating our main results, we need the following hypotheses on the function $M : \mathbb{R}^+ \to \mathbb{R}^+$: The function M is continuous, increasing and there exists $0 < m_0$ such that

$$M(t) \geq m_0 = M(0), \quad \text{for all } t \in \mathbb{R}^+. \tag{1.4}$$

A typical example of a function satisfying this condition is

$$M(t) = m_0 + bt$$

with $b \geq 0$ and for all $t \geq 0$, which is the one considered for (1.2) by Woinowsky-Krieger [42] and for (1.3) by Berger in [8]. However, our hypotheses about the function M include other functions, such as $M(t) = m_0 + ln(1+t)$, $M(t) = m_0 + bt + \sum_{i=1}^k b_i t^{d_i}$ with $b_i \geq 0$ and $d_i \in (0,1)$ for all $i \in \{1, 2, \dots, k\}$ or $M(t) = \exp t$.

The first result is related to the case $1 < q < 2$ with a small positive parameter λ.

Theorem 1.1. *If* $1 < q < 2$ *and* (1.4) *hold, then exists a positive constant* λ^* *such that* (1.1) *has infinitely many solutions, for all* $\lambda \in (0, \lambda^*)$*. Moreover, if* u_λ *is one of these solutions, then* $u_\lambda \in C^{4,\alpha}(\Omega) \cap C^3(\overline{\Omega})$ *with* $\alpha \in (0,1)$ *and*

$$\lim_{\lambda \to 0} \|u_\lambda\| = 0.$$

The second result is related with the case $2 < q < 2^{**}$ with λ large.

Theorem 1.2. *If* $2 < q < 2^{**}$ *and* (1.4) *hold, then exists a positive constant* λ^{**} *such that* (1.1) *has a nontrivial solution* u_λ*, for all* $\lambda \in (\lambda^{**}, +\infty)$*. Moreover,* $u_\lambda \in C^{4,\alpha}(\Omega) \cap C^3(\overline{\Omega})$ *with* $\alpha \in (0,1)$ *and*

$$\lim_{\lambda \to +\infty} \|u_\lambda\| = 0.$$

Problem (1.1) with the function M constant and subcritical growth was exhaustively studied, as can be seen in [1, 11, 13, 25, 27, 37, 44] and references therein. On the other hand, there are only a few works dedicated to equations modeling stationary beam equations or Berger plate equation; that is, problems involving a function M depending on the gradient of the solution of problem. In this direction, we mention the papers [13, 31, 32, 38, 39, 40, 43]. The difficulty that arises in the study of this class of problems is the growth of the operator $\widehat{M}(\|u\|^2) = m_0\|u\|^2 + \frac{b}{2}\|u\|^4$, where $M(t) = \int_0^t M(s)\,ds$ and $m_0, b > 0$. This requires us to impose a 4-superlinear growth on the nonlinearity f; that is, $f(x,t) = t^p$ with $p \in (3, 2^{**} = \frac{2N}{N-4})$. But $2^{**} = \frac{2N}{N-4} \to 2$ as $N \to +\infty$. To circumvent this difficulty, it is common to fix $N \leq 4$ because, in this case, $2^{**} = \infty$ or M bounded or make a truncation on function M. In [31] the author shows some existence results using the Ekeland variational principle and also discuss a numerical example considering a general function M and $N = 1$. In [32] the author gives a necessary and sufficient condition for the existence of solutions when the nonlinearity is increasing considering again a general function M and $N = 1$. In [38] the authors show the existence of nontrivial solution using the Mountain Pass Theorem considering the function M bounded and $N \geq 1$. In [39] the authors show the existence of nontrivial solution using an iterative scheme of Mountain Pass "approximated" solutions considering the case $M(t) = \lambda(a + bt)$, $N \geq 1$, $a, b > 0$ and $\lambda > 0$ small. The paper [40] is a version of [39] in $\mathbb{R}^N$. In [43] the authors analyze from both the physical and the analytical viewpoints problem (1.1) with $N = 1$ and $M(t) = \gamma + t$. In this article, the author consider two cases namely: $\gamma > 0$ and $\gamma < 0$. In [14] the author consider a version of problem (1.1) in $\mathbb{R}^N$ with a general version of M and $N \geq 5$.

In this article, we complement the results found in [31, 32, 38, 39] in the following sense:

- (i) Unlike of [31], [32], [38] and [39], we overcome the difficult of competition between the operator and the critical exponent without consider $N \leq 4$ or M bounded or $M(t) = a + bt$ with a, b small. In our work we use a truncation on function M and we use the size of lambda to show that the solution of truncated problem is a solution of original problem. Of course, the estimates on the operator of the truncated problem was adapted from [38].
- (ii) Moreover, we study the asymptotic behavior of solution of problem (1.1) when $\lambda \to \infty$. This study was not observed in the articles above.

Unfortunately we do not have information on the case $1 < q < 2$ and λ large or on the case $2 < q < 2^{**}$ and λ small.

In the proof of theorem 1.1 we use an argument that can be found in [9] and the proof of Theorem 1.2 we use an argument that can be found in [4], for example. But, due to the presence of the function M and its truncation, some estimates more refined are necessary, such as in Lemmas 3.5 and 4.4.

In recent years, problems involving biharmonic or polyharmonic operators have received a special attention, in particular problems where the nonlinearity has a critical growth. In this interesting book [20], the reader can find a lot of results involving this class of operator and an excellent bibliography about this subject. In addition to this book, we would like to cite the papers [7, 9, 21, 22, 23, 24, 34, 36] and references therein.

The plan of this article is as follows. In Section 2, we define the truncated problem. In section 3, we recall some properties of genus theory, we prove some technical lemmas on truncated problem and we prove the Theorem 1.1. The proof of Theorem 1.2 is made in section 4.

2. AUXILIARY PROBLEM AND VARIATIONAL FRAMEWORK

Since intend to work with $N \geq 5$, we use a truncation argument. Here we are assuming, without loss of generality, that M is unbounded. Otherwise, the truncation of the function M is not necessary. We make a truncation on the function M for the case $1 < q < 2$ and another truncation on function M for the case $2 < q < 2^{**}$ as follows:

From (1.4), there exists $t_0 > 0$ such that $m_0 < M(t_0) < \frac{2^{**}}{2} m_0$ for the case $1 < q < 2$ and $m_0 < M(t_0) < \frac{q}{2} m_0$ for the case $2 < q < 2^{**}$. We set

$$M_0(t) := \begin{cases} M(t), & \text{if } 0 \leq t \leq t_0, \\ M(t_0) & \text{if } t \geq t_0. \end{cases} \tag{2.1}$$

From (1.4) we obtain

$$M_0(t) \leq \frac{2^{**}}{2} m_0 \quad \text{in the case } 1 < q < 2, \tag{2.2}$$

$$M_0(t) \leq \frac{q}{2} m_0 \quad \text{in the case } 2 < q < 2^{**}. \tag{2.3}$$

The proofs of Theorems 1.1 and 1.2 are based on a careful study of solutions of the auxiliary problem

$$\begin{gathered} \Delta^2 u - M_0\Big(\int_\Omega |\nabla u|^2 dx \Big) \Delta u = \lambda |u|^{q-2} u + |u|^{2^{**}-2} u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.4}$$

where N and λ are as in the introduction.

We say that $u \in H := H^2(\Omega) \cap H_0^1(\Omega)$ is a weak solution of problem (2.4) if u satisfies

$$\begin{aligned} &\int_\Omega \Delta u \Delta \phi \, dx + M_0\Big(\int_\Omega |\nabla u|^2 dx \Big) \int_\Omega \nabla u \nabla \phi dx \\ &= \lambda \int_\Omega |u|^{q-2} u \phi \, dx + \int_\Omega |u|^{2^{**}-2} u \phi \, dx, \end{aligned}$$

for all $\phi \in H$.

Note that H is a Hilbert space with the norm

$$\|u\|^2 = \int_\Omega |\Delta u|^2 \, dx + \int_\Omega |\nabla u|^2 dx$$

and we will look for solutions of (2.4) by finding critical points of the C^1-functional $I_\lambda : H \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx + \frac{1}{2} \widehat{M_0}\Big(\int_\Omega |\nabla u|^2 dx \Big) - \frac{\lambda}{q} \int_\Omega |u|^q \, dx - \frac{1}{2^{**}} \int_\Omega |u|^{2^{**}} \, dx,$$

where $\widehat{M_0}(t) = \int_0^t M_0(s) \, ds$. Note that

$$I'_\lambda(u)\phi = \int_\Omega \Delta u \Delta \phi \, dx + M_0\Big(\int_\Omega |\nabla u|^2 dx \Big) \int_\Omega \nabla u \nabla \phi dx$$

$$-\lambda\int_\Omega |u|^{q-2}u\phi\,dx-\int_\Omega |u|^{2^{**}-2}u\phi\,dx,$$

for all $\phi\in H$. Hence critical points of I_λ are weak solutions for (2.4).

To use variational methods, we first derive some results related to the Palais-Smale compactness condition.

We say that a sequence $(u_n)\subset H$ is a Palais-Smale sequence for the functional I_λ if

$$I_\lambda(u_n)\to c_\lambda \quad\text{and}\quad \|I'_\lambda(u_n)\|\to 0 \quad\text{in } H'. \tag{2.5}$$

If (2.5) implies the existence of a subsequence $(u_{n_j})\subset(u_n)$ which converges in H, we say that I_λ satisfies the Palais-Smale condition. If this strongly convergent subsequence exists only for some c_λ values, we say that I_λ satisfies a local Palais-Smale condition.

3. Case $1<q<2$

We start by considering some basic notions on the Krasnoselskii genus that we will use in the proof of Theorem 1.1.

3.1. Genus theory.

Let E be a real Banach space. Let us denote by $\mathfrak{A}$ the class of all closed subsets $A\subset E\setminus\{0\}$ that are symmetric with respect to the origin, that is, $u\in A$ implies $-u\in A$.

Definition 3.1. Let $A\in\mathfrak{A}$. The Krasnoselskii genus $\gamma(A)$ of A is defined as being the least positive integer k such that exists an odd mapping $\phi\in C(A,\mathbb{R}^k)$ such that $\phi(x)\neq 0$ for all $x\in A$. If such a k does not exist we set $\gamma(A)=\infty$. Furthermore, by definition, $\gamma(\emptyset)=0$.

In the sequel we will establish only the properties of the genus that will be used in this work. More information on this subject may be found in the references [2, 15, 17, 29].

Theorem 3.2. *Let $E=\mathbb{R}^N$ and let $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega\subset\mathbb{R}^N$ with $0\in\Omega$. Then $\gamma(\partial\Omega)=N$.*

Corollary 3.3. *$\gamma(S^{N-1})=N$.*

Proposition 3.4. *If $K\in\mathfrak{A}$ and $\gamma(K)\geq 2$, then K has infinitely many points.*

3.2. Proof of Theorem 1.1.

The genus theory requires that the functional I_λ be bounded below. Since this not occur, it is necessary to make other truncation. The plan of the proof is to show that the set of critical points of the truncated functional is compact, symmetric, does not contain the zero and has genus more than 2. Thus, by Proposition 3.4, this functional has infinitely many critical points. With the size of lambda, we show that each critical point of the truncated functional is a solution of the auxiliary problem and solution of the original problem.

Here we adapt arguments from [4]. We make a truncation in the functional I_λ as follows: From (1.4) and Sobolev's embedding, we obtain

$$I_\lambda(u)\geq \frac{k_0}{2}\|u\|^2-\frac{\lambda}{qS_q^{q/2}}\|u\|^q-\frac{1}{2^{**}S^{2^{**}/2}}\|u\|^{2^{**}}=g(\|u\|^2),$$

where $k_0=\min\{1,m_0\}$,

$$S_q:=\inf\big\{\|u\|^2: u\in H \text{ and } \int_\Omega |u|^q\,dx=1\big\},$$

$$S := \inf \big\{ \|u\|^2 : u \in H \text{ and } \int_\Omega |u|^{2^{**}}\,dx = 1 \big\},$$

and

$$g(t) = \frac{k_0}{2}t - \frac{\lambda}{qS_q^{q/2}}t^{q/2} - \frac{1}{2^{**}S^{2^{**}/2}}t^{2^{**}/2}. \tag{3.1}$$

Hence, there exists $\tau_1 > 0$ such that, if $\lambda \in (0, \tau_1)$, then g attains its positive maximum.

Denoting by $R_0(\lambda) < R_1(\lambda)$ the only roots of g. We have the following result.

Lemma 3.5.

$$R_0(\lambda) \to 0 \quad \textit{as } \lambda \to 0. \tag{3.2}$$

Proof. From $g(R_0(\lambda)) = 0$ and $g'(R_0(\lambda)) > 0$, we have

$$CR_0(\lambda) = \frac{\lambda}{qS_q^{q/2}}R_0(\lambda)^{q/2} + \frac{1}{2^{**}S^{2^{**}/2}}R_0(\lambda)^{2^{**}/2}, \tag{3.3}$$

$$C > \frac{\lambda}{2S_q^{q/2}}R_0(\lambda)^{q-2/2} + \frac{1}{2S^{2^{**}/2}}R_0(\lambda)^{2^{**}-2/2}, \tag{3.4}$$

for all $\lambda \in (0, \tau_1)$. From, (3.3) we conclude that $R_0(\lambda)$ is bounded. Suppose that $R_0(\lambda) \to \widetilde{R} > 0$ as $\lambda \to 0$. Then

$$C = \frac{1}{2^{**}S^{2^{**}/2}}\widetilde{R}^{2^{**}-2/2}, \tag{3.5}$$

$$C \geq \frac{1}{2S^{2^{**}/2}}\widetilde{R}^{2^{**}-2/2}, \tag{3.6}$$

which is a contradiction, because $2^{**} > 2$. Therefore $R_0(\lambda) \to 0$ as $\lambda \to 0$. □

We consider τ_1 such that $R_0 \leq M(t_0)$ and we make the following truncation on the functional I_λ:

Take $\phi \in C_0^\infty([0, +\infty))$, $0 \leq \phi(t) \leq 1$, for all $t \in [0, +\infty)$, such that $\phi(t) = 1$ if $t \in [0, R_0]$ and $\phi(t) = 0$ if $t \in [R_1, +\infty)$. Now, we consider the truncated functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}\widehat{M_0}\Big(\int_\Omega |\nabla u|^2 dx\Big) - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \phi(\|u\|^2)\frac{1}{2^{**}}\int_\Omega |u|^{2^{**}}\,dx.$$

Note that $J_\lambda \in C^1(H, \mathbb{R})$ and, as in (3.1), $J_\lambda(u) \geq \overline{g}(\|u\|^2)$, where

$$\overline{g}(t) = \frac{k_0}{2}t - \frac{\lambda}{qS_q^{q/2}}t^{q/2} - \phi(t)\frac{1}{2^{**}S^{2^{**}/2}}t^{2^{**}/2}.$$

Note that if $\|u\|^2 \leq R_0$ then $J_\lambda(u) = I_\lambda(u)$ and if $\|u\|^2 \geq R_1$, then

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}\widehat{M_0}\Big(\int_\Omega |\nabla u|^2 dx\Big) - \frac{\lambda}{q}\int_\Omega |u|^q\,dx.$$

Thus, we conclude that the functional J_λ is coercive and, hence, J_λ is bounded below.

Now, we show that J_λ satisfies the local Palais-Smale condition. For this, we need the following technical result, which is an analogous of [9, Lemma 3.3]. Here, λ_1 is the first eigenvalue of the problem

$$\begin{gathered} \Delta^2 u = \lambda u, \quad \text{in } \Omega \\ u = \Delta u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.7}$$

Lemma 3.6. *Let $(u_n) \subset H$ be a bounded sequence such that*

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty.$$

If

$$c_\lambda < \frac{2}{N} S^{N/4} - \lambda^{\frac{2}{(2-q)}} \big(\frac{1}{q} - \frac{1}{2^{**}}\big)|\Omega|^{\frac{(2-q)}{2}} \big[q\big(\frac{1}{q} - \frac{1}{2^{**}}\big)|\Omega|^{\frac{(2-q)}{2}} \frac{N}{4}\lambda_1^2\big]^{\frac{q}{(2-q)}},$$

then we have that, up to a subsequence, (u_n) is strongly convergent in H.

Proof. Taking a subsequence, we suppose that

$$|\Delta u_n|^2 \rightharpoonup |\Delta u|^2 + \mu, \quad |\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + \gamma,$$
$$|u_n|^{2^{**}} \rightharpoonup |u|^{2^{**}} + \nu \quad \text{(weak}^*\text{-sense of measures)}.$$

Using the concentration compactness-principle by Lions [26, Lemma 2.1], we obtain at most countable index set Λ, sequences $(x_i) \subset \mathbb{R}^N$, $(\mu_i), (\gamma_i), (\nu_i), \subset [0, \infty)$, such that

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \mu \ge \sum_{i\in\Lambda} \mu_i \delta_{x_i}, \quad \gamma \ge \sum_{i\in\Lambda} \gamma_i \delta_{x_i}, \quad S\nu_i^{2/2^{**}} \le \mu_i, \tag{3.8}$$

for all $i \in \Lambda$, where δ_{x_i} is the Dirac mass at $x_i \in \mathbb{R}^N$.

Now we claim that $\Lambda = \emptyset$. Arguing by contradiction, assume that $\Lambda \neq \emptyset$ and fix $i \in \Lambda$. Consider $\psi \in C_0^\infty(\Omega, [0, 1])$ such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\Omega \setminus B_2(0)$ and $|\nabla\psi|_\infty \le 2$. Defining $\psi_\varrho(x) := \psi((x - x_i)/\varrho)$ where $\varrho > 0$, we have that $(\psi_\varrho u_n)$ is bounded. Thus $I'_\lambda(u_n)(\psi_\varrho u_n) \to 0$; that is,

$$\begin{aligned}
&\int_\Omega u_n \Delta u_n \Delta\psi_\varrho \, dx + \int_\Omega \psi_\varrho |\Delta u_n|^2 \, dx + 2\int_\Omega \Delta u_n \nabla\psi_\varrho \nabla u_n \, dx \\
&+ M_0\Big(\int_\Omega |\nabla u_n|^2 \, dx\Big) \int_\Omega u_n \nabla u_n \nabla\psi_\varrho \, dx + M_0\Big(\int_\Omega |\nabla u_n|^2 \, dx\Big) \int_\Omega \psi_\varrho |\nabla u_n|^2 \, dx \\
&= \lambda \int_\Omega |u_n|^q \psi_\varrho \, dx + \int_\Omega \psi_\varrho |u_n|^{2^{**}} \, dx + o_n(1).
\end{aligned}$$

Since the support of ψ_ϱ is contained in $B_{2\varrho}(x_i)$, we obtain

$$\Big|\int_\Omega u_n \Delta u_n \Delta\psi_\varrho \, dx\Big| \le \int_{B_{2\rho}(x_i)} |\Delta u_n||u_n \Delta\psi_\varrho| \, dx.$$

By Hölder inequality and the fact that the sequence (u_n) is bounded in H we have

$$\begin{aligned}
|\int_\Omega u_n \Delta u_n \Delta\psi_\varrho \, dx| &\le C\Big(\int_{B_{2\varrho}(x_i)} |u_n \Delta\psi_\varrho|^2 \, dx\Big)^{1/2} \\
&\le C\Big(\int_{B_{2\varrho}(x_i)} |u_n|^2 |\Delta\psi_\varrho|^2 \, dx\Big)^{1/2}.
\end{aligned}$$

By the Dominated Convergence Theorem $\int_{B_{2\varrho}(x_i)} |u_n \Delta\psi_\varrho|^2 \, dx \to 0$ as $n \to +\infty$ and $\varrho \to 0$. Thus, we obtain

$$\lim_{\varrho\to 0} \big[\lim_{n\to\infty} \int_\Omega u_n \Delta u_n \Delta\psi_\varrho \, dx\big] = 0.$$

Using the same reasoning we obtain

$$\lim_{\varrho\to 0} \big[\lim_{n\to\infty} \int_\Omega u_n \nabla u_n \nabla\psi_\varrho \, dx\big] = 0,$$

$$\lim_{\varrho\to 0}\Big[\lim_{n\to\infty}\int_\Omega \Delta u_n \nabla\psi_\varrho \nabla u_n\,dx\Big]=0,$$
$$\lim_{\varrho\to 0}\lim_{n\to\infty}\Big[\int_\Omega \psi_\varrho |u_n|^q\,dx\Big]=0.$$

Since $0<m_0\le M_0(t)\le M(t_0)$, for all $t\in\mathbb{R}$, we obtain

$$\lim_{\varrho\to 0}\lim_{n\to\infty}\big[M_0(\|u_n\|^2)\int_\Omega u_n\nabla u_n\nabla\psi_\varrho\,dx\big]=0.$$

Thus, we have

$$\int_\Omega \psi_\varrho \mathrm{d}\mu \le \int_\Omega \psi_\varrho \mathrm{d}\mu + m_0\int_\Omega \psi_\varrho \mathrm{d}\gamma \le \int_\Omega \psi_\varrho \mathrm{d}\nu + o_\varrho(1).$$

Letting $\varrho\to 0$ and using standard theory of Radon measures, we conclude that $\mu_i\le\nu_i$. It follows from (3.8) that

$$\mu_i\ge S\nu_i^{2/2^{**}}\ge S\mu_i^{2/2^{**}},$$

where we conclude that

$$\mu_i\ge S^{N/4}. \tag{3.9}$$

Now we shall prove that the above inequality cannot occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that $\mu_i\ge S^{N/4}$, for some $i\in\Lambda$. Thus,

$$c_\lambda = I_\lambda(u_n)-\frac{1}{2^{**}}I'_\lambda(u_n)u_n+o_n(1).$$

Since $M_0(t)\le\frac{2^{**}}{2}m_0$ for all $t\in\mathbb{R}$, we have

$$c_\lambda\ge\frac{2}{N}\int_\Omega|\Delta u_n|^2\,dx-\lambda(\frac{1}{q}-\frac{1}{2^{**}})\int_\Omega|u_n|^q\,dx.$$

Letting $n\to\infty$, we obtain

$$c_\lambda\ge\frac{2}{N}\mu_i+\frac{2}{N}\int_\Omega|\Delta u|^2\,dx-\lambda(\frac{1}{q}-\frac{1}{2^{**}})\int_\Omega|u|^q\,dx.$$

Hence,

$$c_\lambda\ge\frac{2}{N}S^{N/4}+\frac{2}{N}\frac{1}{\lambda_1^2}\int_\Omega|u|^2\,dx-\lambda(\frac{1}{q}-\frac{1}{2^{**}})\int_\Omega|u|^q\,dx.$$

By Hölder's inequality

$$c_\lambda\ge\frac{2}{N}S^{N/4}+\frac{2}{N}\frac{1}{\lambda_1^2}\int_\Omega|u|^2\,dx-\lambda(\frac{1}{q}-\frac{1}{2^{**}})|\Omega|^{\frac{(2-q)}{2}}\Big(\int_\Omega|u|^2\,dx\Big)^{q/2}.$$

Note that

$$f(t)=\frac{2}{N}\frac{1}{\lambda_1^2}t^2-\lambda(\frac{1}{q}-\frac{1}{2^{**}})|\Omega|^{\frac{(2-q)}{2}}t^q$$

is a continuous function that attains its absolute minimum, for $t>0$, at the point

$$\alpha_0=\Big[q\lambda\big(\frac{1}{q}-\frac{1}{2^{**}}\big)|\Omega|^{(2-q)/2}\frac{N}{4}\lambda_1^2\Big]^{\frac{1}{(2-q)}}.$$

Hence,

$$c_\lambda\ge\frac{2}{N}S^{N/4}+\frac{2}{N}\frac{1}{\lambda_1^2}\alpha_0^2-\lambda(\frac{1}{q}-\frac{1}{2^{**}})|\Omega|^{\frac{(2-q)}{2}}\alpha_0^q.$$

So

$$c_\lambda \geq \frac{2}{N}S^{N/4} - \lambda(\frac{1}{q} - \frac{1}{2^{**}})|\Omega|^{\frac{(2-q)}{2}}\alpha_0^q.$$

Thus, we conclude that

$$c_\lambda \geq \frac{2}{N}S^{N/4} - \lambda^{\frac{2}{(2-q)}}(\frac{1}{q} - \frac{1}{2^{**}})|\Omega|^{\frac{(2-q)}{2}}\Big[q\big(\frac{1}{q} - \frac{1}{2^{**}}\big)|\Omega|^{\frac{(2-q)}{2}}\frac{N}{4}\lambda_1^2\Big]^{\frac{q}{(2-q)}},$$

which is a contradiction. Thus Λ is empty and it follows that $u_n \to u$ in $L^{2^{**}}(\Omega)$. Thus, up to a subsequence,

$$\lim_{n\to\infty}\Big[\int_\Omega |\Delta u_n|^2\,dx + M_0\big(\int_\Omega |\nabla u_n|^2\,dx\big)\int_\Omega |\nabla u_n|^2\,dx\Big] = \lambda\int_\Omega |u|^q\,dx + \int_\Omega |u|^{2^{**}}\,dx.$$

Moreover, since $u_n \rightharpoonup u$ in H and $M_0\big(\int_\Omega |\nabla u_n|^2\,dx\big) \to \beta$, for some $\beta \geq 0$, we have

$$\big[\int_\Omega |\Delta u|^2\,dx + \beta\int_\Omega |\nabla u|^2\,dx\big] = \lambda\int_\Omega |u|^q\,dx + \int_\Omega |u|^{2^{**}}\,dx.$$

We claim that

$$\lim_{n\to\infty}\|u_n\|^2 = \|u\|^2$$

because, otherwise, we have

$$\limsup_{n\to\infty}\int_\Omega |\Delta u_n|^2\,dx < \int_\Omega |\Delta u|^2\,dx$$

or

$$\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^2\,dx \ < \int_\Omega |\nabla u|^2\,dx.$$

The second inequality implies

$$\limsup_{n\to\infty} M_0\big(\int_\Omega |\nabla u_n|^2\,dx\big)\int_\Omega |\nabla u_n|^2\,dx \ < \beta\int_\Omega |\nabla u|^2\,dx.$$

Thus, in either of these two cases, we have

$$\begin{aligned}
&\lambda\int_\Omega |u|^q\,dx + \int_\Omega |u|^{2^{**}}\,dx\\
&= \limsup_{n\to\infty}\big[\int_\Omega |\Delta u_n|^2\,dx + M_0\big(\int_\Omega |\nabla u_n|^2\,dx\big)\int_\Omega |\nabla u_n|^2\,dx\big]\\
&< \int_\Omega |\Delta u|^2\,dx + \beta\int_\Omega |\nabla u|^2\,dx\\
&= \lambda\int_\Omega |u|^q\,dx + \int_\Omega |u|^{2^{**}}\,dx,
\end{aligned}$$

which is a contradiction. Hence, $\|u_n - u\|^2 = o_n(1)$. □

By Lemma 3.6 we conclude that, there exists $\tau_2 > 0$ such that, for all $\lambda \in (0, \tau_2)$ we obtain

$$\frac{2}{N}S^{N/4} - \lambda^{\frac{2}{(2-q)}}\big(\frac{1}{q} - \frac{1}{2^{**}}\big)|\Omega|^{\frac{(2-q)}{2}}\Big[q\big(\frac{1}{q} - \frac{1}{2^{**}}\big)|\Omega|^{\frac{(2-q)}{2}}\frac{N}{4}\lambda_1^2\Big]^{\frac{q}{(2-q)}} > 0$$

and, hence, if (u_n) is a bounded sequence such that $I_\lambda(u_n) \to c$, $I'_\lambda(u_n) \to 0$ with $c < 0$, then (u_n) has a subsequence convergent.

Lemma 3.7. *If $J_\lambda(u) < 0$, then $\|u\|^2 < R_0 \leq M(t_0)$ and $J_\lambda(v) = I_\lambda(v)$, for all v in a small enough neighborhood of u. Moreover, J_λ satisfies a local Palais-Smale condition for $c_\lambda < 0$.*

Proof. Since $\lambda \in (0, \tau_1)$ and $J_\lambda(u) < 0$, then by definition of $\overline{g}$, we obtain $\overline{g}(\|u\|^2) \leq J_\lambda(u) < 0$. Consequently, $J_\lambda(u) = I_\lambda(u)$. Hence, we conclude $\|u\|^2 < R_0 \leq M(t_0)$. Moreover, since J_λ is a continuous functional, we derive $J_\lambda(v) = I_\lambda(v)$, for all $v \in B_{R_0/2}(0)$. Besides, if (u_n) is a sequence such that $J_\lambda(u_n) \to c_\lambda < 0$ and $J'_\lambda(u_n) \to 0$, for n sufficiently large, $I_\lambda(u_n) = J_\lambda(u_n) \to c_\lambda < 0$ and $I'_\lambda(u_n) = J'_\lambda(u_n) \to 0$. Since J_λ is coercive, we obtain that (u_n) is bounded in H. From Lemma 3.6, for all $\lambda \in (0, \tau_2)$, we obtain

$$c_\lambda < 0 < \frac{2}{N} S^{N/4} - \lambda^{\frac{2}{(2-q)}} \big(\frac{1}{q} - \frac{1}{2^{**}}\big) |\Omega|^{\frac{(2-q)}{2}} \Big[q \big(\frac{1}{q} - \frac{1}{2^{**}}\big) |\Omega|^{\frac{(2-q)}{2}} \frac{N}{4} \lambda_1^2 \Big]^{\frac{q}{(2-q)}}$$

and, hence, up to a subsequence, (u_n) is strongly convergent in H. □

Now, we construct an appropriate mini-max sequence of negative critical values for the functional J_λ.

Lemma 3.8. *Given $k \in \mathbb{N}$, there exists $\epsilon = \epsilon(k) > 0$ such that*

$$\gamma(J_\lambda^{-\epsilon}) \geq k,$$

where $J_\lambda^{-\epsilon} = \{u \in H : J_\lambda(u) \leq -\epsilon\}$ and γ was given in definition 3.1.

Proof. Fix $k \in \mathbb{N}$, let X_k be a k-dimensional subspace of H. Thus, there exists $C_k > 0$ such that

$$-C(k)\|u\|^q \geq -\int_\Omega |u|^q \, dx,$$

for all $u \in X_k$. We now use the inequality above and (2.2) to conclude that

$$J_\lambda(u) \leq \frac{2^{**}}{4} \|u\|^2 - \frac{C(k)}{q} \|u\|^q = \|u\|^q \Big(\frac{k_1}{2} \|u\|^{2-q} - \frac{C(k)}{q} \Big).$$

Considering $R > 0$ sufficiently small, there exists $\epsilon = \epsilon(R) > 0$ such that

$$J_\lambda(u) < -\epsilon < 0,$$

for all $u \in \mathcal{S}_R = \{u \in X_k ; \|u\| = R\}$. Since X_k and $\mathbb{R}^k$ are isomorphic and $\mathcal{S}_R$ and S^{k-1} are homeomorphic, where S^{k-1} is the sphere of $\mathbb{R}^k$. Then we conclude from Corollary 3.3 that $\gamma(\mathcal{S}_R) = \gamma(S^{k-1}) = k$. Moreover, since $\mathcal{S}_R \subset J_\lambda^{-\epsilon}$ and $J_\lambda^{-\epsilon}$ is symmetric and closed, we have

$$k = \gamma(\mathcal{S}_R) \leq \gamma(J_\lambda^{-\epsilon}).$$

□

Now for each $k \in \mathbb{N}$, we define the sets

$$\Gamma_k = \{C \subset H \backslash \{0\} : C \text{ is closed }, C = -C \text{ and } \gamma(C) \geq k\},$$
$$K_c = \{u \in H \backslash \{0\} : J'_\lambda(u) = 0 \text{ and } J_\lambda(u) = c\}$$

and the number

$$c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\lambda(u).$$

Lemma 3.9. *For each $k \in \mathbb{N}$, the number c_k is negative.*

Proof. From Lemma 3.8, for each $k \in \mathbb{N}$ there exists $\epsilon > 0$ such that $\gamma(J_\lambda^{-\epsilon}) \geq k$. Moreover, $0 \notin J_\lambda^{-\epsilon}$ and $J_\lambda^{-\epsilon} \in \Gamma_k$. On the other hand

$$\sup_{u \in J_\lambda^{-\epsilon}} J_\lambda(u) \leq -\epsilon.$$

Hence,

$$-\infty < c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\lambda(u) \leq \sup_{u \in J_\lambda^{-\epsilon}} J_\lambda(u) \leq -\epsilon < 0.$$

□

The next Lemma allows us to prove the existence of critical points of J_λ.

Lemma 3.10. *If $c = c_k = c_{k+1} = \cdots = c_{k+r}$ for some $r \in \mathbb{N}$, then there exists $\lambda^* > 0$ such that*

$$\gamma(K_c) \geq r + 1,$$

for $\lambda \in (0, \lambda^*)$.

Proof. Since $c = c_k = c_{k+1} = \cdots = c_{k+r} < 0$, for $\lambda^* = \min\{\tau_1, \tau_2\}$ and for all $\lambda \in (0, \lambda^*)$, from Lemma 3.6 and Lemma 3.9, we obtain that K_c is a compact set. Moreover, $K_c = -K_c$. If $\gamma(K_c) \leq r$, there exists a closed and symmetric neighborhood U of K_c such that $\gamma(U) = \gamma(K_c) \leq r$. Note that we can choose $U \subset J_\lambda^0$ because $c < 0$. By the deformation lemma [10] we have an odd homeomorphism $\eta : H \to H$ such that $\eta(J_\lambda^{c+\delta} - U) \subset J_\lambda^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J_\lambda^{c+\delta} \subset J_\lambda^0$ and by definition of $c = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u \in A} < c + \delta$, that is, $A \subset J_\lambda^{c+\delta}$ and

$$\eta(A - U) \subset \eta(J_\lambda^{c+\delta} - U) \subset J_\lambda^{c-\delta}. \tag{3.10}$$

But $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{A - U})) \geq \gamma(\overline{A - U}) \geq k$. Then $\eta(\overline{A - U}) \in \Gamma_k$ and this contradicts (3.10). Hence, the lemma is proved. □

Remark 3.11. If $-\infty < c_1 < c_2 < \cdots < c_k < \cdots < 0$ with $c_i \neq c_j$, since each c_k is a critical value of J_λ, then we obtain infinitely many critical points of J_λ and, hence problem (2.4) has infinitely many solutions.

On the other hand, if there are two constants $c_k = c_{k+r}$, then $c = c_k = c_{k+1} = \cdots = c_{k+r}$ and from Lemma 3.10, there exists $\lambda^* > 0$ such that

$$\gamma(K_c) \geq r + 1 \geq 2$$

for all $\lambda \in (0, \lambda^*)$. From Proposition 3.4, K_c has infinitely many points, that is, problem (2.4) has infinitely many solutions.

Proof of Theorem 1.1. Let λ^* be as in Lemma 3.10 and, for $\lambda < \lambda^*$, let u_λ be the nontrivial solution of problem (2.4) found in remark 3.11. Thus $J_\lambda(u_\lambda) = I_\lambda(u_\lambda) < 0$. Hence,

$$\int_\Omega |\nabla u_\lambda|^2 \, dx \leq \|u_\lambda\|^2 \leq R_0 \leq t_0. \tag{3.11}$$

By the definition of M_0 we obtain

$$M_0\Big(\int_\Omega |\nabla u_\lambda|^2 \, dx\Big) = M\Big(\int_\Omega |\nabla u_\lambda|^2 \, dx\Big),$$

which implies that u_λ is a solution of (1.1). Moreover, from (3.11) and (3.2), we conclude

$$\lim_{\lambda \to 0} \|u_\lambda\| = 0.$$

Since for each solution u_λ we have that $M(\|u_\lambda\|^2) \geq m_0 > 0$ is a positive number, then the regularity of these solutions is a consequence of [9, Theorem 2.1]. □

4. CASE $2 < q < 2^{**}$

In this section, we adapt for our study some ideas from [19]. In the sequel, we prove that the functional I_λ has the Mountain Pass Geometry. This fact is proved in the next lemmas:

Lemma 4.1. *Assume that condition* (1.4) *holds. There exist positive numbers ρ and α such that*

$$I_\lambda(u) \geq \alpha > 0, \quad \forall u \in H : \|u\| = \rho.$$

Proof. From (1.4), we have

$$I_\lambda(u) \geq \frac{k_0}{2}\|u\|^2 - \frac{\lambda}{q}\int_\Omega |u|^q \, dx - \frac{1}{2^{**}}\int_\Omega |u|^{2^{**}} \, dx,$$

where $k_0 = \min\{1, m_0\}$. So, using Sobolev's Embedding Theorem, there exists a positive constant $C > 0$ such that

$$I_\lambda(u) \geq C\|u\|^2 - \lambda C\|u\|^q - C\|u\|^{2^*}.$$

Since $2 < q < 2^{**}$, the result follows by choosing $\rho > 0$ small enough. □

Lemma 4.2. *For all $\lambda > 0$, there exists $e \in H$ with $I_\lambda(e) < 0$ and $\|e\| > \rho$, where ρ was given in Lemma 4.1.*

Proof. Fix $v_0 \in C_0^\infty(\Omega) \setminus \{0\}$ with $v_0 \geq 0$ in Ω and $\|v_0\| = 1$. Using (2.3) we obtain

$$I_\lambda(tv_0) \leq \frac{1}{2}\max\{1, \frac{m_0 q}{2}\}t^2 - \frac{t^{2^{**}}}{2^{**}}\int_\Omega |v_0|^{2^{**}} \, dx.$$

Since $2 < q < 2^{**}$, the result follows by considering $e = \bar{t}v_0$ for some $\bar{t} > 0$ large enough. □

Using a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [2], without (PS) condition (see [41, p.12]), there exists a sequence $(u_n) \subset H$ satisfying

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0,$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0,$$
$$\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, \ I_\lambda(\gamma(1)) < 0\}.$$

Next, we shall prove an estimate for c_λ.

Lemma 4.3. *If condition* (1.4) *holds, then* $\lim_{\lambda \to \infty} c_\lambda = 0$.

Proof. Since the functional I_λ has the Mountain Pass geometry, it follows that there exists $t_\lambda > 0$ satisfying $I_\lambda(t_\lambda v_0) = \max_{t \geq 0} I_\lambda(tv_0)$, where v_0 is the function given by Lemma 4.2, that does not depend of λ. Hence, from (2.3) we obtain

$$t_\lambda^2 \frac{1}{2}\max\{1, \frac{m_0 q}{2}\} \geq \lambda t_\lambda^q \int_\Omega |v_0|^q \, dx + t_\lambda^{2^{**}}\int_\Omega |v_0|^{2^{**}} \, dx \geq t_\lambda^{2^{**}}\int_\Omega |v_0|^{2^{**}} \, dx, \quad (4.1)$$

which implies that (t_λ) is bounded. Thus, there exists a sequence $\lambda_n \to +\infty$ and $\beta_0 \geq 0$ such that $t_{\lambda_n} \to \beta_0$ as $n \to +\infty$. Consequently, exists $D > 0$ such that

$$t^2_{\lambda_n} \frac{1}{2} \max\{1, \frac{m_0 q}{2}\} \leq D \quad \forall n \in \mathbb{N},$$

and so

$$t^q_{\lambda_n} \lambda_n \int_\Omega |v_0|^q \, dx + t^{2^{**}}_{\lambda_n} \int_\Omega |v_0|^{2^{**}} \leq D \quad \forall n \in \mathbb{N}.$$

If $\beta_0 > 0$, the above inequality leads to

$$\lim_{n\to\infty} \lambda_n t^q_{\lambda_n} \int_\Omega |v_0|^q \, dx + t^{2^{**}}_{\lambda_n} \int_\Omega |v_0|^{2^{**}} = +\infty,$$

which is a contradiction. Thus, we conclude that $\beta_0 = 0$. Now, let us consider the path $\gamma_*(t) = te$ for $t \in [0,1]$, to get the estimate

$$0 < c_\lambda \leq \max_{t\in[0,1]} I(\gamma_*(t)) = I(t_\lambda v_0) \leq C t^2_\lambda,$$

for some positive C. In this way, $\lim_{\lambda\to\infty} c_\lambda = 0$. □

Lemma 4.4. *Let $(u_n) \subset H$ be a sequence such that*

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0.$$

Then

$$\|u_n\|^2 \leq t_0, \quad \text{for all } n \in \mathbb{N} \text{ where } t_0 \text{ is given in } (2.1).$$

Proof. Assuming, by contradiction, that, up to a subsequence that $\|u_n\|^2 > t_0$. Thus, from (2.3) we obtain

$$c_\lambda = I_\lambda(u_n) - \frac{1}{q} I'_\lambda(u_n)u_n + o_n(1) \geq \frac{1}{2}\widehat{M}_0(\|u_n\|^2) - \frac{1}{q} M(t_0)\|u_n\|^2 + o_n(1).$$

Thus

$$c_\lambda \geq \big(\frac{1}{2}m_0 - \frac{1}{q}M(t_0)\big)\|u_n\|^2 + o_n(1). \tag{4.2}$$

Since $m_0 < M(t_0) < \frac{q}{2} m_0$, we obtain

$$c_\lambda \geq \big(\frac{1}{2}m_0 - \frac{1}{q}M(t_0)\big)t_0.$$

But this last inequality is in contradiction with Lemma 4.3. Hence (u_n) is bounded in H by constant $\sqrt{t_0}$. □

Proof of Theorem 1.2. From Lemma 4.3 we have $\lim_{\lambda\to+\infty} c_\lambda = 0$. Therefore, there exists $\lambda^{**} > 0$ such that

$$c_\lambda < \frac{2}{N} S^{\frac{N}{4}}, \tag{4.3}$$

for all $\lambda \geq \lambda^{**}$. Now, fix $\lambda \geq \lambda^{**}$ and let us to show that (2.4) admits a positive solution. From Lemmas 4.1 and 4.2, there exists a bounded sequence $(u_n) \subset H$ satisfying

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0.$$

Arguing as in Lemma 3.6 we conclude that $u_n \to u_\lambda$ in $L^{2^{**}}(\Omega)$. This convergence implies that $u_n \to u_\lambda$ in H. Thus, u_λ is a solution of (2.4). Moreover, by Lemma 4.4, u_λ is a solution of Problem (1.1) and from (4.2) and Lemma 4.3 we obtain

$$\lim_{\lambda\to+\infty} \|u_\lambda\| = 0.$$

Since for each solution u_λ we have that $M(\|u_\lambda\|^2) \geq m_0 > 0$ is a positive number, then the regularity of these solutions is a consequence of [9, Theorem 2.1]. □

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